

Euler's Formula on the Graph Theory

Let G be a connected plane graph with V vertices, E edges and F faces, then $V - E + F = 2$.

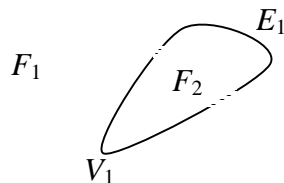
Proof: Case 1: Suppose G has no cycle. Hence G is a tree.

$E = V - 1$ (every vertices are connected.) $F = 1$ (G has no cycle.)

$$V - E + F = V - (V - 1) + 1 = 2$$

Case 2: Suppose G has a cycle, use induction on the number of edges.

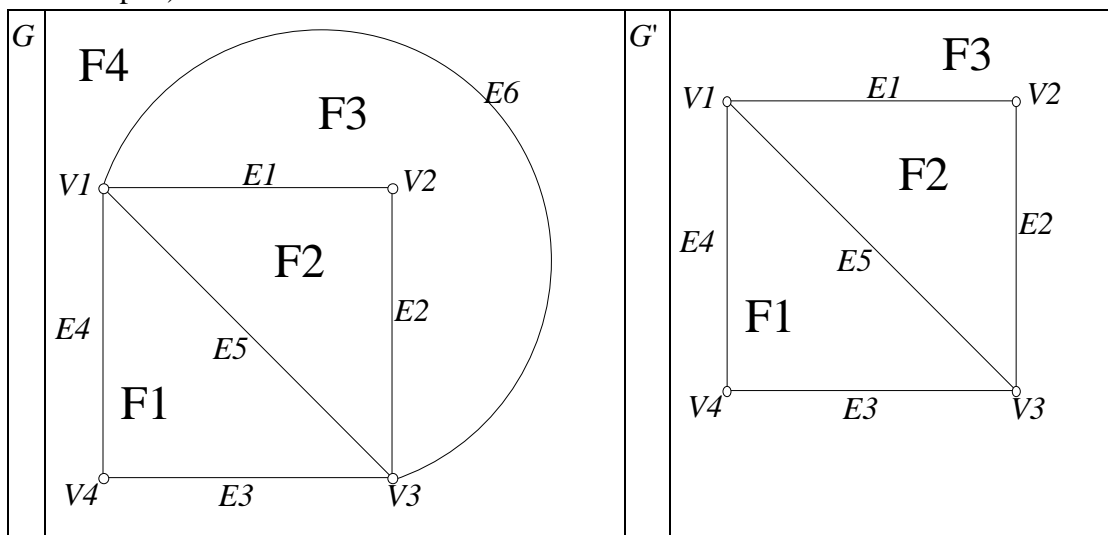
$$E = 1 \Rightarrow V = 1, F = 2 \Rightarrow V - E + F = 1 - 1 + 2 = 2.$$



It is true for $E = 1$

Suppose the formula to true for some positive integer k ($E = k$ edges).

For a connected plane graph with $k + 1$ edges (with at least one cycle). Remove an edge from this cycle to obtain a new graph called G' . (The following graph is an example.)



G' is a connected plane graph with one edge less. ($E' = k$)

$$V' = V, F' = F - 1$$

$V' - E' + F' = 2$ (by induction assumption)

$$V - k + (F - 1) = 2$$

$$V - (k + 1) + F = 2$$

$$V - E + F = 2$$

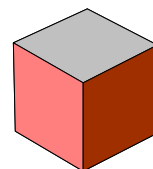
The formula is still true for $E = k + 1$.

By the principle of Mathematical Induction, Euler's formula is true for all number of edges.

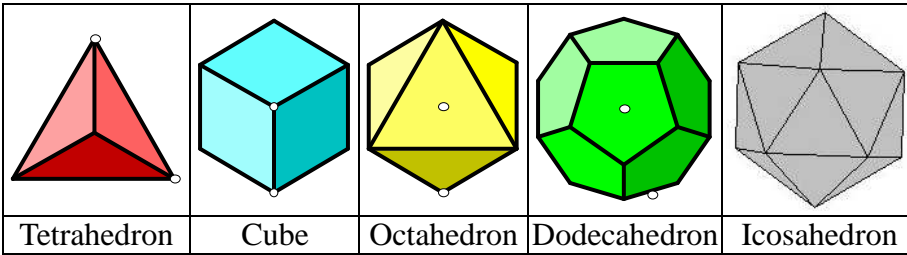
Example on Euler's formula

Given a cube, $V = 8$, $E = 12$, $F = 6$.

then $V - E + F = 2$.



Theorem: There are only five Platonic solids. Each solid is formed by identical regular polygons. They are tetrahedron, cube (=hexahedron), octahedron, icosahedron and dodecahedron.



We shall prove the existence and uniqueness of Platonic solids by Euler's formula.

On each vertex, there are m edges ($m \geq 3$). There are V vertices, so altogether Vm edges. Each edge is connected by two vertices $\Rightarrow Vm = 2E$ (1)

Each face is a regular polygon with n sides (edges) ($n \geq 3$). There are F faces, so altogether nF edges. Each edge is shared by two faces $\Rightarrow nF = 2E$ (2)

From (1) and (2), substitute V and F in terms of E, m, n into Euler's Formula: $V - E + F = 2$

$$\frac{2E}{m} - E + \frac{2E}{n} = 2$$

$$E\left(\frac{2}{m} + \frac{2}{n} - 1\right) = 2$$

$$E(2m + 2n - mn) = 2mn$$
(3)

$\because E$ is a positive integer and also RHS = $2mn$ is a positive integer

$\therefore 2m + 2n - mn$ is a positive integer.

$2m + 2n - mn = 4 - (m - 2)(n - 2)$ which is a positive integer.

$\because m - 2 > 0$ and $n - 2 > 0$,

Possible values of $(m - 2)(n - 2)$ may be 1, 2 or 3

When $(m - 2)(n - 2) = 1$,

$$m - 2 = 1, n - 2 = 1$$

$$m = 3, n = 3$$

Sub. into (3): $E(6 + 6 - 9) = 2(3)(3) \Rightarrow E = 6$

$$V = \frac{2E}{m} = \frac{2 \times 6}{3} = 4$$

$F = \frac{2E}{n} = 4$, the solid is a tetrahedron, each face is an equilateral triangle. ($n = 3$)

When $(m - 2)(n - 2) = 2$,

$$m - 2 = 2, n - 2 = 1 \text{ or } m - 2 = 1, n - 2 = 2$$

$$m = 4, n = 3 \text{ or } m = 3, n = 4$$

Sub. into (3): $E(8 + 6 - 12) = 2(3)(4) \Rightarrow E = 12$

when $m = 3, n = 4$

$$V = \frac{2E}{m} = \frac{2 \times 12}{3} = 8$$

$F = \frac{2E}{n} = 6$, the solid is a cube, each face is a square. ($n = 4$)

when $m = 4, n = 3$

$$V = \frac{2E}{m} = \frac{2 \times 12}{4} = 6$$

$F = \frac{2E}{n} = 8$, the solid is an octahedron, each face is an equilateral triangle. ($n = 3$)

When $(m - 2)(n - 2) = 3$,

$m - 2 = 1, n - 2 = 3$ or $m - 2 = 3, n - 2 = 1$

$m = 3, n = 5$ or $m = 5, n = 3$

Sub. into (3): $E(6 + 10 - 15) = 2(3)(5) \Rightarrow E = 30$

when $m = 3, n = 5$

$$V = \frac{2E}{m} = \frac{2 \times 30}{3} = 20$$

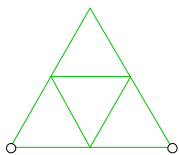
$F = \frac{2E}{n} = 12$, the solid is a dodecahedron, each face is a regular pentagon. ($n = 5$)

when $m = 5, n = 3$

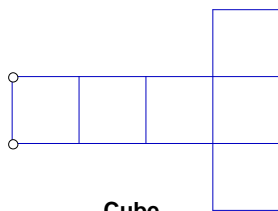
$$V = \frac{2E}{m} = \frac{2 \times 30}{5} = 12$$

$F = \frac{2E}{n} = 20$, the solid is an icosahedron, each face is an equilateral triangle. ($n = 3$)

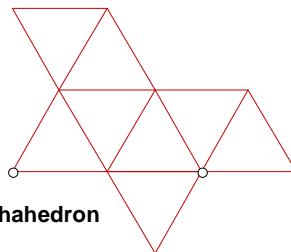
The following diagrams are the plane developments of the five Platonic solids.



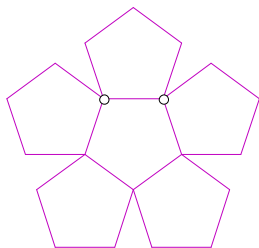
Tetrahedron



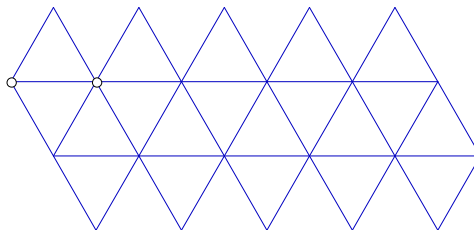
Cube



Octahedron



Dodecahedron (two needed)



Icosahedron

Let one side of the Platonic solids be a . Express the volumes of the figures in terms of a .

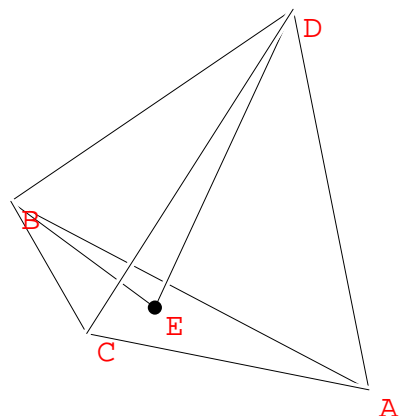
Let $ABCD$ be a regular tetrahedron with side a .

DE is the altitude from D onto the plane ABC .

E is the centroid of $\triangle ABC$.

It is easy to find the length of median of $\triangle ABC$.

$$\text{median} = a \sin 60^\circ = \frac{\sqrt{3}}{2} a$$



$$BE = \frac{2}{3} \text{ median} = \frac{\sqrt{3}}{3} a$$

$$DE = \sqrt{BD^2 - BE^2} = \sqrt{a^2 - \frac{a^2}{3}} = \sqrt{\frac{2}{3}} a$$

$$\text{Volume} = \frac{1}{3} \text{ base area} \times \text{height} = \frac{1}{3} \times \frac{1}{2} a^2 \sin 60^\circ \times \sqrt{\frac{2}{3}} a = \frac{\sqrt{2}}{12} a^3$$

For a square with side a , the volume is a^3

For a regular octahedron with side a ,

it can be cut into two identical right pyramids with square base.

In the figure, $BD = \sqrt{a^2 + a^2} = \sqrt{2} a$

$$\text{Height} = \sqrt{a^2 - \left(\frac{\sqrt{2}a}{2}\right)^2} = \frac{\sqrt{2}}{2} a$$

$$\text{Total volume} = 2 \times \frac{1}{3} \times a^2 \times \frac{\sqrt{2}}{2} a = \frac{\sqrt{2}}{3} a^3$$

The following figure shows a regular dodecahedron and a regular face:

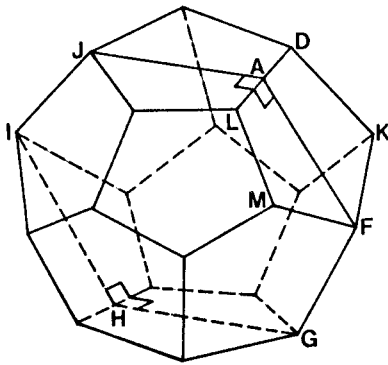
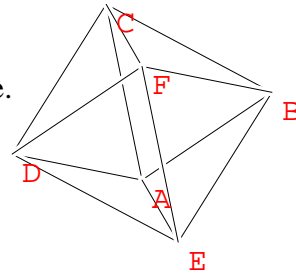


Figure 1 (a)

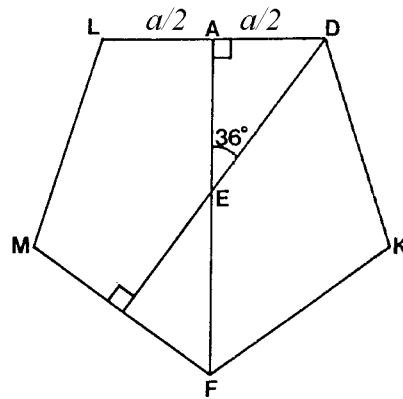


Figure 1 (b)

Let the angle between two adjacent sides be 2θ . We can use figure 1(a) and figure 1(b) to find θ .

$$\text{In figure 1(b), } \Delta AED, AE = \frac{a}{2 \tan 36^\circ}$$

$$EF = ED$$

$$= \frac{a}{2 \sin 36^\circ}$$

$$AF = AE + EF = \frac{a}{2 \tan 36^\circ} + \frac{a}{2 \sin 36^\circ}$$

$$= \frac{a}{2} \left(\frac{1 + \cos 36^\circ}{\sin 36^\circ} \right)$$

$$= \frac{a}{2 \tan 18^\circ}, \text{ by using the formula } t = \tan \frac{\theta}{2}.$$

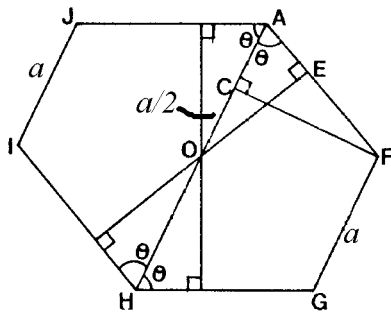


Figure 1(c)

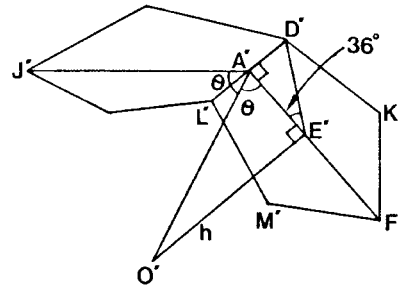


Figure 1(d)

In figure 1(c), which is the cross section of figure 1(a),

$$\begin{aligned} \cos \theta &= \frac{AE}{AO} \\ &= \frac{AE}{AC + CO} \\ &= \frac{AE}{AF \cos \theta + CO} \\ &= \frac{\frac{a}{2 \tan 36^\circ}}{\frac{a \cos \theta}{2 \tan 18^\circ} + \frac{a}{2}} \\ &= \frac{\tan 18^\circ}{\tan 36^\circ \cos \theta + \tan 36^\circ \tan 18^\circ} \end{aligned}$$

$$(\tan 36^\circ) \cos^2 \theta + (\tan 36^\circ \tan 18^\circ) \cos \theta - \tan 18^\circ = 0$$

$$\begin{aligned} \cos \theta &= \frac{-\tan 18^\circ \tan 36^\circ \pm \sqrt{(\tan 18^\circ \tan 36^\circ)^2 + 4 \tan 18^\circ \tan 36^\circ}}{2 \tan 36^\circ} = -\frac{1}{2} (\tan 18^\circ \pm \tan 54^\circ) \\ &= 0.5257 \text{ or } -0.8507 \end{aligned}$$

$$\theta = 58^\circ 17'$$

The regular dodecahedron can be regarded as 12 identical right pyramids with regular pentagon base, side = a . If the area of the pentagonal base is B and the height of the cone is h , then the volume of the regular dodecahedron is

$$\begin{aligned} 12\left(\frac{1}{3}\right)Bh &= 4 \times 5 \times \frac{1}{2} \times a \times AE \times OE \\ &= 10a \times \frac{a}{2 \tan 36^\circ} \times AE \tan \theta = 10a \times \frac{a}{2 \tan 36^\circ} \times AE \tan \theta \\ &= \frac{5a^2}{2 \tan 36^\circ} \times \frac{a}{2 \tan 36^\circ} \times \tan 58^\circ 17' = 3.83a^3 \end{aligned}$$

The following figure shows a regular icosahedron and a regular face:

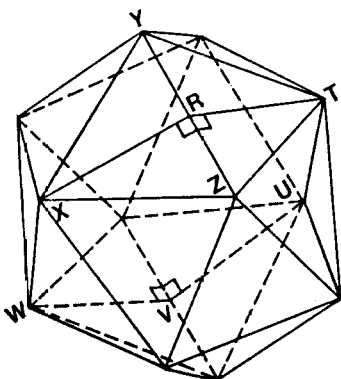


Figure 2 (a)

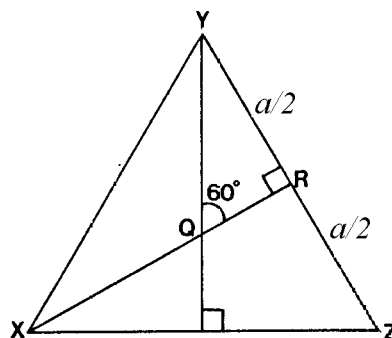


Figure 2 (b)

Let the angle between two adjacent sides be 2θ . We can use figure 2(a) and figure 2(b) to find θ .

$$\begin{aligned} \text{In figure 2(b), } \Delta YQR, RQ &= \frac{a}{2 \tan 60^\circ} = \frac{a}{2\sqrt{3}} \\ QX &= QY \\ &= \frac{a}{2 \sin 60^\circ} = \frac{a}{\sqrt{3}} \\ RX &= \frac{a}{2} \tan 60^\circ = \frac{\sqrt{3}}{2} a \end{aligned}$$

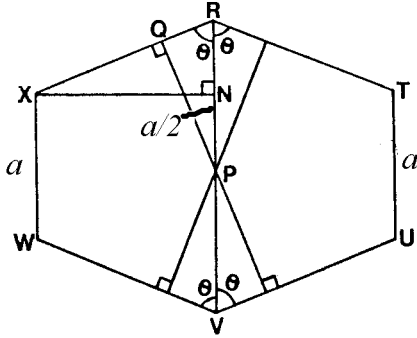


Figure 2(c)

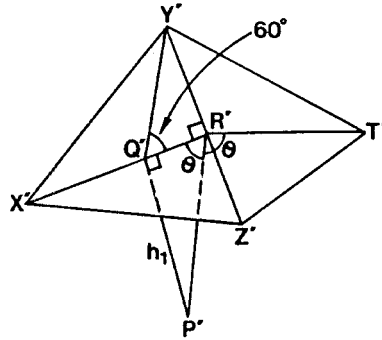


Figure 2(d)

$$\begin{aligned} \text{In the figure 2(c), } \cos \theta &= \frac{RQ}{RP} \\ &= \frac{RQ}{RN + NP} \\ &= \frac{RQ}{RX \cos \theta + NP} \\ &= \frac{\frac{a}{2\sqrt{3}}}{\frac{\sqrt{3}}{2} a \cos \theta + \frac{a}{2}} = \frac{1}{3 \cos \theta + \sqrt{3}} \end{aligned}$$

$$3 \cos^2 \theta + \sqrt{3} \cos \theta - 1 = 0$$

$$\cos \theta = \frac{-\sqrt{3} \pm \sqrt{3+12}}{6}$$

$$= 0.3568 \text{ or } -0.9342$$

$$\theta = 69^\circ 6'$$

The regular icosahedron can be regarded as 20 identical right pyramids with equilateral triangular base, side = a . The volume of the regular icosahedron is

$$\begin{aligned} 20 \times \frac{1}{3} \times \frac{1}{2} \times a^2 \sin 60^\circ \times PQ &= \frac{5a^2}{\sqrt{3}} \times RQ \tan \theta \\ &= \frac{5a^2}{\sqrt{3}} \times \frac{a}{2\sqrt{3}} \times \tan 69^\circ 6' \\ &= 2.18a^3 \end{aligned}$$

參考資料：等周問題 香港時代圖書有限公司，1975 年