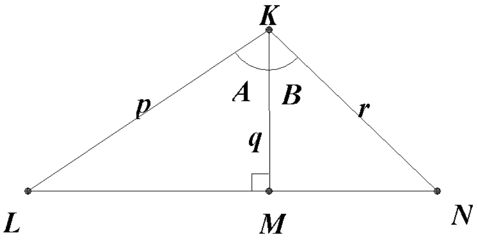
Compound angled formulae

Created by Mr. Francis Hung on 20090205 Last updated on 2009-02-05

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

Consider a triangle KLN with a right angle at M, as shown in the figure.

Let $\angle LKN = A$, $\angle MKN = B$, then $\angle LKM = A + B$.



Area of $\Delta LKM = \frac{1}{2} pq \sin A$

Area of $\Delta MKN = \frac{1}{2} qr \sin B$

Area of $\Delta LKN = \frac{1}{2} pr \sin(A + B)$

Area of ΔLKM + Area of ΔMKN = Area of ΔLKN

$\therefore \quad \frac{1}{2} pq \sin A + \frac{1}{2} qr \sin B = \frac{1}{2} pr \sin(A + B)$

$\sin(A + B) = \frac{q}{r} \sin A + \frac{q}{p} \sin B$

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

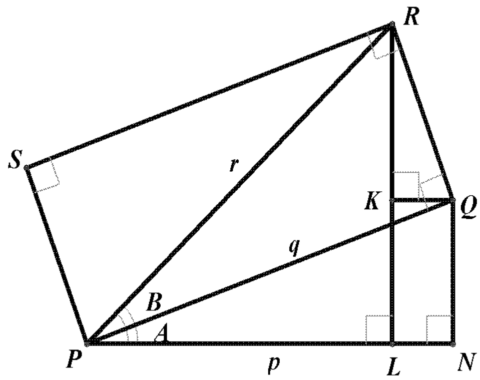
Method 2

In ΔPQN, $\angle PNQ = 90°$, $\angle NPQ = A$

$PQRS$ is a rectangle. $PR = r$, $PQ = q$, $PN = p$.

$\angle RPQ = B$, $\angle RPN = A + B$.

$RL \perp PN$, $KQ \perp RL$.



$\angle KQP = A$ (alt. $\angle$s KQ // PN)

$\angle RQK = 90° - A$

$\angle KRQ = A$ ($\angle$s sum of Δ)

$KQLN$ is a rectangle

$RL = RK + KL = RQ \cos A + QL$

$r \sin(A + B) = r \sin B \cos A + q \sin A$

$r \sin(A + B) = r \sin B \cos A + r \cos B \sin A$

$\therefore \ \sin(A + B) = \sin A \cos B + \cos A \sin B$

Method 3 In ΔABC, $A + B + C = 180°$

By sine rule, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \Rightarrow \sin A = \frac{a}{2R}$; $\sin B = \frac{b}{2R}$; $\sin C = \frac{c}{2R}$

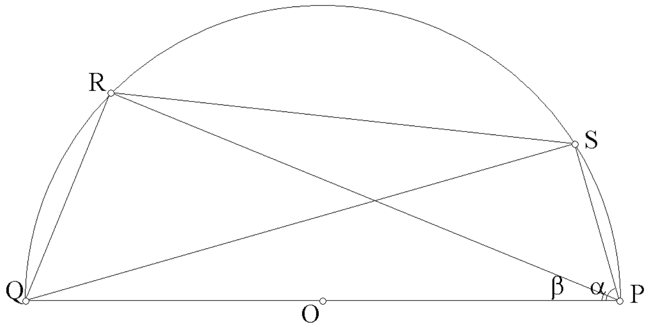
By cosine rule, $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$; $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$

$\sin A \cos B + \cos A \sin B = \frac{a}{2R} \cdot \frac{a^2 + c^2 - b^2}{2ac} + \frac{b^2 + c^2 - a^2}{2bc} \cdot \frac{b}{2R}$

$= \frac{2c^2}{4cR} = \frac{c}{2R} = \sin C = \sin(180° - (A + B))$ ($\angle$s sum of Δ)

$= \sin(A + B)$

The formula sin(α – β) = sin α cos β – cos α sin β



Suppose $PQRS$ is a semi-circle, with diameter $PQ = d$, $\angle QPR = \beta$, $\angle QPS = \alpha$.
$\angle PRQ = 90°$, $\angle PSQ = 90°$ ($\angle$ in semi-circle)
$QR = d \sin\beta$
$QS = d\sin\alpha$
$PR = d\cos\beta$
$PS = d\cos\alpha$
$\angle RPS = \alpha - \beta$

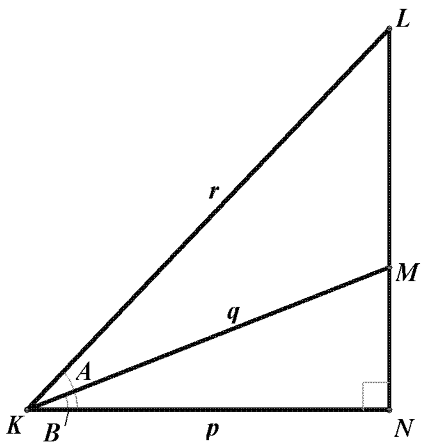
By Sine rule on ΔPRS, $\dfrac{RS}{\sin(\alpha-\beta)} = \dfrac{PS}{\sin\angle PRS}$

$\dfrac{RS}{\sin(\alpha-\beta)} = \dfrac{PS}{\sin\angle PQS}$ ($\because \angle PRS = \angle PQS$, $\angle$s in the same segment)

$\dfrac{RS}{\sin(\alpha-\beta)} = \dfrac{d\cos\alpha}{\sin(90°-\alpha)}$ ($\because \angle PQS = 90° - \alpha$, $\angle$s sum of ΔPQS)

$RS = d\sin(\alpha - \beta)$ ($\because \sin(90° - \alpha) = \cos\alpha$)
By Ptolemy's theorem, i.e. $PR\cdot QS = RS\cdot PQ + QR\cdot PS$
$d\cos\beta\cdot d\sin\alpha = d\sin(\alpha-\beta)\cdot d + d\sin\beta\cdot d\cos\alpha$
$\Rightarrow \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

Method 2 New Trend Additional Mathematics Volume One (2002) p. 142
Consider a triangle KLN with a right angle at N, as shown in the figure.
Let $\angle LKN = A$, $\angle MKN = B$, then $\angle LKM = A - B$.



Area of $\Delta LKM = \dfrac{1}{2}qr\sin(A-B)$

Area of $\Delta MKN = \dfrac{1}{2}pq\sin B$

Area of $\Delta LKN = \dfrac{1}{2}pr\sin A$

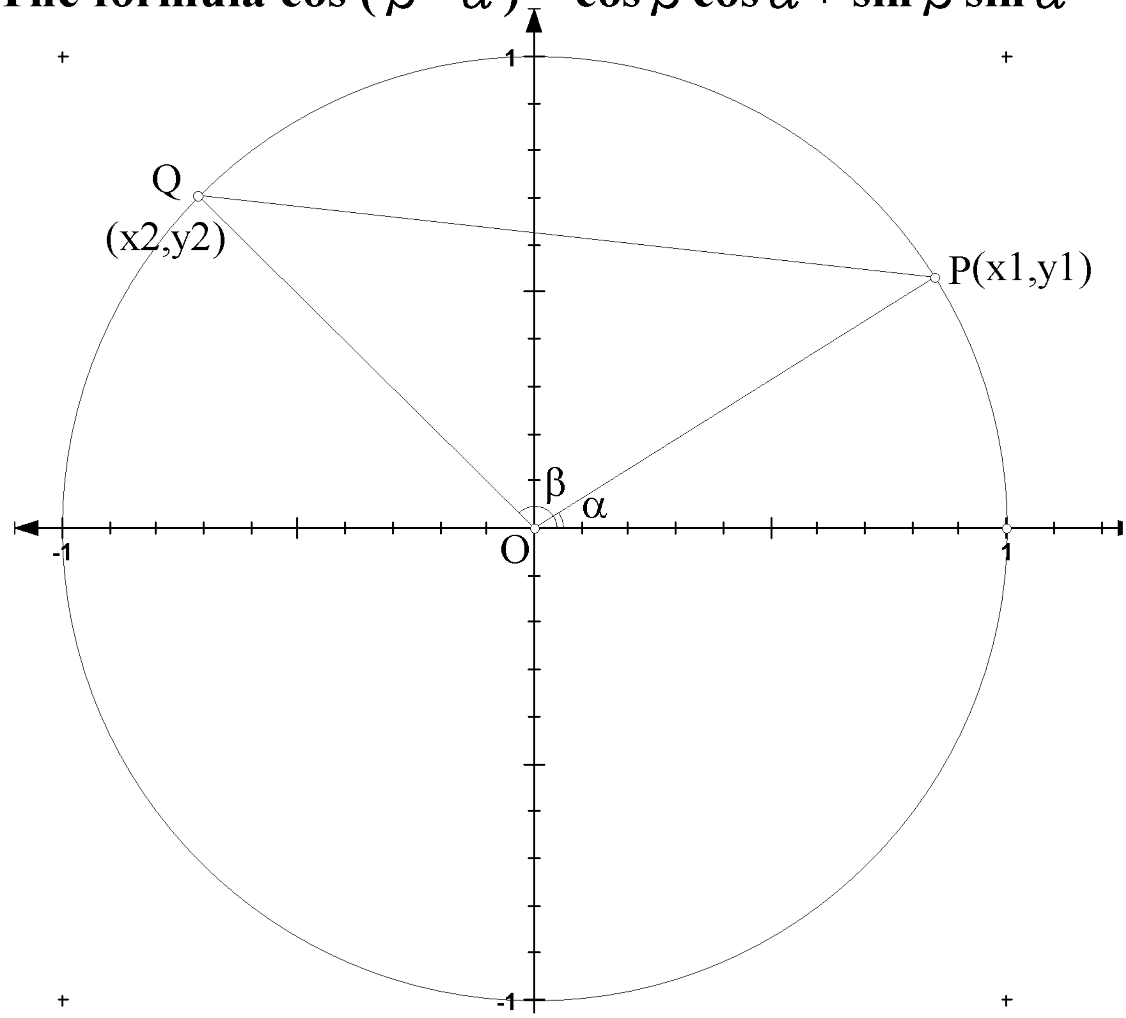
Area of ΔLKM + Area of ΔMKN = Area of ΔLKN

$\therefore\ \dfrac{1}{2}qr\sin(A-B) + \dfrac{1}{2}pq\sin B = \dfrac{1}{2}pr\sin A$

$\sin(A-B) = \dfrac{p}{q}\sin A - \dfrac{p}{r}\sin B$

$\sin(A - B) = \sin A\cos B - \cos A\sin B$

The formula cos ($\beta - \alpha$) = cos β cos α + sin β sin α



Draw a unit circle with centre O and radius 1. Suppose $P(x_1, y_1)$, $Q(x_2, y_2)$ are two points on the circumference. Suppose OP makes an angle α with positive x-axis. OQ makes an angle β with positive axis. $\angle QOP = \beta - \alpha$.

$x_1 = \cos\alpha$, $y_1 = \sin\alpha$; $x_2 = \cos\beta$, $y_2 = \sin\beta$.

By cosine rule, $PQ^2 = OP^2 + OQ^2 - 2OP \cdot OQ \cos \angle QOP$.

$(x_2 - x_1)^2 + (y_2 - y_1)^2 = 1 + 1 - 2\cos(\beta - \alpha)$

$(\cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2 = 1 + 1 - 2\cos(\beta - \alpha)$

$\cos^2\beta - 2\cos\alpha\cos\beta + \cos^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta + \sin^2\alpha = 2 - 2\cos(\beta - \alpha)$

$-2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = -2\cos(\beta - \alpha)$

$\therefore \cos(\beta - \alpha) = \cos\beta\cos\alpha + \sin\beta\sin\alpha$