

Given a triangle with one angle is 120° . If all sides are integers, find all possible solution.

This document was created by Mr. Francis Hung, last updated: February 28, 2008.

$$c^2 = a^2 + b^2 - 2ab \cos 120^\circ$$

$$c^2 = a^2 + b^2 + ab$$

$$c^2 = (a + b)^2 - ab$$

$$ab = (a + b)^2 - c^2$$

$$ab = (a + b + c)(a + b - c)$$

$$\frac{a + b + c}{a} = \frac{b}{a + b - c} = k, \text{ where } k \text{ is a positive constant.}$$

$$a + b + c = ak; b = (a + b - c)k$$

$$\Rightarrow \begin{cases} a(1 - k) + b + c = 0 \dots\dots(1) \\ ak + b(k - 1) - ck = 0 \dots\dots(2) \end{cases}$$

$$\text{From (1): } c = a(k - 1) - b \dots\dots(3)$$

$$\text{Sub. (3) into (2): } ak + b(k - 1) - a(k^2 - k) + bk = 0$$

$$b(2k - 1) = a(k^2 - 2k)$$

Let $a = (2k - 1)p$, $b = (k^2 - 2k)p$, then $c = (k^2 - k + 1)p$; where p is a positive integer.

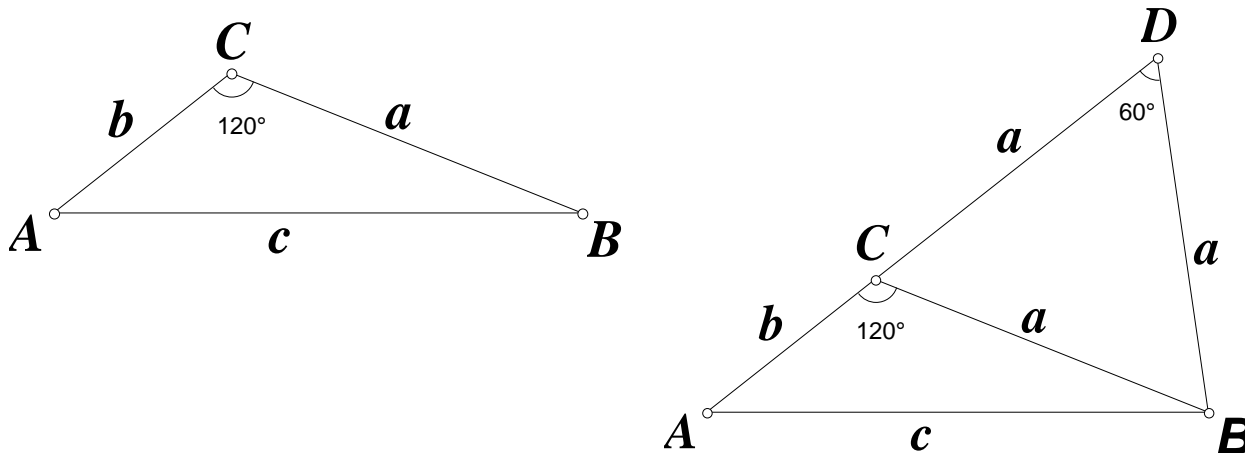
$$a : b : c = (2k - 1) : (k^2 - 2k) : (k^2 - k + 1)$$

Let $a = (2k - 1)p$, $b = k(k - 2)p$, $c = (k^2 - k + 1)p$; where p is a positive integer.

| k | p | a | b | c |
|-----|-----|-----|-----|-----|
| 3 | 1 | 5 | 3 | 7 |
| 4 | 1 | 7 | 8 | 13 |
| 5 | 1 | 9 | 15 | 21 |
| 6 | 1 | 11 | 24 | 31 |

Given a triangle with one angle is 60° . If all sides are integers, find all possible solution.

Given the above triangle with $\angle C = 120^\circ$, we can construct another triangle ABD with $\angle D = 60^\circ$



So, if (a, b, c) is a solution to a 120° triangle, then $(a, a+b, c)$ or $(a+b, b, c)$ is a solution to a $60^\circ \Delta$.

The general solution are: $((2k - 1)p, (k^2 - 1)p, (k^2 - k + 1)p)$ or $((k^2 - 1)p, (k^2 - 2k)p, (k^2 - k + 1)p)$

| k | p | a | $a + b$ | c | $a + b$ | b | c |
|-----|-----|-----|---------|-----|---------|-----|-----|
| 2 | 1 | 3 | 3 | 3 | | | |
| 3 | 1 | 5 | 8 | 7 | 8 | 3 | 7 |
| 4 | 1 | 7 | 15 | 13 | 15 | 8 | 13 |
| 5 | 1 | 9 | 24 | 21 | 24 | 15 | 21 |
| 6 | 1 | 11 | 35 | 31 | 35 | 24 | 31 |

"A cyclic quadrilateral with all 4 sides and 2 diagonals are integers."

- (A) A cyclic quadrilateral with an equilateral triangle and another triangle.

If $\triangle ABC$ is equilateral of side length a , $\triangle BCD$ with $\angle C = 120^\circ$,
 $BC = c$, $CD = b$.

By the notes on 120° -triangle,

$$c = (2k - 1)p, b = (k^2 - 2k)p, a = (k^2 - k + 1)p$$

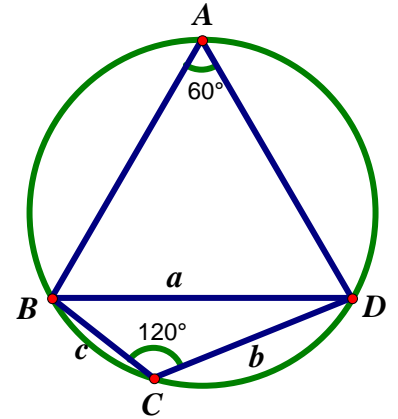
$$\text{or } b = (2k - 1)p, c = (k^2 - 2k)p, a = (k^2 - k + 1)p$$

Apply Ptolemy's theorem, $AC \times BD = ab + ac$

$$\Rightarrow AC = b + c = (k^2 - 1)p, \text{ which is a positive integer.}$$

Hence, all 4 sides and the 2 diagonals are integers.

$$\text{e.g. } k = 3, p = 1, AB = 7 = AD = BD, BC = 5, CD = 3, AC = 8$$

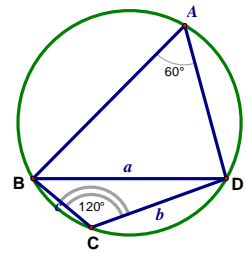


- (B) A cyclic quadrilateral with $\angle A = 60^\circ$, $\angle C = 120^\circ$

By the notes on 60° -triangle,

$$a = (k^2 - k + 1)p, AB = (2k - 1)p, AD = (k^2 - 1)p$$

$$\text{or } a = (k^2 - k + 1)p, AB = (k^2 - 1)p, AD = (k^2 - 2k)p$$



- Case 1 $a = (k^2 - k + 1)p, AB = (2k - 1)p, AD = (k^2 - 1)p, b = (2k - 1)p, c = (k^2 - 2k)p$

$$\angle ADB = \angle CBD \text{ (eq. chords eq. } \angle s)$$

$$\therefore AD \parallel BC \text{ (alt. } \angle s \text{ eq.)}$$

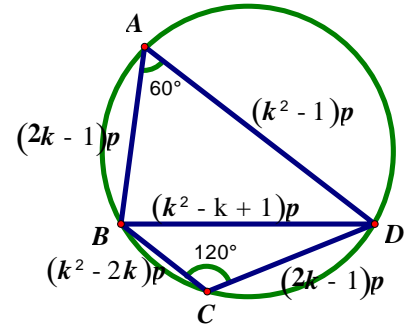
$$\angle ABD = \angle ACD, \angle ABD = \angle CAD \text{ (eq. chords eq. } \angle s)$$

$$\triangle ABD \cong \triangle DCA \text{ (AAS)}$$

$$AC = BD \text{ (corr. sides } \cong \Delta s)$$

Hence, all 4 sides and the 2 diagonals are integers.

$$\text{e.g. } k = 3, p = 1, AB = 5, AD = 8, BD = 7 = AC, BC = 3, CD = 5.$$



- Case 2 $a = (k^2 - k + 1)p, AB = (2k - 1)p, AD = (k^2 - 1)p, c = (2k - 1)p, b = (k^2 - 2k)p$

Apply Ptolemy's Theorem,

$$AC(k^2 - k + 1)p = (2k - 1)p \times (k^2 - 2k)p + (2k - 1)p \times (k^2 - 1)p$$

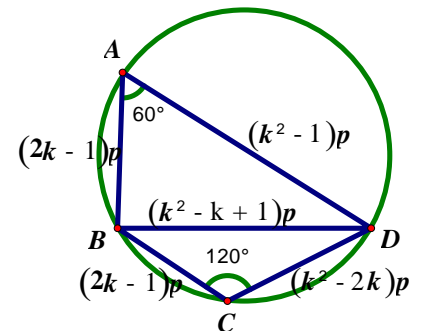
$$AC = \frac{(2k - 1)(2k^2 - 2k - 1)}{k^2 - k + 1} p, \text{ let } p = k^2 - k + 1$$

$$AB = (2k - 1)(k^2 - k + 1) = BC, CD = (k^2 - 2k)(k^2 - k + 1),$$

$$AD = (k^2 - 1)(k^2 - k + 1), AC = (2k - 1)(2k^2 - 2k - 1), BD = (k^2 - k + 1)^2$$

all 4 sides and the 2 diagonals are integers.

$$\text{e.g. } k = 3, AB = 35 = BC, AD = 56, BD = 49, AC = 55, CD = 21.$$



- Case 3 When $BD = (k^2 - k + 1)p, AB = (k^2 - 1)p, AD = (k^2 - 2k)p, BC = (2k - 1)p, CD = (k^2 - 2k)p$

$$AC = \frac{(k^2 - 2k)(k^2 + 2k - 2)}{k^2 - k + 1} p, \text{ let } p = k^2 - k + 1, \text{ then all sides are integers.}$$

$$\text{e.g. } k = 3, AB = 56, BC = 35, AD = 21, BD = 49, AC = 55, CD = 21.$$

- Case 4 When $BD = (k^2 - k + 1)p, AD = (k^2 - 1)p, AB = (k^2 - 2k)p, BC = (2k - 1)p, CD = (k^2 - 2k)p$

$$AC = (k^2 - k + 1)p = BD \text{ (similar to Case 1), then all sides are integers.}$$

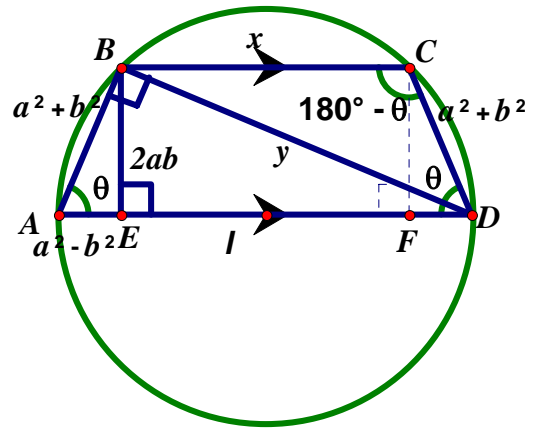
$$\text{e.g. } k = 3, p = 1, AB = 3, AD = 8, BD = 7 = AC, BC = 5, CD = 3.$$

(C) A cyclic quadrilateral $ABCD$ with one pair of equal opposite sides $AB = CD$ and AD is the diameter.

$$\angle ABD = 90^\circ \text{ (}\angle \text{ in semi-circle)}$$

Let $BC = x$, $BD = y = AC$, $AD = l$, $BE \perp AD$ (E lies on AD .)

$$\angle BAD = \theta = \angle CDA \text{ (given } AB = CD)$$



case 1 In $\triangle ABE$, $\angle AEB = 90^\circ$, it can be proved that^[*]

$$BE = 2ab, AE = a^2 - b^2, AB = a^2 + b^2$$

$\triangle ABE \sim \triangle ADB$ (equiangular)

$$y : (a^2 + b^2) : l = 2ab : (a^2 - b^2) : (a^2 + b^2) \text{ (ratio of sides, } \sim \Delta\text{'s)}$$

$$y = \frac{2ab(a^2 + b^2)}{(a^2 - b^2)}; l = \frac{(a^2 + b^2)^2}{(a^2 - b^2)}$$

$$l = AE + EF + FD = x + 2(a^2 - b^2) \Rightarrow x = l - 2(a^2 - b^2)$$

$$x = \frac{(a^2 + b^2)^2}{(a^2 - b^2)} - 2(a^2 - b^2) = \frac{(a^2 + b^2)^2 - 2(a^2 - b^2)^2}{(a^2 - b^2)} = \frac{a^4 + 2a^2b^2 + b^4 - 2(a^4 - 2a^2b^2 + b^4)}{(a^2 - b^2)} = \frac{6a^2b^2 - a^4 - b^4}{(a^2 - b^2)}$$

$$x = \frac{4a^2b^2 - (a^4 - 2a^2b^2 + b^4)}{(a^2 - b^2)} = \frac{(2ab)^2 - (a^2 - b^2)^2}{(a^2 - b^2)} = \frac{(2ab + a^2 - b^2)(2ab - a^2 + b^2)}{(a^2 - b^2)} = \frac{(a^2 + 2ab - b^2)(a^2 - 2ab - b^2)}{(a^2 - b^2)}$$

$$\because x > 0; a > b \therefore -(a^2 + 2ab - b^2)(a^2 - 2ab - b^2) > 0$$

$$[(a + b)^2 - 2b^2][(a - b)^2 - 2b^2] < 0$$

$$(a + b + \sqrt{2}b)(a + b - \sqrt{2}b)(a - b + \sqrt{2}b)(a - b - \sqrt{2}b) < 0$$

$$(a + b - \sqrt{2}b)(a - b - \sqrt{2}b) < 0 \Rightarrow (\sqrt{2} - 1)b < a < (\sqrt{2} + 1)b$$

$$a > b \Rightarrow b < a < (\sqrt{2} + 1)b \text{ to order to ensure that } x > 0$$

Multiply every side by $(a^2 - b^2)$,

$AB = a^4 - b^4 = CD$, $AD = (a^2 + b^2)^2$, $BC = 6a^2b^2 - a^4 - b^4$, $AC = 2ab(a^2 + b^2) = BD$; then all sides are integers.

e.g. $a = 2$, $b = 1$, then $1 < 2 < (\sqrt{2} + 1)$; $AB = 15 = CD$, $AD = 25$, $BC = 7$, $AC = 20 = BD$

case 2 $AE = 2ab$, $BE = a^2 - b^2$, $AB = a^2 + b^2$

$\triangle ABE \sim \triangle ADB$ (equiangular)

$$y : (a^2 + b^2) : l = (a^2 - b^2) : 2ab : (a^2 + b^2) \text{ (ratio of sides, } \sim \Delta\text{'s)}$$

$$y = \frac{(a^2 - b^2)(a^2 + b^2)}{2ab}; l = \frac{(a^2 + b^2)^2}{2ab}$$

$$l = AE + EF + FD = x + 2(2ab) \Rightarrow x = l - 4ab$$

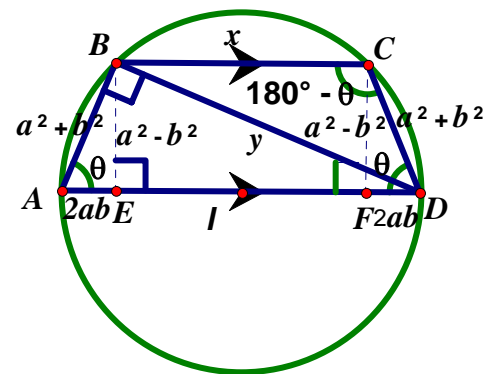
$$x = \frac{(a^2 + b^2)^2}{2ab} - 4ab = \frac{(a^2 + b^2)^2 - 8a^2b^2}{2ab} = \frac{a^4 + b^4 - 6a^2b^2}{2ab}$$

$$(\sqrt{2} + 1)b < a \text{ to order to ensure that } x > 0$$

Multiply every side by $2ab$,

$AB = 2ab(a^2 + b^2) = CD$, $AD = (a^2 + b^2)^2$, $BC = a^4 + b^4 - 6a^2b^2$, $AC = a^4 - b^4 = BD$; then all sides are integers.

e.g. $a = 4$, $b = 1$, then $(\sqrt{2} + 1) < 4$; $AB = 136 = CD$, $AD = 289$, $BC = 161$, $AC = 255 = BD$



[*] http://hk.geocities.com/takwaihung/Number_Theory/Pythagorean_triple.pdf

(D) A cyclic quadrilateral with one pair of adjacent equal sides.

Follow the same steps in (C), $\angle ADB = 90^\circ - \theta$ (\angle sum of Δ)

$$\angle BDC = \angle ADB - \angle BDC = \theta - (90^\circ - \theta) = 2\theta - 90^\circ$$

Let G be a point on the circle so that $BD \perp GD$

$$\angle AGD = 90^\circ \text{ (}\angle \text{ in semi-circle)}$$

$ABDG$ is a rectangle (there are 3 right angles)

$$BG = AD = I \text{ (diagonal of the rectangle)}$$

$$\angle BGD = \angle BAD = \theta \text{ (}\angle \text{s in the same segment)}$$

$$\angle CDG = \angle BCD + \angle BDG = 2\theta - 90^\circ + 90^\circ = 2\theta$$

$$GD = AB = a^2 + b^2 \text{ (opp. sides of rectangle)}$$

ΔCDG is isosceles

$$CG = 2 CD \sin \frac{1}{2} \angle CDG = 2(a^2 + b^2) \sin \theta = 2(a^2 + b^2) \cdot \frac{2ab}{a^2 + b^2} = 4ab \text{ (note that in } \Delta ABE, \sin \theta = \frac{2ab}{a^2 + b^2} \text{)}$$

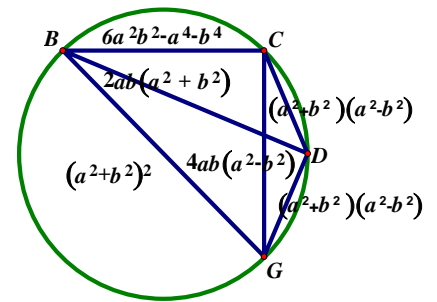
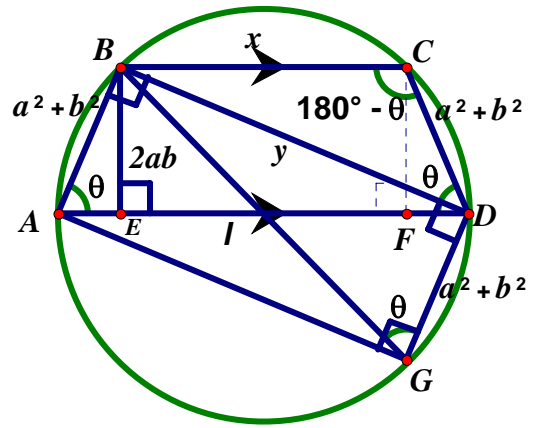
Multiply every side by $(a^2 - b^2)$, then

$$CD = (a^2 + b^2)(a^2 - b^2) = DG, BC = 6a^2b^2 - a^4 - b^4,$$

$$BG = (a^2 + b^2)^2, BD = 2ab(a^2 + b^2), CG = 4ab(a^2 - b^2).$$

All sides are positive integers.

$$\text{e.g. } a = 2, b = 1, CD = 15 = DG, BC = 7, BG = 25, BD = 20, CG = 24$$



(E) A cyclic quadrilateral $ABCD$ with $AD = BC$.

$$\angle ACD = \theta = \angle BAC \text{ (eq. chords eq. } \angle \text{s)}$$

$AB \parallel DC$ (alt. \angle s eq.)

Let $DE \perp AB, CF \perp AB, CD = x = EF, AC = y = BD$.

case 1 Use the method in (C), $DE = CF = 2ab, AE = a^2 - b^2, AD = a^2 + b^2 = BC$

$$AB = x + 2(a^2 - b^2)$$

Apply cosine rule on ΔABC and ΔACD .

$$\cos \theta = \frac{[x + 2(a^2 - b^2)]^2 + y^2 - (a^2 + b^2)^2}{2[x + 2(a^2 - b^2)]y} = \frac{x^2 + y^2 - (a^2 + b^2)^2}{2xy}$$

$$x[x^2 + 4(a^2 - b^2)x + 4(a^2 - b^2)^2] + xy^2 - (a^2 + b^2)^2x = x^2[x + 2(a^2 - b^2)] + y^2[x + 2(a^2 - b^2)] - (a^2 + b^2)^2[x + 2(a^2 - b^2)]$$

$$x^3 + 4(a^2 - b^2)x^2 + 4(a^2 - b^2)^2x - (a^2 + b^2)^2x = x^3 + 2(a^2 - b^2)x^2 + 2(a^2 - b^2)y^2 - (a^2 + b^2)^2x - 2(a^2 + b^2)^2(a^2 - b^2)$$

$$2(a^2 - b^2)x^2 + 4(a^2 - b^2)^2x + 2(a^2 + b^2)^2(a^2 - b^2) = 2(a^2 - b^2)y^2$$

$$x^2 + 2(a^2 - b^2)x + (a^2 + b^2)^2 = y^2$$

$$[x + (a^2 - b^2)]^2 + (a^2 + b^2)^2 - (a^2 - b^2)^2 = y^2$$

$$[x + (a^2 - b^2)]^2 = y^2 - 4a^2b^2 = (y + 2ab)(y - 2ab)$$

$$y^2 = (2ab)^2 + m^2, \text{ where } m \text{ is an integer. } 2ab, m, y \text{ forms a Pythagorean Triple, by the notes } [^*]$$

$$2ab = 2rs, m = r^2 - s^2, y = r^2 + s^2; r > s > 0 \text{ are integers.}$$

$$\frac{a}{r} = \frac{s}{b} = k, a = kr, b = \frac{s}{k}$$

$$x = -(a^2 - b^2) \pm \sqrt{(y + 2ab)(y - 2ab)}, \because x > 0, \therefore x = -(a^2 - b^2) + \sqrt{(y + 2ab)(y - 2ab)}$$

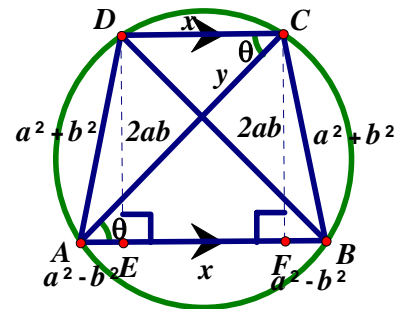
$$x = -(a^2 - b^2) + (r^2 - s^2) = (r^2 - a^2) + (b^2 - s^2) = (r^2 - k^2r^2) + \left(\frac{s^2}{k^2} - s^2\right) = (1 - k^2) \left(r^2 + \frac{s^2}{k^2}\right) > 0 \Rightarrow 0 < k < 1$$

$$s < b < a < r; ab = rs; AD = a^2 + b^2 = BC, AB = (r^2 - s^2) + (a^2 - b^2), CD = (r^2 - a^2) + (b^2 - s^2), AC = r^2 + s^2 = BD$$

$$\text{e.g. } a = 3, b = 2, s = 1, r = 6, AD = 13 = BC, CD = 30, AB = 40, AC = 37 = BD$$

$$\text{e.g. } a = 4, b = 3, s = 2, r = 6, AD = 25 = BC, CD = 25, AB = 39, AC = 40 = BD$$

$$\text{e.g. } a = 4, b = 2, s = 1, r = 8, AD = 20 = BC, CD = 51, AB = 75, AC = 65 = BD$$



case 2 $DE = a^2 - b^2 = CF$, $AE = 2ab = BF$, $AD = a^2 + b^2 = BC$

$CD = x = EF$, $AC = y = BD$, $AB = x + 4ab$

Apply cosine rule on $\triangle ABC$ and $\triangle ACD$.

$$\cos \theta = \frac{(x + 4ab)^2 + y^2 - (a^2 + b^2)^2}{2(x + 4ab)y} = \frac{x^2 + y^2 - (a^2 + b^2)^2}{2xy}$$

$$x^3 + 8abx^2 + 16a^2b^2x + xy^2 - (a^2 + b^2)^2x = x^3 + 4abx^2 + xy^2 + 4aby^2 - (a^2 + b^2)^2x - 4ab(a^2 + b^2)^2$$

$$4abx^2 + 16a^2b^2x + 4ab(a^2 + b^2)^2 = 4aby^2$$

$$x^2 + 4abx + (a^2 + b^2)^2 = y^2$$

$(x + 2ab)^2 = y^2 - (a^2 - b^2)^2 = m^2$, where m is an integer.

by the notes ^{1*}: $y = r^2 + s^2$, $a^2 - b^2 = r^2 - s^2$, $m = 2rs$, where $r > s > 0$ are integers

$$(a + b)(a - b) = (r + s)(r - s)$$

$$\frac{a + b}{r + s} = \frac{r - s}{a - b} = k$$

$$a + b = k(r + s) \dots \dots \dots (1), \quad a - b = \frac{1}{k}(r - s) \dots \dots \dots (2)$$

$$((1)+(2)) \div 2: a = \frac{1}{2k} [(k^2 + 1)r + (k^2 - 1)s]; \quad ((1)-(2)) \div 2: b = \frac{1}{2k} [(k^2 - 1)r + (k^2 + 1)s]$$

$$x = m - 2ab = 2rs - 2ab = 2 \left\{ rs - \frac{1}{2k} [(k^2 + 1)r + (k^2 - 1)s] \cdot \frac{1}{2k} [(k^2 - 1)r + (k^2 + 1)s] \right\}$$

$$x = 2 \left\{ rs - \frac{1}{4k^2} [k^2(r + s) + (r - s)][k^2(r + s) - (r - s)] \right\} = \frac{2}{4k^2} \{ 4k^2rs - [k^4(r + s)^2 - (r - s)^2] \}$$

$$x = -\frac{1}{2k^2} [k^4(r + s)^2 - 4k^2rs - (r - s)^2] = -\frac{1}{2k^2} [k^2(r + s)^2 + (r - s)^2](k^2 - 1) > 0 \Rightarrow 0 < k < 1$$

$$0 < \frac{a + b}{r + s} = \frac{r - s}{a - b} = k < 1 \Rightarrow a + b < r + s, \quad r - s < a - b, \quad r > s, \quad a > b \text{ and } (a + b)(a - b) = (r + s)(r - s)$$

$$CD = 2(rs - ab), \quad AD = a^2 + b^2 = BC, \quad AC = r^2 + s^2 = BD, \quad AB = 2(ab + rs)$$

e.g. $a = 7, b = 1, s = 4, r = 8, AD = 50 = BC, CD = 50, AB = 78, AC = 80 = BD$

Example 1 Given $ABCD$ is a quadrilateral such that $AB = 7, BC = CD = 15, DA = 25$. Find the maximum area of the quadrilateral.

By cosine law, $AC^2 = 7^2 + 15^2 - 2 \cdot 7 \cdot 15 \cos D = 25^2 + 15^2 - 2 \cdot 25 \cdot 15 \cos B$
 $576 = 2(125 \cos B - 105 \cos D)$

$$288 = 375 \cos B - 105 \cos D \dots \dots \dots (1)$$

Let $K = \text{area of } ABCD = \text{area of } \triangle ABC + \text{area of } \triangle ACD$

$$= \frac{1}{2} 25 \cdot 15 \sin B + \frac{1}{2} 7 \cdot 15 \sin D$$

$$4K^2 = (375 \sin B + 105 \sin D)^2 \dots \dots \dots (2)$$

$$(1)^2 + (2): 4K^2 + 288^2 = 375^2 + 105^2 - 2 \times 375 \times 105 (\cos B \cos D - \sin B \sin D)$$

$$4K^2 = 68706 - 78750 \cos(B + D)$$

K is a maximum when $\cos(B + D)$ is a minimum

$$-1 \leq \cos(B + D) \leq 1, \text{ maximum area} = \frac{1}{2} \sqrt{68706 + 78750} = 192$$

Example 2 Given $ABCD$ is a quadrilateral such that $AB = 7, BC = CD = 15, DA = 25, AC = 20$. Find BD .

$$AC^2 + BC^2 = 20^2 + 15^2 = 25^2 = AB^2, \quad \angle ACB = 90^\circ \text{ (converse, Pyth. Thm)}$$

$$\text{In } \triangle ACD, \cos \angle ACD = \frac{15^2 + 20^2 - 7^2}{2 \cdot 15 \cdot 20} = \frac{24}{25}$$

$$\text{In } \triangle BCD, BD^2 = 15^2 + 15^2 - 2 \times 15 \times 15 \cos(90^\circ + \angle ACD)$$

$$BD^2 = 450 + 450 \sin \angle ACD = 450 \cdot \left[1 + \sqrt{1 - \left(\frac{24}{25} \right)^2} \right] = 450 \cdot \left(1 + \frac{7}{25} \right) = 576$$

$$BD = 24$$

