Supplementary Exercise on Trigonometry

1. Prove that $\cos^2(A+\theta)+\cos^2(B+\theta)+2\cos(A-B)\sin(A+\theta)\sin(B+\theta)$ is independent of θ.
2. Prove that, if $\sin(\alpha+\beta)=k\sin(\alpha-\beta)$, then $(k+1)\cot\alpha=(k-1)\cot\beta$.
3. If $x\sin\theta+y\cos\theta=\sin\varphi$ and $x\cos\theta-y\sin\theta=\cos\varphi$, express x in terms of θ and φ as simple as possible.
4. Without using tables or calculators, find the values of $\tan 15°$ and $\tan 22\frac{1}{2}°$.
5. If $\cos\theta+\cos\varphi=x$ and $\sin\theta+\sin\varphi=y$, prove that $\cos\frac{1}{2}(\theta-\varphi)=\pm\frac{1}{2}\sqrt{x^2+y^2}$.
6. Prove the identities: $\sin^2 A+\sin^2 B-\sin^2(A-B)=2\sin A\sin B\cos(A-B)$,
$$\frac{\tan 3A-2\tan 2A+\tan A}{4(\tan 3A-\tan 2A)}=\sin 2A.$$
7. In any triangle ABC, prove that
   (*a*) $b^2\sin(C-A)=(c^2-a^2)\sin B$,
   (*b*) $a^2-(b-c)^2\cos^2\frac{A}{2}=(b+c)^2\sin^2\frac{A}{2}$,
   (*c*) $\tan(\frac{A}{2}+B)=\frac{c+b}{c-b}\tan\frac{A}{2}$.
8. Solve completely the triangle ABC in which $a=2.718$, $b=3.142$, $A=54°18'$. Show that there are two possible triangles and find their areas.
9. If A, B, C are the angles of a triangle, prove that
$$(\sin B-\cos B)^2+(\sin C-\cos C)^2-(\sin A-\cos A)^2=1-4\sin A\cos B\cos C.$$
10. Given that $(1+\cos A)(1+\cos B)(1+\cos C)(1+\cos D)=p\sin A\sin B\sin C\sin D$,
prove that $(1-\cos A)(1-\cos B)(1-\cos C)(1-\cos D)=\frac{1}{p}\sin A\sin B\sin C\sin D$.
11. If A, B, C are the angles of a triangle, using sine rule to prove $\begin{cases}a=b\cos C+c\cos B\\ b=c\cos A+a\cos C\ ...(*)\\ c=a\cos B+b\cos A\end{cases}$
Hence, solve the system (*) and express $\cos A$, $\cos B$, $\cos C$ in terms of a, b and c.
12. If $\theta+\varphi=\frac{1}{4}\pi$, prove that $(1+\tan\theta)(1+\tan\varphi)=2$. Deduce the value of $\tan\frac{1}{8}\pi$.
13. Establish the identity $\sin\theta(\cos 2\theta+\cos 4\theta+\cos 6\theta)=\sin 3\theta\cos 4\theta$.
Prove that, if $x=\cos 3\theta+\sin 3\theta$ and $y=\cos\theta-\sin\theta$, then $x-y=2y\sin 2\theta$.
14. By expressing $(3+\cos\theta)\operatorname{cosec}\theta$ in terms of $\tan\frac{1}{2}\theta$ $(=t)$, show that this expression cannot have any value between $-2\sqrt{2}$ and $2\sqrt{2}$.
15. By projection of the sides of an equilateral triangle onto a certain line, or otherwise, prove that
$$\cos\theta+\cos(\theta+\tfrac{2}{3}\pi)+\cos(\theta+\tfrac{4}{3}\pi)=0,$$
and find the value of $\sin\theta+\sin(\theta+\frac{2}{3}\pi)+\sin(\theta+\frac{4}{3}\pi)$.
16. If $\tan\alpha=k\tan\beta$, show that $(k-1)\sin(\alpha+\beta)=(k+1)\sin(\alpha-\beta)$.
Show that, if the equation $\tan x=k\tan(x-\alpha)$ has real solutions (in x), $(k-1)^2$ is not less than $(k+1)^2\sin^2\alpha$.
Solve the equation when $k=-2$ and $\alpha=30°$, give your answer in general solution.
17. (*a*) If $\sin\alpha+\cos\alpha=2a$, form the quadratic equation whose roots are $\sin\alpha$ and $\cos\alpha$.
    (*b*) Solve the equation and find the general solution of x:
$$\cos^2 x+\cos x-\sin x-\sin^2 x=0.$$
18. Let $y=\sin\theta(3\sin\theta-\sin 2\alpha)+\cos\theta(3\cos\theta-\cos 2\alpha)$ $(0°<\alpha<90°)$.
    (*a*) Find the general solution of θ such that y has a minimum value and find this value.
    (*b*) Find the general solution of θ such that y has a maximum value and find this value.
    (*c*) Find also the maximum and minimum value of the expression
$$\sin\alpha(3\sin\alpha-\sin 2\theta)+\cos\alpha(3\cos\alpha-\cos 2\theta)\ (0°<\alpha<90°).$$

19. If α, β are two distinct roots of the equation $a\cos\theta + b\sin\theta = c$, prove that

$$\frac{a}{b}\sin(\alpha+\beta) - \cos(\alpha+\beta) = 1.$$

20. Prove the identity $\dfrac{\cos(2\theta+\varphi)+\cos(2\varphi+\theta)}{2\cos(\theta+\varphi)-1} = \dfrac{\cos(2\theta-\varphi)+\cos(2\varphi-\theta)}{2\cos(\theta-\varphi)-1}$.

21. Prove the identities

(*a*) $\sin^2(2\theta+\varphi) + \sin^2(2\varphi+\theta) - \sin^2(\theta-\varphi) = 2\cos(\theta-\varphi)\sin(2\theta+\varphi)\sin(2\varphi+\theta)$,

(*b*) $\tan(3A - \frac{3}{4}\pi)\tan(A + \frac{1}{4}\pi) = \dfrac{1+2\sin 2A}{1-2\sin 2A}$,

(*c*) $(\cos A + \cos B + \cos C)^2 + (\sin A + \sin B + \sin C)^2 = 1 + 8\cos\frac{1}{2}(B-C)\cos\frac{1}{2}(C-A)\cos\frac{1}{2}(A-B)$,

(*d*) $\dfrac{\tan 3A}{\tan A} = \dfrac{2\cos 2A+1}{2\cos 2A-1}$,

(*e*) $\sin^2(A+\theta) + \sin^2(B+\theta) = 1 - \cos(A-B)\cos(A+B+2\theta)$.

1. $\cos^2(A+\theta) + \cos^2(B+\theta) + 2\cos(A-B)\sin(A+\theta)\sin(B+\theta)$

$= \frac{1}{2}[1+\cos 2(A+\theta)] + \frac{1}{2}[1+2\cos 2(B+\theta)] - \cos(A-B)[\cos(A+B+2\theta) - \cos(A-B)]$

$= 1 + \frac{1}{2}[\cos 2(A+\theta) + \cos 2(B+\theta)] - \cos(A-B)\cos(A+B+2\theta) + \cos^2(A-B)$

$= 1 + \cos(A+B+2\theta)\cos(A-B) - \cos(A-B)\cos(A+B+2\theta) + \cos^2(A-B)$

$= 1 + \cos^2(A-B)$

After simplification, the expression does not carry θ, which is independent of θ.

2. $\sin(\alpha+\beta) = k\sin(\alpha-\beta)$, then $(k+1)\cot\alpha = (k-1)\cot\beta$.

$\sin\alpha\cos\beta + \cos\alpha\sin\beta = k\sin\alpha\cos\beta - k\cos\alpha\sin\beta$

$k\cos\alpha\sin\beta + \cos\alpha\sin\beta = k\sin\alpha\cos\beta - \sin\alpha\cos\beta$

$(k+1)\cos\alpha\sin\beta = (k-1)\sin\alpha\cos\beta$

$(k+1)\cot\alpha = (k-1)\cot\beta$

3. $x\sin\theta + y\cos\theta = \sin\varphi$ ....(1)

$x\cos\theta - y\sin\theta = \cos\varphi$ ....(2)

(1) × sin θ: $x\sin^2\theta + y\sin\theta\cos\theta = \sin\theta\sin\varphi$

+) (2) × cos θ: $x\cos^2\theta - y\sin\theta\cos\theta = \cos\theta\cos\varphi$

$x = \sin\theta\sin\varphi + \cos\theta\cos\varphi$

$= \cos(\varphi - \theta)$

4. Using the formula $\tan x = \dfrac{\sin 2x}{1+\cos 2x}$ (proof: $\dfrac{\sin 2x}{1+\cos 2x} = \dfrac{2\sin x\cos x}{1+2\cos^2 x - 1} = \tan x$)

$$\tan 15^\circ = \frac{\sin 30^\circ}{1+\cos 30^\circ} = \frac{\frac{1}{2}}{1+\frac{\sqrt{3}}{2}} = \frac{1}{2+\sqrt{3}} = \frac{1}{2+\sqrt{3}}\cdot\frac{2-\sqrt{3}}{2-\sqrt{3}} = 2-\sqrt{3}$$

$$\tan 22\frac{1}{2}^\circ = \frac{\sin 45^\circ}{1+\cos 45^\circ} = \frac{\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}+1} = \frac{1}{\sqrt{2}+1}\cdot\frac{\sqrt{2}-1}{\sqrt{2}-1} = \sqrt{2}-1$$

Alternatively, using the formula $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

$\tan 30^\circ = \dfrac{2\tan 15^\circ}{1-\tan^2 15^\circ}$

$\dfrac{1}{\sqrt{3}} = \dfrac{2\tan 15^\circ}{1-\tan^2 15^\circ}$

$1 - \tan^2 15^\circ = 2\sqrt{3}\tan 15^\circ$

$\tan^2 15^\circ + 2\sqrt{3}\tan 15^\circ - 1 = 0$

$\tan 15^\circ = -\sqrt{3} \pm \sqrt{\left(\sqrt{3}\right)^2 + 1}$

$= -\sqrt{3} + 2$ or $-\sqrt{3} - 2$

$\because 0^\circ < 15^\circ < 90^\circ \therefore \tan 15^\circ > 0 \Rightarrow -\sqrt{3} - 2$ is rejected

$\tan 15^\circ = 2 - \sqrt{3}$

$\tan 22\frac{1}{2}^\circ$ can be found in a similar way.

5. $\cos\theta + \cos\varphi = x$ ....(1)

$\sin\theta + \sin\varphi = y$ ....(2)

$(1)^2$ $\cos^2\theta + \cos^2\varphi + 2\cos\theta\cos\varphi = x^2$

+) $(2)^2$ $\sin^2\theta + \sin^2\varphi + 2\sin\theta\sin\varphi = y^2$

$1 + 1 + 2\cos(\theta - \varphi) = x^2 + y^2$

$2[1 + \cos(\theta - \varphi)] = x^2 + y^2$

$2[1 + 2\cos^2\frac{1}{2}(\theta-\varphi) - 1] = x^2 + y^2$

$4\cos^2\frac{1}{2}(\theta-\varphi) = x^2 + y^2$

$\cos\frac{1}{2}(\theta-\varphi) = \pm\frac{1}{2}\sqrt{x^2+y^2}$

6. RHS = $2\sin A\sin B\cos(A-B)$

$= -[\cos(A+B) - \cos(A-B)]\cos(A-B)$

$= -\cos(A+B)\cos(A-B) + \cos^2(A-B)$

$= -\frac{1}{2}(\cos 2A + \cos 2B) + 1 - \sin^2(A-B)$

$= -\frac{1}{2}(1 - 2\sin^2 A + 1 - 2\sin^2 B) + 1 - \sin^2(A-B)$

$= \sin^2 A + \sin^2 B - \sin^2(A-B)$ = LHS

You may try to prove the identity from the left side.

LHS $= \dfrac{\tan 3A - 2\tan 2A + \tan A}{4(\tan 3A - \tan 2A)}$

$= \dfrac{1}{4} - \dfrac{\tan 2A - \tan A}{4(\tan 3A - \tan 2A)}$

$= \dfrac{1}{4} - \dfrac{\frac{\sin(2A-A)}{\cos 2A\cos A}}{\frac{4\sin(3A-2A)}{\cos 3A\cos 2A}}$ (note that $\tan\alpha - \tan\beta = \dfrac{\sin\alpha}{\cos\alpha} - \dfrac{\sin\beta}{\cos\beta} = \dfrac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta}$

$= \dfrac{1}{4} - \dfrac{\sin A\cos 3A}{4\sin A\cos A}$ $\qquad= \dfrac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}$)

$= \dfrac{1}{4} - \dfrac{4\cos^3 A - 3\cos A}{4\cos A}$

$= \frac{1}{4}[1 - (4\cos^2 A - 3)]$

$= \frac{1}{4}(4 - 4\cos^2 A)$

$= \sin^2 A$ = RHS

7. In ΔABC, $A + B + C = 180°$, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k \Rightarrow a = k\sin A$, $b = k\sin B$, $c = k\sin C$.

(*a*) $\dfrac{b^2}{c^2 - a^2} = \dfrac{(k\sin B)^2}{(k\sin C)^2 - (k\sin A)^2}$

$= \dfrac{\sin^2 B}{\sin^2 C - \sin^2 A}$

$= \dfrac{\sin^2 B}{(\sin C - \sin A)(\sin C + \sin A)}$

$= \dfrac{\sin^2 B}{2\cos\frac{C+A}{2}\sin\frac{C-A}{2}\cdot 2\sin\frac{A+C}{2}\cos\frac{A-C}{2}}$

$= \dfrac{\sin^2 B}{2\sin\frac{A+C}{2}\cos\frac{A+C}{2}\cdot 2\sin\frac{C-A}{2}\cos\frac{C-A}{2}}$

$= \dfrac{\sin^2 B}{\sin(A+C)\sin(C-A)}$

$$=\frac{\sin^2 B}{\sin(180^\circ - B)\sin(C-A)}$$

$$=\frac{\sin^2 B}{\sin B\sin(C-A)}$$

$$=\frac{\sin B}{\sin(C-A)}$$. Hence result follows.

(*b*) $a^2-(b-c)^2\cos^2\frac{1}{2}A$

$$= a^2-(b-c)^2\cdot\frac{1+\cos A}{2}$$

$$=\frac{1}{2}\left[2b^2+2c^2-4bc\cos A-\left(b^2-2bc+c^2\right)\left(1+\cos A\right)\right]$$, by cosine rule

$$=\frac{1}{2}\left[2b^2+2c^2-4bc\cos A-\left(b^2-2bc+c^2+b^2\cos A-2bc\cos A+c^2\cos A\right)\right]$$

$$=\frac{1}{2}\left[b^2+2bc+c^2-\left(b^2\cos A+2bc\cos A+c^2\cos A\right)\right]$$

$$=\frac{1}{2}\left[(b+c)^2-(b+c)^2\cos A\right]$$

$$=(b+c)^2\frac{1-\cos A}{2}$$

$$=(b+c)^2\sin^2\frac{1}{2}A$$

(*c*) $\frac{c+b}{c-b}\tan\frac{A}{2}$

$$=\frac{k\sin C+k\sin B}{k\sin C-k\sin B}\cdot\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}$$

$$=\frac{2\sin\frac{B+C}{2}\cos\frac{C-B}{2}}{2\cos\frac{B+C}{2}\sin\frac{C-B}{2}}\cdot\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}$$

$$=\frac{2\cos\frac{A}{2}\cos\frac{C-B}{2}}{2\sin\frac{A}{2}\sin\frac{C-B}{2}}\cdot\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}$$, note that $\sin\frac{B+C}{2}=\sin\frac{180^\circ-A}{2}=\cos\frac{A}{2}$,

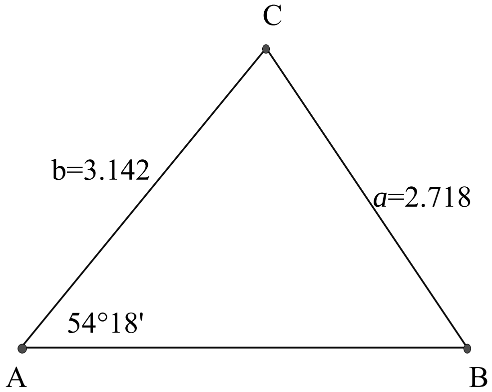
$$=\frac{1}{\tan\frac{C-B}{2}}$$ $\cos\frac{B+C}{2}$ can be simplified similarly.

$$=\frac{1}{\tan\frac{180^\circ-A-B-B}{2}}$$

$$=\frac{1}{\tan\left(90^\circ-\frac{A}{2}-B\right)}$$

$= \tan(\frac{1}{2}A+B)$, this is known as tangent rule.

8. $a = 2.718$, $b = 3.142$, $A = 54°18'$: SSA



By cosine rule $a^2 = b^2 + c^2 - 2bc\cos A$

$2.718^2 = 3.142^2 + c^2 - 2\times 3.142\, c\cos 54°18'$

$c^2 - 3.666973\, c + 2.48464 = 0$

$c = 2.77$ or 0.90 ($B = 69.8°$ or $110.2°$, $C = 55.9°$ or $15.6°$)

So, there are two possible triangles.

$c = 2.77$, area $= \dfrac{1}{2} bc\sin A = \dfrac{1}{2}\times 3.142\times 2.77\sin 54.3° = 3.53$

$c = 0.90$, area $= \dfrac{1}{2}\times 3.142\times 0.90\sin 54.3° = 1.14$

9. $(\sin B - \cos B)^2 + (\sin C - \cos C)^2 - (\sin A - \cos A)^2$

$= \sin^2 B - 2\sin B\cos B + \cos^2 B + \sin^2 C - 2\sin C\cos C + \cos^2 B - \sin^2 A + 2\sin A\cos A - \cos^2 A$

$= 1 - \sin 2B + 1 - 2\sin C\cos C - (1 - \sin 2A)$

$= 1 + \sin 2A - \sin 2B - 2\sin C\cos C$

$= 1 + 2\cos(A + B)\sin(A - B) - 2\sin C\cos C$

$= 1 - 2\cos C\sin(A - B) - 2\sin C\cos C$

$= 1 - 2\cos C[\sin(A - B) + \sin C]$

$= 1 - 4\cos C\sin\dfrac{A - B + C}{2}\cos\dfrac{A - B - C}{2}$

$= 1 - 4\cos C\sin(90° - B)\cos(A - 90°)$

$= 1 - 4\sin A\cos B\cos C.$

10. Given that $(1 + \cos A)(1 + \cos B)(1 + \cos C)(1 + \cos D) = p\sin A\sin B\sin C\sin D$

$(1 - \cos A)(1 - \cos B)(1 - \cos C)(1 - \cos D)$

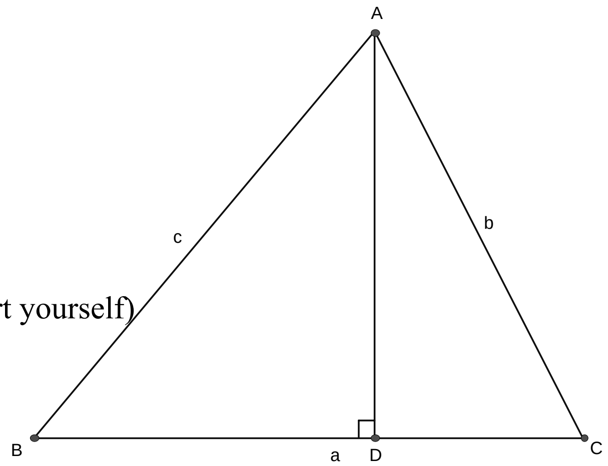
$= \dfrac{(1-\cos A)(1-\cos B)(1-\cos C)(1-\cos D)(1+\cos A)(1+\cos B)(1+\cos C)(1+\cos D)}{(1+\cos A)(1+\cos B)(1+\cos C)(1+\cos D)}$

$= \dfrac{(1-\cos^2 A)(1-\cos^2 B)(1-\cos^2 C)(1-\cos^2 D)}{p\sin A\sin B\sin C\sin D}$

$= \dfrac{\sin^2 A\sin^2 B\sin^2 C\sin^2 D}{p\sin A\sin B\sin C\sin D}$

$= \dfrac{1}{p}\sin A\sin B\sin C\sin D.$

11. In the triangle on the right,



$a = BC = BD + DC = c\cos B + b\cos C$ ....(1)

$b = AC = AE + EC = c\cos A + a\cos C$ ....(2)

Similarly $c = a\cos B + b\cos A$ ....(3) (do this part yourself)

These equations are known as projection formulae.

In (2) $\cos C = \dfrac{b - c\cos A}{a}$ ....(4)

In (3) $\cos B = \dfrac{c - b\cos A}{a}$ ....(5)

Put (4) and (5) into (1)

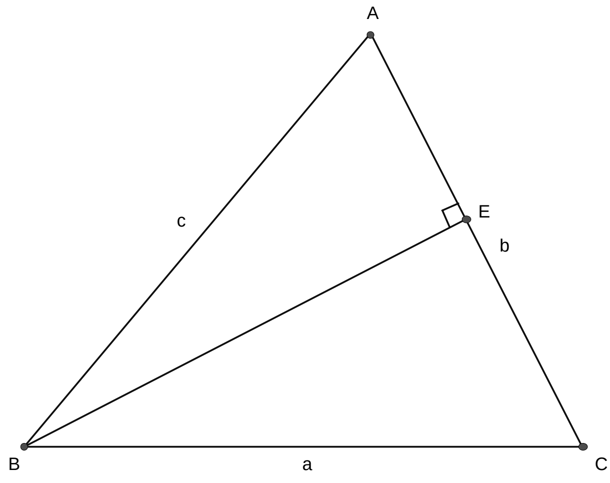
$a = c\cdot\dfrac{c - b\cos A}{a} + b\cdot\dfrac{b - c\cos A}{a}$

$\Rightarrow a^2 = c^2 + b^2 - 2bc\cos A$

Similarly $b^2 = a^2 + c^2 - 2ac\cos B$

$c^2 = a^2 + b^2 - 2ab\cos C$

(Do the substitution yourself.)



12. $\theta + \varphi = \frac{1}{4}\pi$, $\tan(\theta + \varphi) = \tan \frac{1}{4}\pi = 1$

$\frac{\tan\theta + \tan\varphi}{1 - \tan\theta\tan\varphi} = 1$

$\tan\theta + \tan\varphi = 1 - \tan\theta\tan\varphi$

$1 + \tan\theta + \tan\varphi + \tan\theta\tan\varphi = 2$

$(1 + \tan\theta)(1 + \tan\varphi) = 2$

Let $\theta = \varphi = \frac{1}{8}\pi$, then $\theta + \varphi = \frac{1}{4}\pi$

By the above result, $(1 + \tan\theta)(1 + \tan\varphi) = 2$

$\Rightarrow (1 + \tan\theta)^2 = 2$

$1 + \tan\theta = \pm\sqrt{2}$

$\tan \frac{1}{8}\pi = \sqrt{2} - 1$ (reject $-\sqrt{2} - 1$)

13. $\sin\theta(\cos 2\theta + \cos 4\theta + \cos 6\theta)$

$= \sin\theta(\cos 4\theta + 2\cos 4\theta\cos 2\theta)$

$= \cos 4\theta \sin\theta(2\cos 2\theta + 1)$

$= \cos 4\theta\,[2(1 - 2\sin^2\theta) + 1]\sin\theta$

$= \cos 4\theta(3\sin\theta - 4\sin^3\theta)$

$= \sin 3\theta\cos 4\theta$

if $x = \cos 3\theta + \sin 3\theta$ and $y = \cos\theta - \sin\theta$,

$x - y = \cos 3\theta + \sin 3\theta - \cos\theta + \sin\theta$

$= 4\cos^3\theta - 3\cos\theta + 3\sin\theta - 4\sin^3\theta - \cos\theta + \sin\theta$

$= 4\cos^3\theta - 4\cos\theta + 4\sin\theta - 4\sin^3\theta$

$= 4\cos\theta(\cos^2\theta - 1) + 4\sin^2\theta(1 - \sin^2\theta)$

$= -4\cos\theta\sin^2\theta + 4\sin\theta\cos^2\theta$

$= 4\sin\theta\cos\theta(\cos\theta - \sin\theta)$

$= 2y\sin 2\theta$.

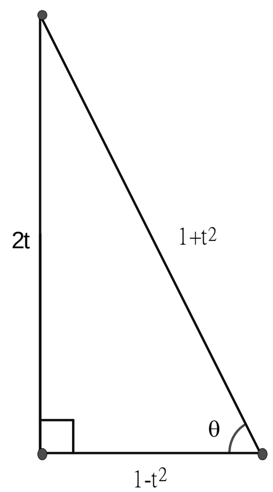
14. Let $E = (3 + \cos\theta)\operatorname{cosec}\theta$

$= (3 + \frac{1-t^2}{1+t^2})\frac{1+t^2}{2t}$

$= \frac{3 + 3t^2 + 1 - t^2}{2t}$

$= \frac{4 + 2t^2}{2t}$

$= \frac{2 + t^2}{t}$



$Et = 2 + t^2$

$t^2 - Et + 2 = 0$

$\because t = \tan\frac{\theta}{2}$ can be any real number, $\Delta \geq 0$

$E^2 - 4\times 2 \geq 0$

$(E + 2\sqrt{2})(E - 2\sqrt{2}) \geq 0$

$\Rightarrow E \leq -2\sqrt{2}$ or $E \geq 2\sqrt{2}$

This expression cannot have any value between $-2\sqrt{2}$ and $2\sqrt{2}$.

15. ΔPQR is an equilateral with QR inclined to the horizontal at an angle θ.

Suppose $PQ = QR = RP = 1$ unit.

The projection of QR on horizontal is MT, the projection of PQ on horizontal is MN, the projection of PR on horizontal is NT.

$MT = QR\cos\theta = \cos\theta$

$MN = PQ\cos(\frac{1}{3}\pi + \theta) = \cos(\frac{1}{3}\pi + \theta)$

$NT = PR\cos(\frac{1}{3}\pi - \theta)$

$\because MT = MN + NT$

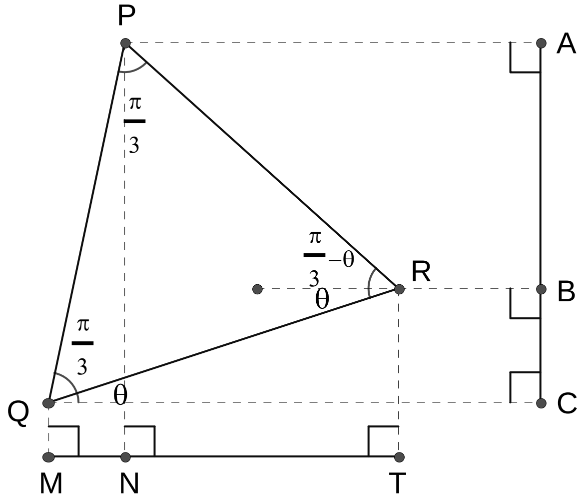
$\cos\theta = \cos(\frac{1}{3}\pi + \theta) + \cos(\frac{1}{3}\pi - \theta)$

$\cos\theta - \cos(\frac{1}{3}\pi + \theta) - \cos(\frac{1}{3}\pi - \theta) = 0$

$\cos\theta + \cos(\frac{1}{3}\pi + \theta + \pi) + \cos(\frac{1}{3}\pi - \theta - \pi) = 0$

$\cos\theta + \cos(\frac{4}{3}\pi + \theta) + \cos(-\frac{2}{3}\pi - \theta) = 0$

$\cos\theta + \cos(\theta + \frac{2}{3}\pi) + \cos(\theta + \frac{4}{3}\pi) = 0$



Using the projection of ΔPQR onto the vertical,

$AB = PR\sin(\frac{1}{3}\pi - \theta) = \sin(\frac{1}{3}\pi - \theta)$

$BC = QR\sin\theta = \sin\theta$

$AC = PQ\sin(\frac{1}{3}\pi + \theta) = \sin(\frac{1}{3}\pi + \theta)$

$AB + BC = AC$

$\sin(\frac{1}{3}\pi - \theta) + \sin\theta = \sin(\frac{1}{3}\pi + \theta)$

$\sin(\frac{1}{3}\pi - \theta) + \sin\theta - \sin(\frac{1}{3}\pi + \theta) = 0$

$\sin\theta + \sin[\pi - (\frac{1}{3}\pi - \theta)] + \sin(\frac{1}{3}\pi + \theta + \pi) = 0$

$\sin\theta + \sin(\theta + \frac{2}{3}\pi) + \sin(\theta + \frac{4}{3}\pi) = 0$

16. $\tan\alpha = k\tan\beta$

$\frac{\sin\alpha}{\cos\alpha} = \frac{k\sin\beta}{\cos\beta}$

$\sin\alpha\cos\beta = k\sin\beta\cos\alpha$

$\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)] = \frac{k}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$

$\sin(\alpha - \beta) + k\sin(\alpha - \beta) = k\sin(\alpha + \beta) - \sin(\alpha + \beta)$

$(k - 1)\sin(\alpha + \beta) = (k + 1)\sin(\alpha - \beta)$

Given $\tan x = k\tan(x - \alpha)$

$(k - 1)\sin(2x - \alpha) = (k + 1)\sin\alpha$

$\sin(2x-\alpha) = \frac{(k+1)}{(k-1)}\sin\alpha$

It has real solutions (in x) $\Rightarrow -1 \le \frac{(k+1)}{(k-1)}\sin\alpha \le 1$

$\frac{(k+1)^2}{(k-1)^2}\sin^2\alpha \le 1$

$(k + 1)^2\sin^2\alpha \le (k - 1)^2$

$(k - 1)^2$ is not less than $(k + 1)^2\sin^2\alpha$.

When $k = -2$ and $\alpha = 30°$,

$(-2 - 1)\sin(2x - 30°) = (-2 + 1)\sin 30°$

$\sin(2x - 30°) = \frac{1}{6}$

$2x - 30° = 180°n + (-1)^n\, 9.594°$

$2x = 180°n + (-1)^n\, 9.594° + 30°$

$x = 90°n + (-1)^n\, 4.797° + 15°$, $n = 0, \pm1, \pm2, \pm3$, ...

17. (*a*) If $\sin\alpha + \cos\alpha = 2a$

squaring $\sin^2\alpha + \cos^2\alpha + 2\sin\alpha\cos\alpha = 4a^2$

$2\sin\alpha\cos\alpha = 4a^2 - 1$

$\sin\alpha\cos\alpha = 2a^2 - \frac{1}{2}$

sum of roots = $2a$, product of roots = $2a^2 - \frac{1}{2}$

quadratic equation whose roots are $\sin\alpha$, $\cos\alpha$ is $x^2 - 2ax + 2a^2 - \frac{1}{2} = 0$

$2x^2 - 4ax + 4a^2 - 1 = 0$

(*b*) $\cos^2 x + \cos x - \sin x - \sin^2 x = 0$
$\cos^2 x - \sin^2 x + \cos x - \sin x = 0$
$(\cos x + \sin x)(\cos x - \sin x) + \cos x - \sin x = 0$
$(\cos x - \sin x)(\cos x + \sin x + 1) = 0$
$\cos x - \sin x = 0$ or $\cos x + \sin x = -1$
$\tan x = 1$ or $\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x = -\frac{1}{\sqrt{2}}$
$x = 180^\circ n + 45^\circ$ or $\cos(x - 45^\circ) = \cos 135^\circ$
$x = 180^\circ n + 45^\circ$ or $x - 45^\circ = 360^\circ n \pm 135^\circ$
$x = 180^\circ n + 45^\circ$ or $x = 360^\circ n + 180^\circ$ or $360^\circ n - 90^\circ$, $n = 0, \pm1, \pm2, \pm3, \ldots\ldots$

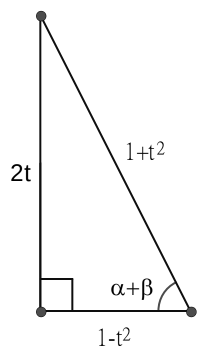
18. Let $y = \sin\theta(3\sin\theta - \sin 2\alpha) + \cos\theta(3\cos\theta - \cos 2\alpha)$ $(0^\circ < \alpha < 90^\circ)$.
$= 3\sin^2\theta - \sin\theta\sin 2\alpha + 3\cos^2\theta - \cos\theta\cos 2\alpha$
$= 3 - (\sin\theta\sin 2\alpha + \cos\theta\cos 2\alpha)$
$= 3 - \cos(\theta - 2\alpha)$

(*a*) when y has a minimum value, $\cos(\theta - 2\alpha) = 1$
$\theta - 2\alpha = 360^\circ n$
$\theta = 360^\circ n + 2\alpha$, $n = 0, \pm1, \pm2, \pm3, \ldots\ldots$

(*b*) when y has a maximum value, $\cos(\theta - 2\alpha) = -1$
$\theta - 2\alpha = 360^\circ n + 180^\circ$
$\theta = 360^\circ n + 180^\circ + 2\alpha$, $n = 0, \pm1, \pm2, \pm3, \ldots\ldots$

(*c*) Interchange the role of α and θ, we have
$\sin\alpha(3\sin\alpha - \sin 2\theta) + \cos\alpha(3\cos\alpha - \cos 2\theta) = 3 - \cos(\alpha - 2\theta)$
maximum value is $3 - (-1) = 4$
minimum value is $3 - 1 = 2$

19. $a\cos\theta + b\sin\theta = c$, α, β are two distinct roots.
$a\cos\alpha + b\sin\alpha = c$ ....(1)
$a\cos\beta + b\sin\beta = c$ ....(2)
(1) = (2) $a\cos\alpha + b\sin\alpha = a\cos\beta + b\sin\beta$
$a(\cos\alpha - \cos\beta) = b(\sin\beta - \sin\alpha)$
$-2a\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} = -2b\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$
$\because \alpha, \beta$ are distinct $\therefore \sin\frac{\alpha-\beta}{2} \neq 0$
$a\sin\frac{\alpha+\beta}{2} = b\cos\frac{\alpha+\beta}{2}$
$\tan\frac{\alpha+\beta}{2} = \frac{b}{a}$



Using the formula for half angle: $t = \tan\frac{\alpha+\beta}{2}$

$$\frac{a}{b}\sin(\alpha+\beta) - \cos(\alpha+\beta) = \frac{a}{b}\cdot\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}$$
$$= \frac{a}{b}\cdot\left[\frac{2\left(\frac{b}{a}\right)}{1+\left(\frac{b}{a}\right)^2}\right] - \frac{1-\left(\frac{b}{a}\right)^2}{1+\left(\frac{b}{a}\right)^2}$$
$$= \frac{a}{b}\cdot\frac{2ab}{a^2+b^2} - \frac{a^2-b^2}{a^2+b^2}$$
$$= \frac{2a^2 - a^2 + b^2}{a^2+b^2}$$
$$= 1$$

20. $\dfrac{\cos(2\theta+\varphi)+\cos(2\varphi+\theta)}{2\cos(\theta+\varphi)-1}=\dfrac{\cos(2\theta-\varphi)+\cos(2\varphi-\theta)}{2\cos(\theta-\varphi)-1}$ is equivalent to

$[2\cos(\theta-\varphi)-1][\cos(2\theta+\varphi)+\cos(2\varphi+\theta)] = [2\cos(\theta+\varphi)-1][\cos(2\theta-\varphi)+\cos(2\varphi-\theta)]$

LHS $= 2\cos(2\theta+\varphi)\cos(\theta-\varphi) + 2\cos(2\varphi+\theta)\cos(\theta-\varphi) - [\cos(2\theta+\varphi)+\cos(2\varphi+\theta)]$

$= \cos 3\theta + \cos(\theta+2\varphi) + \cos(2\theta+\varphi) + \cos 3\varphi - \cos(2\theta+\varphi) - \cos(2\varphi+\theta)$

$= \cos 3\theta + \cos 3\varphi$

RHS $= 2\cos(2\theta-\varphi)\cos(\theta+\varphi) + 2\cos(2\varphi-\theta)\cos(\theta+\varphi) - [\cos(2\theta-\varphi)+\cos(2\varphi-\theta)]$

$= \cos 3\theta + \cos(\theta-2\varphi) + \cos 3\varphi + \cos(\varphi-2\theta) - \cos(2\theta-\varphi) - \cos(2\varphi-\theta)$

$= \cos 3\theta + \cos 3\varphi$

$\therefore$ LHS = RHS

21. (*a*) $2\cos(\theta-\varphi)\sin(2\theta+\varphi)\sin(2\varphi+\theta)$

$= -\cos(\theta-\varphi)[\cos(3\theta+3\varphi)-\cos(\theta-\varphi)]$

$= -\cos(\theta-\varphi)\cos(3\theta+3\varphi)+\cos^2(\theta-\varphi)$

$= -\dfrac{1}{2}[\cos(4\theta+2\varphi)+\cos(2\theta+4\varphi)] + 1 - \sin^2(\theta-\varphi)$

$= -\dfrac{1}{2}[1-2\sin^2(2\theta+\varphi)+1-2\sin^2(\theta+2\varphi)] + 1 - \sin^2(\theta-\varphi)$

$= \sin^2(2\theta+\varphi)+\sin^2(2\varphi+\theta)-\sin^2(\theta-\varphi)$ (similar to Q6(*a*))

(*b*) $\tan(3A-\tfrac{3}{4}\pi)\tan(A+\tfrac{1}{4}\pi)$

$=\dfrac{\sin\left(3A-\frac{3\pi}{4}\right)\sin\left(A+\frac{\pi}{4}\right)}{\cos\left(3A-\frac{3\pi}{4}\right)\cos\left(A+\frac{\pi}{4}\right)}$

$=\dfrac{-\frac{1}{2}\left[\cos\left(4A-\frac{\pi}{2}\right)-\cos(2A-\pi)\right]}{\frac{1}{2}\left[\cos\left(4A-\frac{\pi}{2}\right)+\cos(2A-\pi)\right]}$

$=\dfrac{-(\sin 4A+\cos 2A)}{\sin 4A-\cos 2A}$

$=-\dfrac{2\sin 2A\cos 2A+\cos 2A}{2\sin 2A\cos 2A-\cos 2A}$

$=-\dfrac{2\sin 2A+1}{2\sin 2A-1}$

$=\dfrac{1+2\sin 2A}{1-2\sin 2A}$

(*c*) $(\cos A+\cos B+\cos C)^2+(\sin A+\sin B+\sin C)^2$

$= \cos^2 A+\cos^2 B+\cos^2 C+\sin^2 A+\sin^2 B+\sin^2 C+2(\cos A\cos B+\sin A\sin B$
$+\cos B\cos C+\sin B\sin C+\cos C\cos A+\sin A\sin C)$

$= 3+2[\cos(A-B)+\cos(B-C)+\cos(C-A)]$

$= 3+4\cos\dfrac{A-C}{2}\cos\dfrac{A-2B+C}{2}+4\cos^2\dfrac{C-A}{2}-2$

$= 1+4\cos\dfrac{C-A}{2}\left[\cos\dfrac{A-2B+C}{2}+\cos\dfrac{C-A}{2}\right]$

$= 1+8\cos\dfrac{C-A}{2}\cos\dfrac{C-B}{2}\cos\dfrac{A-B}{2}$ = RHS

(*d*) $\dfrac{\tan 3A}{\tan A}$

$=\dfrac{\sin 3A}{\cos 3A}\cdot\dfrac{\cos A}{\sin A}$

$=\dfrac{3\sin A-4\sin^3 A}{4\cos^3 A-3\cos A}\cdot\dfrac{\cos A}{\sin A}$

$=\dfrac{3-4\sin^2 A}{4\cos^2 A-3}$

$= \dfrac{3 - 2(1 - \cos 2A)}{2(1 + \cos 2A) - 3}$

$= \dfrac{2\cos 2A + 1}{2\cos 2A - 1}$,

(*e*) $\sin^2(A + \theta) + \sin^2(B + \theta)$

$= \dfrac{1}{2}[1 - \cos(2A + 2\theta)] + \dfrac{1}{2}[1 - \cos(2B + 2\theta)]$

$= 1 - \dfrac{1}{2}[\cos(2A + 2\theta) + \cos(2B + 2\theta)]$

$= 1 - \cos(A - B)\cos(A + B + 2\theta)$