Theorem

Let A, B, C be three angles such that $0 \le A, B, C \le \frac{\pi}{2}$ and $A + B + C = \frac{\pi}{2}$

Then $\sin A + \sin B + \sin C \le \frac{3}{2}$, equality holds when $A = B = C = \frac{\pi}{6}$.

Proof: $C = \frac{\pi}{2} - (A + B)$, $\sin C = \cos (A + B)$

$\sin A + \sin B + \sin C = \sin A + \sin B + \cos (A + B)$

$$= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} + 1 - 2\sin^2\frac{A+B}{2}$$

$$= 2\sin\frac{A+B}{2}\left(\cos\frac{A-B}{2} - \sin\frac{A+B}{2}\right) + 1$$

$$\le 2\left[\frac{\sin\frac{A+B}{2} + \cos\frac{A-B}{2} - \sin\frac{A+B}{2}}{2}\right]^2 + 1 \quad \because ab \le \left(\frac{a+b}{2}\right)^2$$

equality holds when $a = b$; i.e. $\cos\frac{A-B}{2} = 2\sin\frac{A+B}{2}$ …(1)

$$= \frac{1}{2}\cos^2\frac{A-B}{2} + 1$$

$$\le \frac{1}{2} + 1 = \frac{3}{2}$$, equality holds when $\frac{A-B}{2} = 0$, i.e. $A = B$ ……(2)

Combine (1) and (2), equality holds when $A = B = C = \frac{\pi}{6}$

Question: If $0 < x < y < \frac{\pi}{2}$, prove that $\sin x + \cos y - \sin (x - y) \le \frac{\pi}{2}$

Let $A = x$, $B = \frac{\pi}{2} - y$, $C = y - x$, then $0 \le A, B, C \le \frac{\pi}{2}$ and $A + B + C = \frac{\pi}{2}$

$\sin x + \cos y - \sin (x - y) = \sin x + \sin(\frac{\pi}{2} - y) + \sin(y - x)$

$= \sin A + \sin B + \sin C$

$\le \frac{3}{2}$ by the above theorem

$\le \frac{\pi}{2}$

The question is proved.