Supplementary Problem on Root Approximation

In this problem you may assumed that if $-1 < M < 1$, then $\lim_{n\to\infty} M^n = 0$

(*a*) If $a < b$, let $g(x)$ be a continuous differentiable function defined on $[a, b]$. Suppose $y = g(x)$ satisfies the following 2 conditions:

(1) $a \le g(x) \le b$ for all x: $a \le x \le b$

(2) $|g'(x)| \le M < 1$ for all x: $a \le x \le b$

If $x_{n+1} = g(x_n)$, prove that $\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} x_n$ and $a \le \lim_{n\to\infty} g(x_n) \le b$.

(*b*) Let $g(x) = \frac{2}{x}$, $1 \le x \le 2$. Show that fixed-point method fails.

(*c*) Let $g(x) = \frac{1}{x^2} - \frac{10}{x}$, $3.1 \le x \le 3.2$. Show that fixed-point method also fails.

(*d*) Given that a root of the equation $x + \ln x = 0$ is α. Show that $0.4 \le \alpha \le 0.8$.

Here are three iterative formulae:

(*i*) $x_{r+1} = -\ln x_r$,

(*ii*) $x_{r+1} = e^{-x_r}$

(*iii*) $x_{r+1} = \frac{x_r + e^{-x_r}}{2}$

(α) Which of the formula can be used?

(β) Which formula is better? Use it to find the root correct to 3 decimal places.

Solution

(*a*) Let $h(x) = g(x) - x$

then $h(a) = g(a) - a \ge 0$

$h(b) = g(b) - b \ge 0$

$\therefore$ there is a root α: $a < \alpha < b$ such that $h(\alpha) = 0$

$\Rightarrow g(\alpha) = \alpha$

By mean value theorem, $g(x_n) - g(\alpha) = (x_n - \alpha)g'(c)$, $\alpha < c < x_n$

$|x_{n+1} - \alpha| = |g(x_n) - g(\alpha)|$

$= |x_n - \alpha||g'(c)|$

$|x_{n+1} - \alpha| \le |x_n - \alpha|M$

$\therefore |x_n - \alpha| \le |x_{n-1} - \alpha|M$

$|x_{n-1} - \alpha| \le |x_{n-2} - \alpha|M$

....

$\times) |x_1 - \alpha| \le |x_0 - \alpha|M$

$|x_n - \alpha| \le |x_0 - \alpha| M^n$

Take limit $n \to \infty$

$\lim_{n\to\infty} |x_n - \alpha| \le |x_0 - \alpha| \lim_{n\to\infty} M^n$

$\left|\lim_{n\to\infty}(x_n - \alpha)\right| \le |x_0 - \alpha| \cdot 0 = 0$

$\Rightarrow \lim_{n\to\infty} x_n = \alpha = g(\alpha) = g(\lim_{n\to\infty} x_n)$

$\Rightarrow \lim_{n\to\infty} g(x_n) = \alpha$ ($\because g$ is continuous.)

$\Rightarrow \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} x_n$

$\because a \le \alpha \le b$

$\therefore a \le \lim_{n\to\infty} g(x_n) \le b$.

(*b*) $g(x) = \frac{2}{x}$ $1 \le x \le 2$

$g'(x) = -\frac{2}{x^2} < 0 \Rightarrow g$ is strictly decreasing on $[1, 2]$

$g(1) = 2$, $g(2) = 1$, $\therefore 1 \le g(x) \le 2$

$\therefore$ condition (1) is satisfied.

$|g'(x)| = \dfrac{2}{x^2}$

$\dfrac{2}{4} \le |g'(x)| \le \dfrac{2}{1}$

$\Rightarrow \dfrac{1}{2} \le |g'(x)| \le 2$, In particular, $|g'(1.25)| = 1.28 > 1$

condition (2) fails.

(*c*) $g(x) = \dfrac{1}{x^2} - \dfrac{10}{x}$, $3.1 \le x \le 3.2$.

$g(3.1) = -3.12 \notin [3.1, 3.2]$

$g(3.2) = -3.027 \notin [3.1, 3.2]$

condition (1) fails

$g'(x) = -\dfrac{2}{x^3} + \dfrac{10}{x^2}$

$g''(x) = \dfrac{6}{x^4} - \dfrac{20}{x^3}$

Let $g''(x) = 0$

$\Rightarrow \dfrac{6}{x^4} - \dfrac{20}{x^3} = 0$

$\Rightarrow 6x^3 = 20x^4$

$\Rightarrow 6 = 20x$

$\Rightarrow x = 0.3$

$\because 3.1 \le x \le 3.2$

$\therefore g''(x) \ne 0 \ \forall x \in [3.1, 3.2]$

$g''(3.1) = -0.61 < 0$

$\therefore g'(x)$ is strictly decreasing

$g'(3.2) < g'(x) < g'(3.1)$

$g'(3.1) = 0.97$, $g'(3.2) = 0.92$

$\therefore 0.92 < g(x) < 0.97$

$\Rightarrow |g'(x)| \le 0.98 \ \forall x \in [3.1, 3.2]$

condition (2) is satisfied.

(*d*) $f(x) = x + \ln x$

$f(0.4) = -0.52 < 0$

$f(0.8) = 0.58 > 0$

$\therefore$ there is a root α: $0.4 < \alpha < 0.8$ such that $f(\alpha) = 0$

$f'(x) = 1 + \dfrac{1}{x}$

$\therefore f'(x) > 0 \ \forall x \in [0.4, 0.8]$

$\therefore f(x)$ is strictly increasing and so that root α is unique.

(α) (*i*) $x_{r+1} = -\ln x_r$

$g(x) = -\ln x$

$g'(x) = -\dfrac{1}{x}$

$g'(0.4) = -2.5$; $g'(0.8) = -1.25$

$\therefore$ condition (2) fails

(*ii*) $x_{r+1} = e^{-x_r}$

$g(x) = e^{-x}$

$g'(x) = -e^{-x} < 0 \Rightarrow g(x)$ is strictly decreasing

$g'(0.4) = -0.67$

$g'(0.8) = -0.45$

$g'(0.8) < g'(x) < g'(0.4)$ for x: $0.4 < x < 0.8$

$|g'(x)| \le 0.7 < 1$
condition (2) is satisfied.
$g(0.4) = 0.67$, $g(0.8) = 0.45$
$0.4 < g(x) < 0.8$ for all x: $0.4 < x < 0.8$
condition (1) is satisfied.
Fixed-point method can be applied.

(*iii*) $x_{r+1} = \dfrac{x_r + e^{-x_r}}{2}$

$g(x) = \dfrac{x + e^{-x}}{2}$

$g(0.4) = 0.54$; $g(0.8) = 0.62$

$g'(x) = \dfrac{1}{2}\left(1 - e^{-x}\right) > 0$ for all $x \in [0.4, 0.8]$

$g(x)$ is strictly increasing.
$g(0.4) \le g(x) \le g(0.8)$
$0.54 \le g(x) \le 0.62$
$0.4 \le g(x) \le 0.8$
condition (1) is satisfied.
$g'(0.4) = 0.16$; $g'(0.8) = 0.28$

$g''(x) = \dfrac{1}{2} e^{-x} > 0$ for all x

$g'(x)$ is strictly increasing on $[0.4, 0.8]$
$g'(0.4) \le g'(x) \le g'(0.8)$
$0.16 \le g'(x) \le 0.28$
$\therefore |g'(x)| \le 0.3 < 1$
condition (2) is satisfied
Fixed-point method can be applied.

(β) For formula (*ii*), $M = 0.7$; for formula (*iii*), $M = 0.3$.
$\therefore$ Formula (*iii*) is better.
Here is a comparison:

Let $x_0 = \dfrac{0.4 + 0.8}{2} = 0.6$

Formula (*ii*)			Formula (*iii*)	
n	$x_{r+1} = e^{-x_r}$		n	$x_{r+1} = \dfrac{x_r + e^{-x_r}}{2}$
0	0.6		0	0.6
1	0.5488		1	0.5744
2	0.5776		2	0.5687
3	0.5612		3	0.5674
4	0.5705		4	0.5672
5	0.5652			
6	0.5682			
7	0.5665			
8	0.5675			
9	0.5669			
10	0.5673			
converges to 0.567 (3 d.p.) in 10 steps			converges to 0.567 (3 d.p.) in 4 steps	