Individual Events

SI	k	2	I1	A	15	I2	P	3	I3	A	2	I4	P	4	IS	a	2
	d	1		B	3		Q	4		B	3		Q	300		b	2
	a	–6		C	4		R	10		C	45		R	2		c	–3
	t	$\frac{50}{11}$		D	8		S	112.5		D	7		S	$2\sqrt{3}$		d	35

Group Events

SG	W	$\frac{1+\sqrt{5}}{2}$	G1	m	8	G2	z	540	G3	k	$\sqrt{33}$	G4	m	13	GS	value	–1
	T	29		h	$\sqrt{13}$		R	6		v	6		n	6		$x^4+\frac{1}{x^4}$	4036079
	S	106		$x+y+z$	11		k	5		value	106		$abc+def$	72		$\cot\alpha$	$\frac{99}{20}$
	k	4		Number	72		xyz	1		r	27405		$p+q$	2		value	$\frac{6023}{6022}$

Sample Individual Event

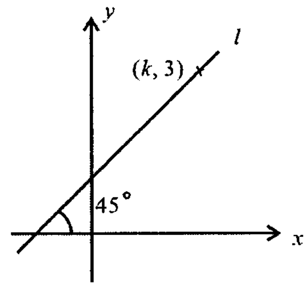
SI.1 Let $\sqrt{k}=\sqrt{7+\sqrt{13}}-\sqrt{7-\sqrt{13}}$, find the value of k.

$$\sqrt{k}^2=\left(\sqrt{7+\sqrt{13}}-\sqrt{7-\sqrt{13}}\right)^2$$

$$k=7+\sqrt{13}-2\sqrt{7^2-\sqrt{13}^2}+7-\sqrt{13}$$

$$=14-2\sqrt{36}=2$$

SI.2 In Figure 1, the straight line ℓ passes though the point (k, 3) and makes an angle 45° with the x-axis. If the equation of ℓ is $x + by + c = 0$ and $d = |1 + b + c|$, find the value of d.



$$\ell:\frac{y-3}{x-2}=\tan 45^\circ$$

$y - 3 = x - 2$

$x - y + 1 = 0$, $b = -1$, $c = 1$

$d = |1 - 1 + 1| = 1$

SI.3 If $x - d$ is a factor of $x^3 - 6x^2 + 11x + a$, find the value of a.

$f(x) = x^3 - 6x^2 + 11x + a$

$f(1) = 1 - 6 + 11 + a = 0$

$a = -6$

SI.4 If $\cos x + \sin x = -\frac{a}{5}$ and $t = \tan x + \cot x$, find the value of t.

$$\cos x+\sin x=\frac{6}{5}$$

$$(\cos x+\sin x)^2=\frac{36}{25}$$

$$1+2\sin x\cos x=\frac{36}{25}$$

$$2\sin x\cos x=\frac{11}{25}\Rightarrow \sin x\cos x=\frac{11}{50}$$

$$d=\tan x+\cot x=\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}=\frac{\sin^2 x+\cos^2 x}{\sin x\cos x}=\frac{1}{\sin x\cos x}=\frac{50}{11}$$

Individual Event 1

I1.1 Let A = 15×tan 44°×tan45°×tan46°, find the value of A.

$$A = 15\times\tan 44^\circ \times 1\times\frac{1}{\tan 44^\circ} = 15$$

I1.2 Let n be a positive integer and $\overbrace{20082008\cdots2008}^{n\ 2008's}15$ is divisible by A. If the least possible value of n is B, find the value of B.

The given number is divisible by 15. Therefore it is divisible by 3 and 5.

The last 2 digits of the given number is 15, which is divisible by 15.

The necessary condition is: $\overbrace{20082008\cdots2008}^{n\ 2008's}$ must be divisible by 3.

2 + 0 + 0 + 8 = 10 which is not divisible by 3.

The least possible n is 3: 2+0+0+8+2+0+0+8+2+0+0+8 = 30 which is divisible by 3.

I1.3 Given that there are C integers that satisfy the equation $|x-2|+|x+1|=B$, find the value of C

$|x-2|+|x+1|=3$

If $x<-1$, $2-x-x-1=3 \Rightarrow x=-1$ (rejected)

If $-1\le x\le 2$, $2-x+x+1=3 \Rightarrow 3=3$, always true $\therefore -1\le x\le 2$

If $2<x$, $x-2+x+1=3 \Rightarrow x=2$ (reject)

$\therefore -1\le x\le 2$ only

$\because x$ is an integer, $\therefore x=-1, 0, 1, 2$; $C=4$

I1.4 In the coordinate plane, the distance from the point $(-C, 0)$ to the straight line $y=x$ is $\sqrt{D}$, find the value of D.

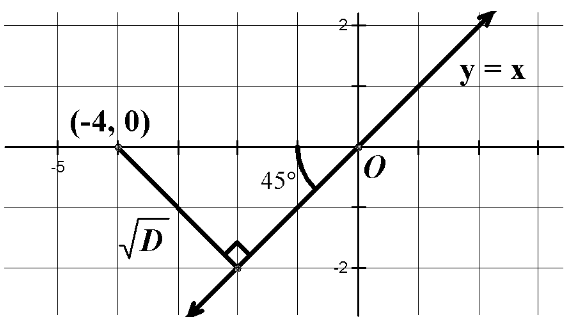
The distance from $P(x_0, y_0)$ to the straight line $Ax+By+C=0$ is $\left|\frac{Ax_0+By_0+C}{\sqrt{A^2+B^2}}\right|$.

The distance from $(-4, 0)$ to $x-y=0$ is $\sqrt{D}=\left|\frac{-4-0+0}{\sqrt{1^2+(-1)^2}}\right|=\frac{4}{\sqrt{2}}=2\sqrt{2}=\sqrt{8}$; $D=8$

Method 2

$\sqrt{D}=4\sin 45^\circ=\frac{4}{\sqrt{2}}$

$D=\frac{16}{2}=8$



Individual Event 2

I2.1 Given that $P=\left[\sqrt[3]{6}\times\left(\sqrt[3]{\frac{1}{162}}\right)\right]^{-1}$, find the value of P.

$P=\left[\sqrt[3]{\frac{6}{162}}\right]^{-1}=\sqrt[3]{\frac{162}{6}}=\sqrt[3]{27}=3$

I2.2 Let a, b and c be real numbers with ratios $b:(a+c)=1:2$ and $a:(b+c)=1:P$. If $Q=\frac{a+b+c}{a}$, find the value of Q.

$2b=a+c$ …… (1), $3a=b+c$ …… (2)

(1) – (2): $2b-3a=a-b\Rightarrow 3b=4a\Rightarrow a:b=3:4$

Let $a=3k$, $b=4k$, sub. into (1): $2(4k)=3k+c\Rightarrow c=5k$

$Q=\frac{a+b+c}{a}=\frac{3k+4k+5k}{3k}=4$

I2.3 Let $R=\left(\sqrt{\sqrt{3}+\sqrt{2}}\right)^{Q}+\left(\sqrt{\sqrt{3}-\sqrt{2}}\right)^{Q}$. Find the value of R.

$R=\left(\sqrt{\sqrt{3}+\sqrt{2}}\right)^{4}+\left(\sqrt{\sqrt{3}-\sqrt{2}}\right)^{4}$

$=\left(\sqrt{3}+\sqrt{2}\right)^{2}+\left(\sqrt{3}-\sqrt{2}\right)^{2}$

$=3+2\sqrt{6}+2+3-2\sqrt{6}+2=10$

I2.4 Let $S=(x-R)^2+(x+5)^2$, where x is a real number. Find the minimum value of S.

$S=(x-10)^2+(x+5)^2$

$=x^2-20x+100+x^2+10x+25$

$=2x^2-10x+125$

$=2(x^2-5x)+125$

$=2(x-2.5)^2+125-2\times 2.5^2$

$=2(x-2.5)^2+112.5\geq 112.5$

The minimum value of S is 75.

Method 2

$S=(10-x)^2+(x+5)^2$

Let $a=10-x$, $b=x+5$

$a+b=15$, which is a constant

$\therefore a^2+b^2$ reaches its minimum when $a=b=7.5$

$\therefore$ Minimum $S=7.5^2+7.5^2=112.5$

Individual Event 3

I3.1 Given that $\frac{1-\sqrt{3}}{2}$ satisfies the equation $x^2 + px + q = 0$, where p and q are rational numbers. If $A = |p| + 2|q|$, find the value of A.

For an equation with rational coefficients, conjugate roots occur in pairs.

That is, the other root is $\frac{1+\sqrt{3}}{2}$.

$\frac{1-\sqrt{3}}{2}+\frac{1+\sqrt{3}}{2}=-p \Rightarrow p=-1$; $\frac{1-\sqrt{3}}{2}\times\frac{1+\sqrt{3}}{2}=q \Rightarrow q=-\frac{1}{2}$

$A = |-1| + 2\left|-\frac{1}{2}\right| = 2$

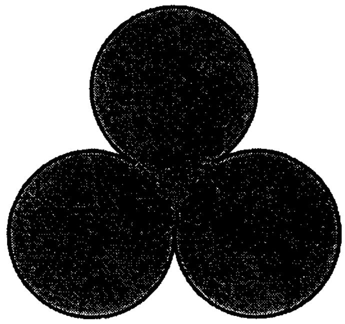
I3.2 Two bags U_1 and U_2 contain identical red and white balls. U_1 contains A red balls and 2 white balls. U_2 contains 2 red balls and B white balls. Take two balls out of each bag. If the probability of all four balls are red is $\frac{1}{60}$, find the value of B.

U_1 contains 2 red and 2 white, total 4 balls. U_2 contains 2 red and B white, total $2 + B$ balls.

P(all 4 are red) $=\frac{2}{4}\times\frac{1}{3}\times\frac{2}{2+B}\times\frac{1}{1+B}=\frac{1}{60}$

$20 = (2 + B)(1 + B) \Rightarrow B^2 + 3B - 18 = 0 \Rightarrow (B - 3)(B + 6) = 0; B = 3$

I3.3 Figure 1 is formed by three identical circles touching one another, the radius of each circle is B cm. If the perimeter of the shaded region is C cm, find the value of C. (Take $\pi = 3$)



Let the centres of the circles be P, Q and R respectively.

Then $PQ = 2B = 6 = QR = PR$

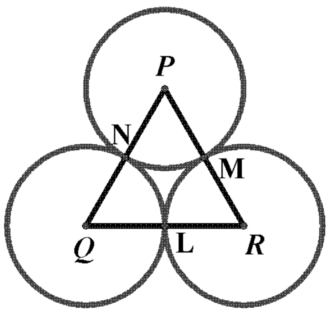
ΔPQR is an equilateral Δ.

$\angle P = \angle Q = \angle R = 60°$

Perimeter $= 3\times 2\pi\times 3 - 3\times\overset{\frown}{MN}$

$= 18\pi - 3\times 2\pi\times 3\times\frac{60}{360}$

$= 15\pi = 45$



I3.4 Let D be the integer closest to $\sqrt{C}$, find the value of D.

$6=\sqrt{36}<\sqrt{45}<\sqrt{49}=7 \Rightarrow 6<D<7$

$6.5^2 = 42.25 < 45 \Rightarrow 6.5 < D < 7$, $D = 7$

Method 2

$6=\sqrt{36}<\sqrt{45}<\sqrt{49}=7 \Rightarrow 6<D<7$

$\because |45 - 36| = 9, |45 - 49| = 4$

$\therefore$ 45 is closer to 49

$\therefore \sqrt{45}$ is closer to 7

$D = 7$

Individual Event 4

I4.1 Given that x and y are real numbers such that $|x| + x + y = 10$ and $|y| + x - y = 10$. If $P = x + y$, find the value of P.

If $x \geq 0, y \geq 0$; $\begin{cases} 2x + y = 10 \\ x = 10 \end{cases}$, solving give $x = 10, y = -10$ (reject)

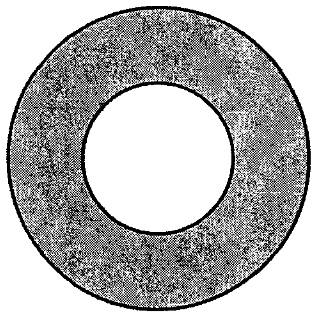
If $x \geq 0, y < 0$, $\begin{cases} 2x + y = 10 \\ x - 2y = 10 \end{cases}$, solving give $x = 6, y = -2$

If $x < 0, y \geq 0$, $\begin{cases} y = 10 \\ x = 10 \end{cases}$, reject

If $x < 0, y < 0$; $\begin{cases} y = 10 \\ x - 2y = 10 \end{cases}$, solving give $x = 30, y = 10$ (reject)

$\therefore x = 6, y = -2; P = x + y = 6 - 2 = 4$

I4.2 In Figure 1, the shaded area is formed by two concentric circles and has area 96π cm^2. If the two radii differ by $2P$ cm and the large circle has area Q cm^2, find the value of Q. (Take $\pi = 3$)



Let the radii of the large and small circles be R and r respectively.

$\pi(R^2 - r^2) = 96\pi$ and $R - r = 8 \ldots (1)$

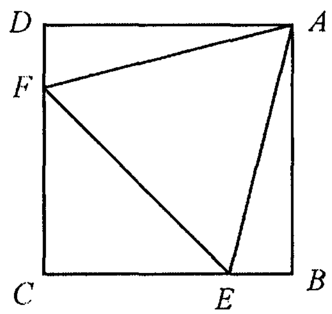
$(R + r)(R - r) = 96 \Rightarrow (R + r)\times 8 = 96 \Rightarrow R + r = 12 \ldots (2)$

$(2) + (1)$: $2R = 20 \Rightarrow R = 10 \Rightarrow Q = \pi(10)^2 = 300$

I4.3 Let R be the largest integer such that $R^Q < 5^{200}$, find the value of R.

$R^{300} < 5^{200} \Rightarrow R^3 < 25$, the largest integer is 2.

I4.4 In Figure 2, there are a square $ABCD$ with side length $(R - 1)$ cm and an equilateral triangle AEF. (E and F are points on BC and CD respectively). If the area of ΔAEF is $(S - 3)$ cm^2, find the value of S.



Reference: 1994-95 Heat Group Event Q7

Let $AF = x$ cm $= FE = AE$

$\angle FAE = 60°$, $\angle DAF = \angle BAE = 15°$

$AD = 1 = AF \cos 15° = x \cos 15° \Rightarrow x = \sec 15°$

$$S - 3 = \frac{1}{2}x^2 \sin 60° = \frac{1}{2}\cdot\frac{1}{\cos^2 15°}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}\cdot\frac{1}{1+\cos 30°} = \frac{\sqrt{3}}{2}\cdot\frac{1}{1+\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{2}\cdot\frac{2}{2+\sqrt{3}} = 2\sqrt{3} - 3\text{ ; } S = 2\sqrt{3}$$

Method 2

Let $CE = CF = x$ cm

Then $BE = DF = (1 - x)$ cm

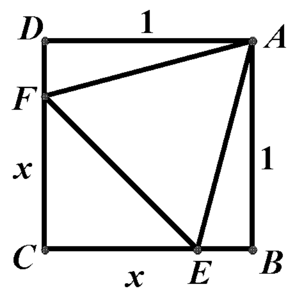
By Pythagoras' Theorem,

$AF = FE \Rightarrow 1 + (1 - x)^2 = x^2 + x^2$

$2 - 2x + x^2 = 2x^2$

$x^2 + 2x - 2 = 0 \Rightarrow x^2 = 2 - 2x$

$x = -1 + \sqrt{3}$



Area of ΔAFE = Area of square – area of ΔCEF – 2 area of ΔADF

$$= 1 - \frac{x^2}{2} - 2\times\frac{1\times(1-x)}{2} = x - \frac{x^2}{2} = x - \frac{2-2x}{2} = 2x - 1$$

$$= 2\left(-1+\sqrt{3}\right) - 1 = 2\sqrt{3} - 3$$

$S - 3 = 2\sqrt{3} - 3$; $S = 2\sqrt{3}$

Individual Spare

IS.1 If all the positive factors of 28 are $d_1, d_2, \ldots, d_n$ and $a = \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_n}$, find the value of a.

Positive factors of 28 are 1, 2, 4, 7, 14, 28. $a = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{7} + \frac{1}{14} + \frac{1}{28} = 2$

IS.2 Given that x is a negative real number that satisfy $\frac{1}{x + \frac{1}{x+2}} = a$. If $b = x + \frac{7}{2}$, find the value of b

$\frac{1}{x + \frac{1}{x+2}} = 2 \Rightarrow \frac{x+2}{x(x+2)+1} = 2 \Rightarrow x + 2 = 2x^2 + 4x + 2 \Rightarrow 2x^2 + 3x = 0 \Rightarrow x = -1.5$ or 0 (reject)

$b = -1.5 + 3.5 = 2$

IS.3 Let α and β be the two roots of the equation $x^2 + cx + b = 0$, where $c < 0$ and $\alpha - \beta = 1$. Find the value of c.

$\alpha\beta = b = 2$; $(\alpha - \beta)^2 = 1 \Rightarrow (\alpha + \beta)^2 - 4\alpha\beta = 1 \Rightarrow (-c)^2 - 4\times 2 = 1$, $c = -3$

IS.4 Let d be the remainder of $(196c)^{2008}$ divided by 97. Find the value of d.

$[196\times(-3)]^{2008} = 588^{2008} = (97\times 6 + 6)^{2008}$

$= (97\times 6)^{2008} + {}_{2008}C_1\cdot(97\times 6)^{2007}\times 6 + \ldots + 6^{2008} = 97m + 6^{2008}$, where m is an integer.

Note that $2^5\times 3 = 96 = 97 - 1 \equiv -1 \pmod{97}$; $2\times 3^5 = 486 = 97\times 5 + 1 \equiv 1 \pmod{97}$;

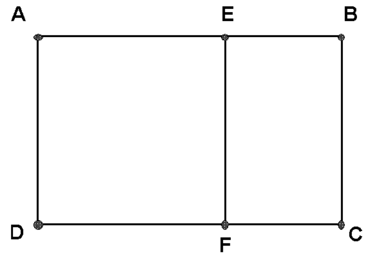
$\therefore 6^6 = (2^5\times 3)\times(2\times 3^5) \equiv -1 \pmod{97}$

$6^{2008} = (6^6)^{334}\times 6^4 \equiv (-1)^{334}\times 1296 \equiv 97\times 13 + 35 \equiv 35 \pmod{97}$

$\therefore d = 35$

Sample Group Event

SG.1 In Figure 1, *AEFD* is a unit square. The ratio of the length of the rectangle *ABCD* to its width is equal to the ratio of the length of the rectangle *BCFE* to its width. If the length of *AB* is *W* units, find the value of *W*.

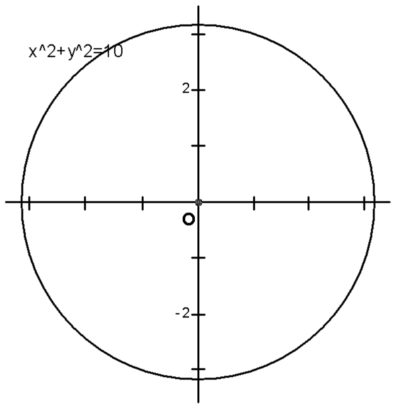


$\frac{W}{1}=\frac{1}{W-1}$

$W^2 - W - 1 = 0 \Rightarrow W=\frac{1+\sqrt{5}}{2}$

SG.2 On the coordinate plane, there are *T* points (*x*, *y*), where *x*, *y* are integers, satisfying $x^2 + y^2 < 10$, find the value of *T*.

T = number of integral points inside the circle $x^2 + y^2 = 10$.



We first count the number of integral points in the first quadrant:

$x = 1; y = 1, 2$

$x = 2; y = 1, 2$

Next, the number of integral points on the *x*-axis and *y*-axis

$= 3 + 3 + 3 + 3 + 1 = 13$

$T = 4\times4 + 3 + 3 + 3 + 3 + 1$

$= 29$

SG.3 Let *P* and *P* + 2 be both prime numbers satisfying $P(P + 2) \le 2007$. If *S* represents the sum of such possible values of *P*, find the value of *S*.

$P^2 + 2P - 2007 \le 0$

$(P + 1)^2 - 2008 \le 0$

$(P + 1 + \sqrt{2008})(P + 1 - \sqrt{2008}) \le 0$

$(P + 1 + 2\sqrt{502})(P + 1 - 2\sqrt{502}) \le 0$

$-1 - 2\sqrt{502} \le P \le -1 + 2\sqrt{502}$

P is a prime $\Rightarrow 0 < P \le -1 + 2\sqrt{502}$

$22 = \sqrt{484} < \sqrt{502} < \sqrt{529} = 23$

$43 < -1 + 2\sqrt{502} < 45$

$\therefore (P, P + 2) = (3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43)$

$S = 3 + 5 + 11 + 17 + 29 + 41 = 106$

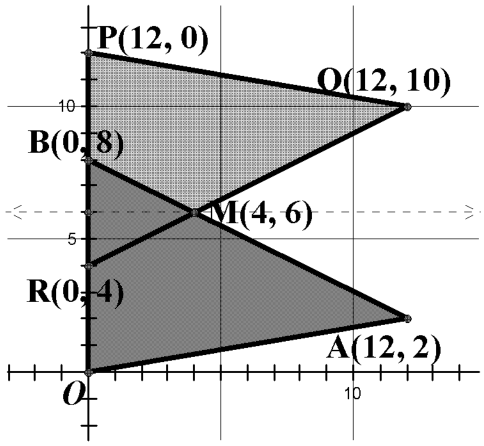
SG.4 It is known that $\log_{10}(2007^{2006} \times 2006^{2007}) = a\times10^k$, where $1 \le a < 10$ and *k* is an integer. Find the value of *k*.

$$\begin{aligned}2006 \log 2007 + 2007 \log 2006 &= 2006\times(\log 2007 + \log 2006) + \log 2006\\ &> 2006\times(\log 2006 + \log 2006) + \log 2006\\ &= 4013 \log 2006\\ &= 4013 \log(2.006\times10^3)\\ &= 4013 (\log 2.006 + 3)\\ &> 4013 \log 2 + 4013\times3 = 4013(\log 2 + 3)\\ &> 4013\times(0.3 + 3) = 13242.9 = 1.32429\times10^4\end{aligned}$$

$k = 4$

Group Event 1

G1.1 Given that there are three points on the coordinate plane: $O(0, 0)$, $A(12, 2)$ and $B(0, 8)$. A reflection of ΔOAB along the straight line $y = 6$ creates ΔPQR. If the overlapped area of ΔOAB and ΔPQR is m square units, find the value of m.



$P(12, 0)$, $Q(12, 10)$, $R(0, 4)$.

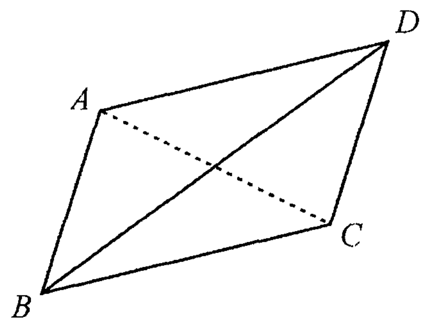
Suppose AB intersects QR at $M(x, 6)$.

Slope of MR = slope of QR

$\frac{6-4}{x} = \frac{10-4}{12} \Rightarrow x = 4$; $M(4, 6)$

Area of the overlap $\Delta BMR = \frac{1}{2}\cdot(8-4)\times 4 = 8$; $m = 8$

G1.2 In Figure 1, $ABCD$ is a parallelogram with $BA = 3$ cm, $BC = 4$ cm and $BD = \sqrt{37}$ cm. If $AC = h$ cm, find the value of h.



$CD = BA = 3$ cm (opp. sides, //-gram)

In ΔBCD, $\cos C = \frac{4^2 + 3^2 - \sqrt{37}^2}{2\times 3\times 4} = -\frac{1}{2}$

$\cos B = \cos(180° - C) = -\cos C = \frac{1}{2}$ (int. $\angle$s AB // DC)

In ΔABC, $AC = \sqrt{3^2 + 4^2 - 2\cdot 3\cdot 4\cos B} = \sqrt{13}$; $h = \sqrt{13}$

G1.3 Given that x, y and z are positive integers and the fraction $\frac{151}{44}$ can be written in the form of $3 + \cfrac{1}{x + \cfrac{1}{y + \cfrac{1}{z}}}$. Find the value of $x + y + z$.

$$\frac{151}{44} = 3 + \frac{19}{44} = 3 + \frac{1}{\frac{44}{19}} = 3 + \frac{1}{2 + \frac{6}{19}} = 3 + \cfrac{1}{2 + \cfrac{1}{\frac{19}{6}}} = 3 + \cfrac{1}{2 + \cfrac{1}{3 + \frac{1}{6}}}$$

$x = 2$, $y = 3$, $z = 6$; $x + y + z = 11$

G1.4 When 491 is divided by a two-digit integer, the remainder is 59. Find this two-digit integer.

Let the number be $10x + y$, where $0 < x \le 9$, $0 \le y \le 9$.

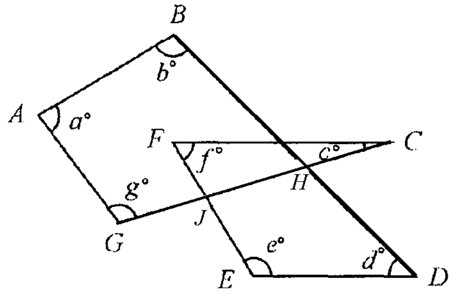
$491 = (10x + y)\cdot Q + 59$; $59 < 10x + y$

$491 - 59 = 432 = (10x + y)\cdot Q$; $432 = 72\times 6$; the number is 72.

Group Event 2

G2.1 In Figure 1, BD, FC, GC and FE are straight lines.

If $z = a + b + c + d + e + f + g$, find the value of z.



$a° + b° + g° + \angle BHG = 360°$ (∠s sum of polygon $ABHG$)

$c° + f° = \angle CJE$ (ext. ∠ of ΔCFJ)

$c° + f° + e° + d° + \angle JHD = 360°$ (∠s sum of polygon $JHDE$)

$a° + b° + g° + \angle BHG + c° + f° + e° + d° + \angle JHD = 720°$

$a° + b° + c° + d° + e° + f° + g° + 180° = 720°$

$z = 540$

G2.2 If R is the remainder of $1^6 + 2^6 + 3^6 + 4^6 + 5^6 + 6^6$ divided by 7, find the value of R.

$x^6 + y^6 = (x + y)(x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5) + 2y^6$

$6^6 + 1^6 = 7Q_1 + 2$; $5^6 + 2^6 = 7Q_2 + 2\times 2^6$; $4^6 + 3^6 = 7Q_3 + 2\times 3^6$

$2 + 2\times 2^6 + 2\times 3^6 = 2(1 + 64 + 729) = 1588 = 7\times 226 + 6$; $R = 6$

Method 2 $1^6 + 2^6 + 3^6 + 4^6 + 5^6 + 6^6 \equiv 1^6 + 2^6 + 3^6 + (-3)^6 + (-2)^6 + (-1)^6 \mod 7$

$\equiv 2(1^6 + 2^6 + 3^6) \equiv 2(1 + 64 + 729) \mod 7$

$\equiv 2(1 + 1 + 1) \mod 7$

$\equiv 6 \mod 7$

G2.3 If 14! is divisible by 6^k, where k is an integer, find the largest possible value of k.

We count the number of factors of 3 in 14!. They are 3, 6, 9, 12. So there are 5 factors of 3.

$k = 5$

G2.4 Let x, y and z be real numbers that satisfy $x + \frac{1}{y} = 4$, $y + \frac{1}{z} = 1$ and $z + \frac{1}{x} = \frac{7}{3}$. Find the value of xyz.

Method 1

From (1), $x = 4 - \frac{1}{y} = \frac{4y-1}{y}$

$\Rightarrow \frac{1}{x} = \frac{y}{4y-1}$ …… (4)

Sub. (4) into (3): $z + \frac{y}{4y-1} = \frac{7}{3}$

$z = \frac{7}{3} - \frac{y}{4y-1}$ …… (5)

From (2): $\frac{1}{z} = 1 - y$

$z = \frac{1}{1-y}$ …… (6)

(5) = (6): $\frac{1}{1-y} = \frac{7}{3} - \frac{y}{4y-1}$

$\frac{1}{1-y} = \frac{28y-7-3y}{3(4y-1)}$

$3(4y - 1) = (1 - y)(25y - 7)$

$12y - 3 = -25y^2 - 7 + 32y$

$25y^2 - 20y + 4 = 0$

$(5y - 2)^2 = 0$

$y = \frac{2}{5}$

Sub. $y = \frac{2}{5}$ into (6): $z = \frac{1}{1-\frac{2}{5}} = \frac{5}{3}$

Sub. $y = \frac{2}{5}$ into (1): $x + \frac{5}{2} = 4 \Rightarrow x = \frac{3}{2}$

$xyz = \frac{2}{5}\times\frac{5}{3}\times\frac{3}{2} = 1$

Method 2

$$\begin{cases} x + \frac{1}{y} = 4 \cdots\cdots (1) \\ y + \frac{1}{z} = 1 \cdots\cdots (2) \\ z + \frac{1}{x} = \frac{7}{3} \cdots\cdots (3) \end{cases}$$

(1)×(2): $xy + 1 + \frac{x}{z} + \frac{1}{yz} = 4$

$x\left(y + \frac{1}{z}\right) + \frac{1}{yz} = 3$

Sub. (2) into the eqt.: $x + \frac{x}{xyz} = 3$

Let $a = xyz$, then $x + \frac{x}{a} = 3 \cdots\cdots (4)$

(2)×(3): $y\left(\frac{7}{3}\right) + \frac{y}{a} = \frac{4}{3} \Rightarrow y\left(\frac{7}{3} + \frac{1}{a}\right) = \frac{4}{3} \cdots\cdots (5)$

(1)×(3): $z(4) + \frac{z}{a} = \frac{25}{3} \Rightarrow z\left(4 + \frac{1}{a}\right) = \frac{25}{3} \cdots\cdots (6)$

(4)×(5)×(6): $a\left(1 + \frac{1}{a}\right)\left(\frac{7}{3} + \frac{1}{a}\right)\left(4 + \frac{1}{a}\right) = \frac{100}{3}$

$\frac{(a+1)(7a+3)(4a+1)}{3a^2} = \frac{100}{3}$

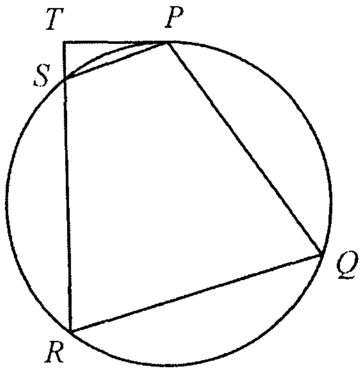
which reduces to $28a^3 - 53a^2 + 22a + 3 = 0$

$\Rightarrow (a - 1)^2(28a + 3) = 0$

$\therefore a = 1$

Group Event 3

G3.1 In Figure 1, $PQRS$ is a cyclic quadrilateral, where S is on the straight line RT and TP is tangent to the circle. If RS = 8 cm, RT = 11 cm and $TP = k$ cm, find the value of k.



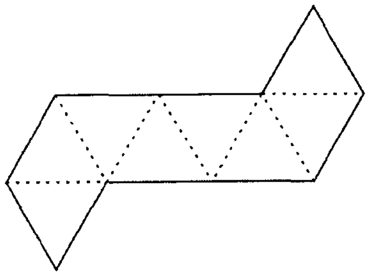
Join PR. $\angle SPT = \angle PRS$ ($\angle$ in alt. seg.)

$\angle STP = \angle PTR$ (common $\angle$)

$\therefore \Delta STP \sim \Delta PTR$ (equiangular)

$\frac{TP}{TR} = \frac{TS}{TP}$ (ratio of sides, ~ Δ) $\frac{k}{11} = \frac{11-8}{k}$; $k = \sqrt{33}$

G3.2 The layout in Figure 2 can be used to fold a polyhedron. If this polyhedron has v vertices, find the value of v.



There are 8 faces. $f = 8$.

There are 8 equilateral Δs, no of sides = $8\times3 = 24$

Each side is shared by 2 faces. Number of edge $e = 12$

By Euler formula, $v - e + f = 2$

$v - 12 + 8 = 2 \Rightarrow v = 6$

G3.3 For arbitrary real number x, define $[x]$ to be the largest integer less than or equal to x. For instance, $[2] = 2$ and $[3.4] = 3$. Find the value of $[1.008^8\times100]$.

$1.008^8\times100 = (1 + 0.008)^8\times100 = 100(1 + 8\times0.008 + 28\times0.008^2 + \ldots) \approx 106.4$

The integral value = 106

G3.4 When choosing, without replacement, 4 out of 30 labelled balls that are marked from 1 to 30, there are r combinations. Find the value of r.

$r = {}_{30}C_4 = \frac{30\times29\times28\times27}{1\times2\times3\times4} = 27405$

Group Event 4

G4.1 Regular tessellation is formed by identical regular m-polygons for some fixed m. Find the sum of all possible values of m.

Each interior angle $= \frac{180^\circ(m-2)}{m}$ ($\angle$s sum of polygon)

Suppose n m-polygons tessellate the space. $\frac{180^\circ(m-2)}{m}\cdot n = 360^\circ$ ($\angle$s at a point)

$n(m-2) = 2m \Rightarrow n(m-2) - 2m + 4 = 4 \Rightarrow (n-2)(m-2) = 4$

$m - 2 = 1$, 2 or 4; $m = 3$, 4 or 6; sum of all possible $m = 3 + 4 + 6 = 13$

G4.2 Amongst the seven numbers 3624, 36024, 360924, 3609924, 36099924, 360999924 and 3609999924, there are n of them that are divisible by 38. Find the value of n.

$38 = 2\times19$, we need to investigate which number is divisible by 19.

$19^2 = 361$, $3624 = 3610 + 13$; $36024 = 36100 - 76 = 100(19^2) - 19\times4$

$360924 = 361000 - 76$; $3609924 = 3610000 - 76$; $36099924 = 36100000 - 76$

$360999924 = 361000000 - 76$; $3609999924 = 3610000000 - 76$; $n = 6$

G4.3 If $208208 = 8^5a + 8^4b + 8^3c + 8^2d + 8e + f$, where a, b, c, d, e, and f are integers and $0 \le a, b, c, d, e, f \le 7$, find the value of $a\times b\times c + d\times e\times f$.

```
8|2 0 8 2 0 8
  8|2 6 0 2 6 ......0
    8|3 2 5 3 ......2
      8|4 0 6 ......5
        8|5 0 ......6
            6 ......2
```

$a = 6$, $b = 2$, $c = 6$, $d = 5$, $e = 2$, $f = 0$; $a\times b\times c + d\times e\times f = 72$

G4.4 In the coordinate plane, rotate point $A(6, 8)$ about the origin $O(0, 0)$ counter-clockwise for 20070° to point $B(p, q)$. Find the value of $p + q$.

$20070^\circ = 360^\circ\times55 + 270^\circ$, $\therefore B(8, -6)$; $p + q = 2$

Group Spare

GS.1 Calculate the value of $\left(\sqrt{2008}+\sqrt{2007}\right)^{2007}\times\left(\sqrt{2007}-\sqrt{2008}\right)^{2007}$.

$\left(\sqrt{2008}+\sqrt{2007}\right)^{2007}\times\left(\sqrt{2007}-\sqrt{2008}\right)^{2007}=(2007-2008)^{2007} = (-1)^{2007} = -1$

GS.2 If $x-\frac{1}{x}=\sqrt{2007}$, find the value of $x^4+\frac{1}{x^4}$.

$\left(x-\frac{1}{x}\right)^2 = 2007$

$x^2-2+\frac{1}{x^2} = 2007$

$x^2+\frac{1}{x^2} = 2009$

$\left(x^2+\frac{1}{x^2}\right)^2 = 2009^2 = 4036081$

$x^4 + 2 + \frac{1}{x^4} = 4036081$

$x^4+\frac{1}{x^4} = 4036079$

GS.3 Given that $\cos\alpha = -\frac{99}{101}$ and $180° < \alpha \le 270°$. Find the value of $\cot\alpha$.

$\sec\alpha = -\frac{101}{99}$, $\tan^2\alpha = \sec^2\alpha - 1 = \left(-\frac{101}{99}\right)^2 - 1 = \frac{101^2-99^2}{99^2} = \frac{(101-99)\cdot(101+99)}{99^2} = \frac{400}{99^2}$

$\tan\alpha = \frac{20}{99}$; $\cot\alpha = \frac{99}{20}$ $(= 4.95)$

GS.4 Calculate the value of $\frac{2008^3+4015^3}{2007^3+4015^3}$.

Let $x = 2007.5$, then $2x = 4015$

$$\frac{2008^3+4015^3}{2007^3+4015^3} = \frac{(x+0.5)^3+(2x)^3}{(x-0.5)^3+(2x)^3} = \frac{8\left(x+\frac{1}{2}\right)^3+8(2x)^3}{8\left(x-\frac{1}{2}\right)^3+8(2x)^3} = \frac{(2x+1)^3+(4x)^3}{(2x-1)+(4x)^3}$$

$$= \frac{(2x+1+4x)\left[(2x+1)^2-4x(2x+1)+(4x)^2\right]}{(2x-1+4x)\left[(2x-1)^2-4x(2x-1)+(4x)^2\right]}$$

$$= \frac{(6x+1)\left(4x^2+4x+1-8x^2-4x+16x^2\right)}{(6x-1)\left(4x^2-4x+1-8x^2+4x+16x^2\right)}$$

$$= \frac{(6x+1)\left(12x^2+1\right)}{(6x-1)\left(12x^2+1\right)} = \frac{6x+1}{6x-1} = \frac{6023}{6022}$$