

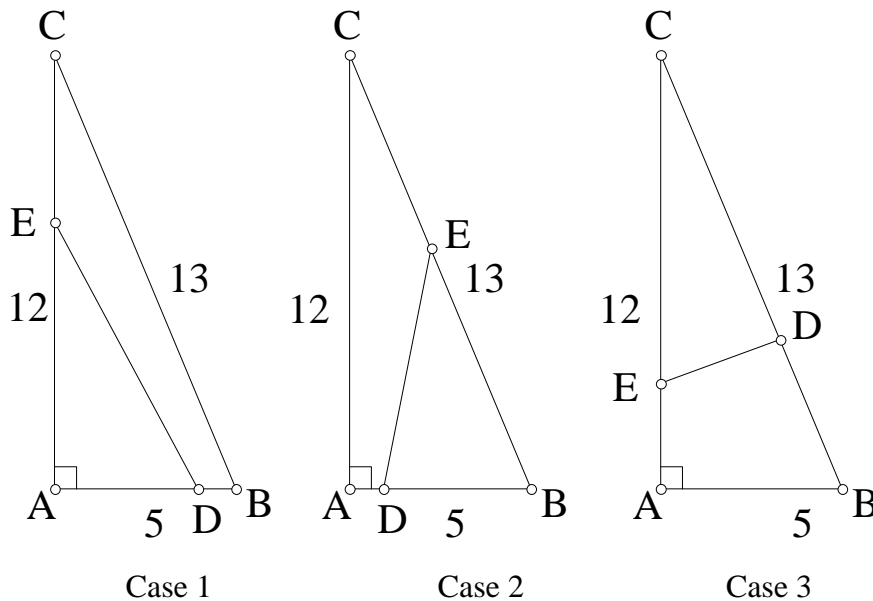
Problem on a 5-12-13 triangle:

Given a triangle ABC . $AB = 5$, $AC = 12$, $BC = 13$. D and E are points on the sides of the triangle such that DE separates $\triangle ABC$ into 2 parts of equal areas. Determine the minimum length of DE .

Solution: Clearly $\angle BAC = 90^\circ$ (Converse Pythagoras' Theorem) and Area of $\triangle ABC = \frac{1}{2} \cdot 5 \times 12 = 30$

$$\sin B = \frac{12}{13}, \cos B = \frac{5}{13}; \sin C = \frac{5}{13}, \cos C = \frac{12}{13}$$

There are three different cases:



Case 1: D lies on AB and E lies on AC .

$$\text{Let } AD = x, AE = y. \text{ Area of } \triangle ADE = \frac{1}{2} \cdot xy = \frac{1}{2} \text{ Area of } \triangle ABC = 15 \Rightarrow xy = 30$$

In $\triangle ADE$, $DE^2 = x^2 + y^2$ (Pythagoras' Theorem)

$$DE^2 \geq 2xy = 60 \text{ (AM } \geq \text{ GM)}$$

Case 2: D lies on AB and E lies on BC .

$$\text{Let } BD = x, BE = y. \text{ Area of } \triangle BDE = \frac{1}{2} \cdot xy \sin B = \frac{1}{2} \text{ Area of } \triangle ABC = 15 \Rightarrow xy = \frac{65}{2}$$

Apply cosine formula on $\triangle BDE$:

$$DE^2 = x^2 + y^2 - 2xy \cos B = x^2 + y^2 - 2 \times \frac{65}{2} \times \frac{5}{13} = x^2 + y^2 - 25$$

$$DE^2 \geq 2xy - 25 = 65 - 25 = 40$$

Case 3: D lies on BC and E lies on AC .

$$\text{Let } CD = x, CE = y. \text{ Area of } \triangle CDE = \frac{1}{2} \cdot xy \sin C = \frac{1}{2} \text{ Area of } \triangle ABC = 15 \Rightarrow xy = 78$$

Apply cosine formula on $\triangle CDE$:

$$DE^2 = x^2 + y^2 - 2xy \cos C = x^2 + y^2 - 2 \times 78 \times \frac{12}{13} = x^2 + y^2 - 144$$

$$DE^2 \geq 2xy - 144 = 156 - 144 = 12$$

Combine the 3 cases, minimum of $DE = \sqrt{12} = 2\sqrt{3}$