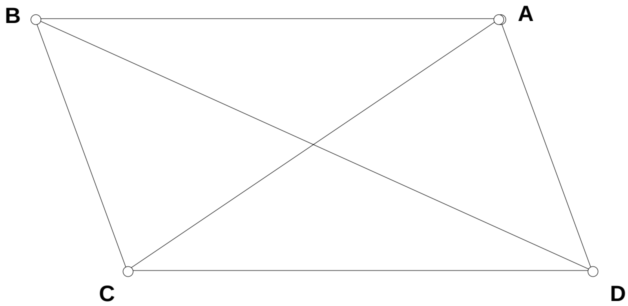
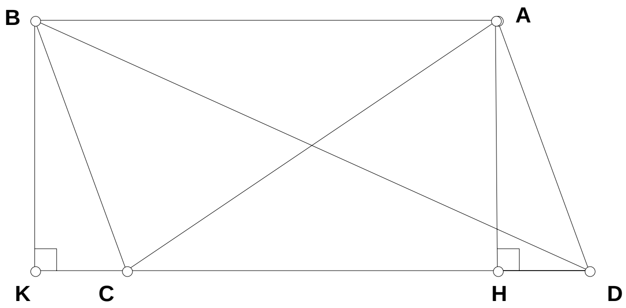
The **Apollonius Theorem**: a theorem about parallelogram.

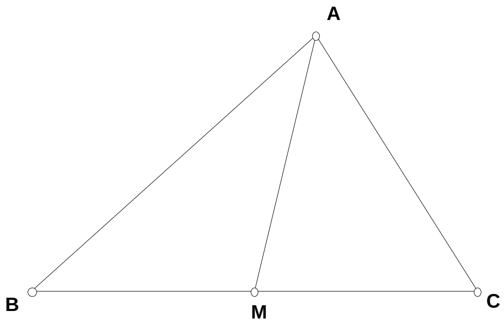


Let ABCD is a parallelogram.
Let AB = CD = p; AD = BC = q.
Let AC = x; BD = y.
Then $2(p^2 + q^2) = x^2 + y^2$
Proof: Let H, and K be points on DC such that BK$\perp$CD, AH$\perp$CD.
Then ΔAHD $\cong$ ΔBKC (AAS)
Let BK = AH = h; CK = HD = t.



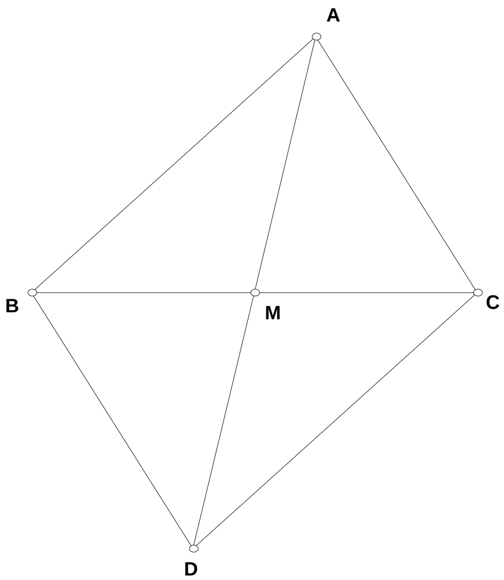
In ΔAHD, $h^2 + t^2 = q^2$ … (1)
In ΔAHC, $h^2 + (p - t)^2 = x^2$ … (2)
In ΔBKD, $h^2 + (p + t)^2 = y^2$ … (3)
(2) + (3): $2h^2 + 2p^2 + 2t^2 = x^2 + y^2$
sub (1) $2p^2 + 2q^2 = x^2 + y^2$
Hence the theorem is proved.

The **median** of a triangle ABC.



In ΔABC, let BC = a, AC = b, AB = c.
Let M be the mid point of BC.
Then the line AM is called a median.

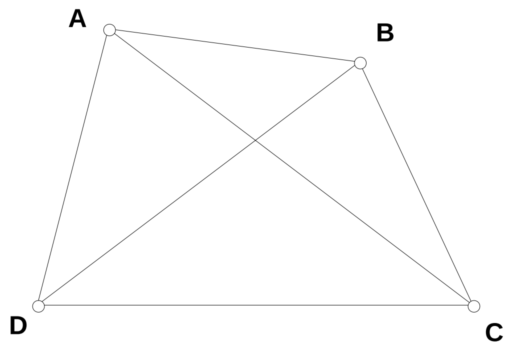
AM = $\frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$



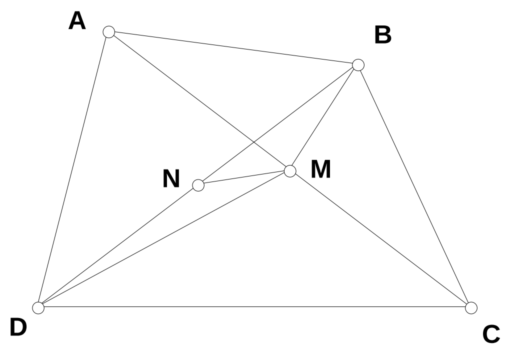
Proof: Produce AM to its own length to D.
AM = MD (by construction)
BM = MC (given)
ABDC is a parallelogram (diagonals bisect each other)
By the above Apollonius Theorem,
$2(b^2 + c^2) = a^2 + (2\text{ AM})^2$

AM = $\frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$

Converse of Apollonius Theorem



Let ABCD be a quadrilateral.
Let AB = p, BC = q, CD = r, DA = s;
Let AC = x, BD = y.
If $p^2 + q^2 + r^2 + s^2 = x^2 + y^2$,
then ABCD is a parallelogram.
Proof: Let M be the mid point of AC,
Let N be the mid point of BD.
By the above theorem on median,



In Δ ABC, $BM^2 = \dfrac{2p^2 + 2q^2 - x^2}{4}$ …(1)

In Δ ADC, $DM^2 = \dfrac{2r^2 + 2s^2 - x^2}{4}$ …(2)

In Δ BMD, $MN^2 = \dfrac{2\,BM^2 + 2\,DM^2 - y^2}{4}$

sub (1) and (2):

$$MN^2 = \frac{p^2 + q^2 - \frac{x^2}{2} + r^2 + s^2 - \frac{x^2}{2} - y^2}{4} = 0$$

Therefore, M = N and hence the two diagonals bisect each other at M (= N).
ABCD is a parallelogram