Given a 5-digits integer $x = \overline{abcde}$. If $a + c + e - (b + d) = 11k$, where k is an integer, prove that x is divisible by 11.

Proof: $x = 10000a + 1000b + 100c + 10d + e$

$= 9999a + a + 1001b - b + 99c + c + 11d - d + e$

$= 11(909a + 91b + 9c + d) + a - b + c - d + e$

$= 11(900a + 100b + 9c + d) + 11k$

which is divisible by 11.

HKHLE General Mathematics 1976 Q8(b)

Prove that, for any positive integer n, the integer $10^n + (-1)^{n-1}$ is divisible by 11.

Hence deduce a necessary and sufficient condition for an integer to be divisible by 11 by considering only the sum and difference of the digits of the integer.

Solution

Induction on n.

$n = 1$, $10^1 + (-1)^0 = 11$ which is obviously divisible by 11.

Suppose $10^k + (-1)^{k-1} = 11m$, where m is an integer, for some positive integer k.

$10^{k+1} + (-1)^k = 10(10^k) + (-1)^k = 10[11m - (-1)^{k-1}] + (-1)^k = 110m + (-1)^k(1 + 10) = 11[10m + (-1)^k]$ which is divisible by 11.

So, by M.I., $10^n + (-1)^{n-1}$ is divisible by 11 for any positive integer n.

The necessary and sufficient condition is: Let S_1 be the sum of all odd digits of an integer N, S_2 be the sum of all even digits of N. $S_1 - S_2$ is divisible by 11 if and only if N is divisible by 11.

Proof: Let $N = a_n\times 10^n + a_{n-1}\times 10^{n-1} + \ldots\ldots + a_1\times 10 + a_0$, where $0 \le a_r \le 9$ and a_r are integers, $0\le r\le n$.

$S_1 - S_2 = (-1)^n a_n + (-1)^{n-1}a_{n-1} + \ldots\ldots - a_1 + a_0$

If N is divisible by 11, $a_n\times 10^n + a_{n-1}\times 10^{n-1} + \ldots\ldots + a_1\times 10 + a_0 = 11m$, where m is an integer.

$N = 11\,m = a_n\times[10^n + (-1)^{n-1}] + a_{n-1}\times[10^{n-1} + (-1)^{n-2}] + \ldots\ldots + a_1\times[10 + 1] + a_0\,(1 - 1)$

$+ [(-1)^n a_n + (-1)^{n-1}a_{n-1} + \ldots\ldots - a_1 + a_0]$

$= a_n\times 11k_n + a_{n-1}\times 11k_{n-1} + \ldots\ldots + a_1\times 11k_1 + a_0\,11k_0 + S_1 - S_2$, where k_r are integers, $0\le r\le n$

$\Rightarrow S_1 - S_2 = 11m - [a_n\times 11k_n + a_{n-1}\times 11k_{n-1} + \ldots\ldots + a_1\times 11k_1 + a_0\,11k_0]$

$= 11[m - a_nk_n + a_{n-1}k_{n-1} + \ldots\ldots + a_1k_1 + a_0k_0]$, which is divisible by 11.

If $S_1 - S_2$ is divisible by 11, then $(-1)^n a_n + (-1)^{n-1}a_{n-1} + \ldots\ldots - a_1 + a_0 = 11m$, where m is an integer.

$N = a_n\times 10^n + a_{n-1}\times 10^{n-1} + \ldots\ldots + a_1\times 10 + a_0$

$= a_n\times[10^n + (-1)^{n-1}] + a_{n-1}\times[10^{n-1} + (-1)^{n-2}] + \ldots\ldots + a_1\times[10 + 1] + a_0\,(1 - 1)$

$+ [(-1)^n a_n + (-1)^{n-1}a_{n-1} + \ldots\ldots - a_1 + a_0]$

$= a_n\times 11k_n + a_{n-1}\times 11k_{n-1} + \ldots\ldots + a_1\times 11k_1 + a_0\,11k_0 + S_1 - S_2$

$= 11\,[a_nk_n + a_{n-1}k_{n-1} + \ldots\ldots + a_1k_1 + a_0k_0] + 11m$, which is divisible by 11.

The question is proved.