Game on numbers

Let x, y and z be three 3-digit numbers such that $x : y : z = 1 : 2 : 3$. If 1, 2, ..., 7, 8, 9 appear exactly once on all digits of x, y and z, find all possible numbers.

Let $x = 100a + 10b + c$, $y = 100d + 10e + f$, $z = 100g + 10h + i$.

Note that c, f, i cannot be 5, otherwise, the digits 5 are repeated.

$\because z = 3x \le 999 \Rightarrow 101 \le x \le 333 \Rightarrow a = 1, 2$ or 3

We shall divide x into 7 intervals and find the possible solution from each interval:

(1) $101 \le x \le 133$
(2) $134 \le x \le 166$
(3) $167 \le x \le 199$
(4) $200 \le x \le 233$
(5) $234 \le x \le 266$
(6) $267 \le x \le 299$
(7) $300 \le x \le 333$

But all digits must be different and cannot be 0, $\Rightarrow$ c, f, i cannot be 5.

(1) $124 \le x \le 132$
(2) $134 \le x \le 164$
(3) $167 \le x \le 198$
(4) $213 \le x \le 231$
(5) $234 \le x \le 264$
(6) $267 \le x \le 298$
(7) $312 \le x \le 329$

(1) $124 \le x \le 132$, $248 \le 2x \le 264$, $372 \le 3x \le 396$

$\because d = 2 \Rightarrow 130 \le x \le 132 \Rightarrow x = 132$, $y = 264$ the digit "2" is repeated, no solution

(2) $134 \le x \le 164$, $268 \le 2x \le 328$, $402 \le 3x \le 492$

$g = 4 \Rightarrow 136 \le x \le 163$, $268 \le 2x \le 326$, $423 \le 3x \le 489$

$141 \le x \le 163$, $282 \le 2x \le 326$, $423 \le 3x \le 489$

$g = 4 \Rightarrow 152 \le x \le 163$, $304 \le 2x \le 326$, $456 \le 3x \le 489$

$a = 1$, $e \ne 0, 1$, $g = 4$ and $d = 3 \Rightarrow 152 \le x \le 162$, $2x = 326$, $456 \le 3x \le 489$

$\Rightarrow x = 163$, $2x = 326$, the digit "3" is repeated, no solution

(3) $167 \le x \le 198$, $334 \le 2x \le 396$, $501 \le 3x \le 594$

$h \ne 0, 1$, $i \ne 3$ and $e \ne 3 \Rightarrow 167 \le x \le 198$, $342 \le 2x \le 396$, $524 \le 3x \le 594$

$c \ne 5 \Rightarrow 176 \le x \le 198$, $352 \le 2x \le 396$, $528 \le 3x \le 594$

$e \ne 5 \Rightarrow 176 \le x \le 198$, $362 \le 2x \le 396$, $528 \le 3x \le 594$

$181 \le x \le 198$, $362 \le 2x \le 396$, $542 \le 3x \le 594$

$c \ne 1 \Rightarrow 182 \le x \le 198$, $364 \le 2x \le 396$, $546 \le 3x \le 594$

If $b = 8$, $182 \le x \le 189$, $364 \le 2x \le 378$, $546 \le 3x \le 567$

$f \ne 8$ and $e \ne f \Rightarrow 182 \le x \le 189$, $364 \le 2x \le 376$, $546 \le 3x \le 567$

$c \ne 8 \Rightarrow 182 \le x \le 187$, $364 \le 2x \le 374$, $546 \le 3x \le 561$

$i \ne 1$, $h \ne 5 \Rightarrow 182 \le x \le 187$, $364 \le 2x \le 374$, $546 \le 3x \le 549$

$182 \le x \le 183$, $364 \le 2x \le 366$, $546 \le 3x \le 549$

when $x = 182$, $2x = 364$, $3x = 546$, the digits "4", "6" are repeated

when $x = 183$, $2x = 366$, the digit "6" is repeated, no solution

If $b = 9$, $192 \le x \le 198$, $384 \le 2x \le 396$, $576 \le 3x \le 594$

$e \ne 9 \Rightarrow 192 \le x \le 198$, $384 \le 2x \le 386$, $576 \le 3x \le 587$

$192 \le x \le 193$, $384 \le 2x \le 386$, $576 \le 3x \le 579$

$i \ne 9 \Rightarrow x = 192$, $2x = 384$, $3x = 576$, accepted

(4) $213 \le x \le 231$, $426 \le 2x \le 462$, $639 \le 3x \le 693$

$d \ne 2$, $f \ne 2, 6 \Rightarrow 213 \le x \le 231$, $438 \le 2x \le 458$, $639 \le 3x \le 693$

$219 \le x \le 229$, $438 \le 2x \le 458$, $657 \le 3x \le 687$

$b \ne 2 \Rightarrow x = 219$, $2x = 438$, $3x = 657$ accepted

(5) $234 \le x \le 264$, $468 \le 2x \le 528$, $702 \le 3x \le 792$

$e \ne 2$, $h \ne 0$, $i \ne 2 \Rightarrow 234 \le x \le 264$, $468 \le 2x \le 518$, $714 \le 3x \le 789$

$238 \le x \le 259$, $476 \le 2x \le 518$, $714 \le 3x \le 777$

$e \ne 7$, $h \ne 7$, $i \ne 7$, $f \ne 2, 4 \Rightarrow 238 \le x \le 259$, $486 \le 2x \le 518$, $714 \le 3x \le 768$

$243 \le x \le 256$, $486 \le 2x \le 512$, $729 \le 3x \le 768$

$f \ne 2$, $h \ne 2 \Rightarrow 243 \le x \le 256$, $486 \le 2x \le 498$, $738 \le 3x \le 768$

$246 \le x \le 249$, $486 \le 2x \le 498$, $738 \le 3x \le 768$

$b = 4 = d$, contradiction, no solution

(6) $267 \le x \le 298$, $534 \le 2x \le 596$, $801 \le 3x \le 894$

$c \ne 8$, $h \ne 0 \Rightarrow 267 \le x \le 297$, $534 \le 2x \le 596$, $813 \le 3x \le 894$

$271 \le x \le 297$, $542 \le 2x \le 594$, $813 \le 3x \le 891$

$f \ne 2, 4 \Rightarrow 271 \le x \le 297$, $546 \le 2x \le 594$, $813 \le 3x \le 891$

$273 \le x \le 297$, $546 \le 2x \le 594$, $819 \le 3x \le 891$

If $x = 270 + c$, $273 \le x \le 279$, $546 \le 2x \le 558$, $819 \le 3x \le 837$

$e \ne 5$, $i \ne 7 \Rightarrow 273 \le x \le 279$, $546 \le 2x \le 546$, $819 \le 3x \le 834$

$x = 273$, $2x = 546$, $3x = 819$, accepted

If $x = 280 + c$, $281 \le x \le 289$, $562 \le 2x \le 578$, $843 \le 3x \le 867$, the digit "8" is repeated

If $x = 290 + c$, $291 \le x \le 297$, $582 \le 2x \le 594$, $873 \le 3x \le 891$

$e \ne 8, 9$, $i \ne 9$ no solution

(7) $312 \le x \le 329$, $624 \le 2x \le 658$, $936 \le 3x \le 987$

$h \ne 3 \Rightarrow 312 \le x \le 329$, $624 \le 2x \le 658$, $942 \le 3x \le 987$

$314 \le x \le 329$, $628 \le 2x \le 658$, $942 \le 3x \le 987$

If $x = 310 + c$, $314 \le x \le 318$, $628 \le 2x \le 636$, $942 \le 3x \le 954$

$e \ne 3 \Rightarrow 314 \le x \le 314$, $628 \le 2x \le 628$, $942 \le 3x \le 942$, the digits "2", "4" are repeated

If $x = 320 + c$, $321 \le x \le 329$, $642 \le 2x \le 658$, $963 \le 3x \le 987$

$f \ne 2, 4, 6$, $h \ne 6$, $i \ne 2, 5 \Rightarrow 321 \le x \le 329$, $648 \le 2x \le 658$, $978 \le 3x \le 987$

$326 \le x \le 329$, $652 \le 2x \le 658$, $978 \le 3x \le 987$

$c \ne 6, 9$, $f \ne 2 \Rightarrow 327 \le x \le 328$, $654 \le 2x \le 656$, $981 \le 3x \le 984$

when $x = 327$, $2x = 654$, $3x = 981$, accepted

when $x = 328$, $2x = 656$, the digit "6" is repeated

Conclusion

$x = 192$, $2x = 384$, $3x = 576$

$x = 219$, $2x = 438$, $3x = 657$

$x = 273$, $2x = 546$, $3x = 819$

$x = 327$, $2x = 654$, $3x = 981$