Bolzano's Theorem

Let $f:[a, b] \to \mathbb{R}$ be a continuous function. Suppose $a_1 < b_1 \in (a, b)$ with $f(a_1)\times f(b_1)<0$

then $\exists\, t \in (a, b)$ such that $f(t) = 0$

Proof: WLOG assume $a_1 < b_1$ and $f(a_1) < 0$, $f(b_1) > 0$

Let $E = \{x\in[a_1,b_1]: f(x) \le 0\}$

$\because a_1 \in E \quad \therefore E \ne \phi$

$\forall x \in E, x \le b_1 \quad \therefore$ E is bounded above.

$\therefore \sup E$ exists.

Let $t = \sup E$, the least upper bound of E.

claim 1 $t \ne b_1$

If $t= b_1$, then since $f(b_1) > 0$, $\exists\, \delta > 0$ such that $\forall\, x \in (b_1 - \delta, b_1] \Rightarrow f(x)>0$

In particular, let $x = b_1 - \dfrac{\delta}{2} \in (b_1 - \delta, b_1]$, then $f(x) > 0$ and $x < t$

$\therefore x$ is an upper bound for E, contradict that t is the least upper bound.

claim 2 $t < b_1$

claim 2.1 If $f(t) > 0$, then we shall prove that there is a contradiction.

$\exists\, \delta > 0$ s.t. $\forall\, x \in (t - \delta, t + \delta) \subset (a_1, b_1) \Rightarrow f(x) > 0$

In particular, let $x = t - \dfrac{\delta}{2} \in (t - \delta, t + \delta)$, then $f(x) > 0$and $x < t$

$\therefore x$ is an upper bound for E, contradict that t is the least upper bound.

claim 2.2 If $f(t) < 0$, then we shall prove that there is a contradiction.

claim 2.2.1 $t \ne a_1$

If $t = a_1$, $\exists\, \delta > 0$ s.t. $\forall\, x \in [a_1, a_1 + \delta) \Rightarrow f(x) < 0$

In particular, let $x = a_1 + \dfrac{\delta}{2}$, then $f(x) < 0$

but $t < x$, $\therefore t$ is not an upper bound, contradiction.

case 2.2.2 $a_1 < t$

$\exists\, \delta > 0$ s.t. $\forall\, x \in (t - \delta, t + \delta) \subset (a_1, b_1) \Rightarrow f(x) < 0$

In particular, let $x = t + \dfrac{\delta}{2} \in (t - \delta, t + \delta)$, then $f(x) < 0$

but $t < x$, $\therefore$ same contradiction.

Therefore claim 2.2 is proved.

From claim 2.1 and claim 2.2, it is impossible to have $f(t) > 0$ or $f(t) < 0$; therefore, $f(t) = 0$, where $t = \sup E$, the least upper bound of E.