05-06 Individual	1	250	2	$\frac{147\sqrt{3}}{242}$	3	–6	4	2006	5	60
	6	165	7	0	8	3	9	9	10	$\sqrt{5}$

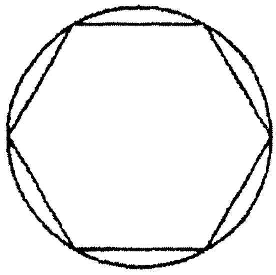
05-06 Group	1	2	2	10^{10}	3	$\frac{\sqrt{2}}{4}$	4	1	5	59
	6	0	7	4	8	328	9	$7\sqrt{2}$	10	$\frac{1}{3}$

Individual

I1 Let $\sqrt{20+\sqrt{300}}=\sqrt{x}+\sqrt{y}$, where x and y are rational numbers and $w = x^2 + y^2$, find the value of w.

$\sqrt{20+\sqrt{300}}=\sqrt{5+15+2\sqrt{5\times 15}}=\sqrt{\left(\sqrt{5}+\sqrt{15}\right)^2}=\sqrt{5}+\sqrt{15}$, $x = 5$, $y = 15$. $w = 5^2 + 15^2 = 250$

I2 In Figure 1, a regular hexagon is inscribed in a circle with circumference 4 m. If the area of the regular is A m^2, find the value of A. (Take $\pi = \frac{22}{7}$)



圖一
Figure 1

Let the radius be r m. $2\pi r = 4 \Rightarrow r = \frac{2}{\pi} = \frac{7}{11}$

A = area of 6 equilateral triangles each with side $= \frac{7}{11}$ m

$= 6\times\frac{1}{2}\times\frac{7}{11}\times\frac{7}{11}\times \sin 60° = \frac{147}{121}\cdot\frac{\sqrt{3}}{2}=\frac{147\sqrt{3}}{242}$

I3 Given that $\dfrac{1}{2+\dfrac{3}{1+\dfrac{1}{x}}}=\dfrac{5}{28}$, find the value of x.

$2+\dfrac{3}{1+\dfrac{1}{x}}=\dfrac{28}{5}\Rightarrow\dfrac{3}{1+\dfrac{1}{x}}=\dfrac{18}{5}\Rightarrow 1+\dfrac{1}{x}=\dfrac{5}{6}\Rightarrow\dfrac{1}{x}=-\dfrac{1}{6}\Rightarrow x=-6$

I4. Let $A=\dfrac{2006}{20052005^2-20052004\times 20052006}$, find the value of A.

Let $x = 2005$, then $20052005 = 2005\times 10001 = 10001x$

$20052004 = 20052005 - 1 = 10001x - 1$, $20052006 = 20052005 + 1 = 10001x + 1$,

$A=\dfrac{2006}{(10001x)^2-(10001x-1)\times(10001x+1)}=\dfrac{2006}{(10001x)^2-(10001x)^2+1}=2006$

I5 Given that $4\sec^2\theta° - \tan^2\theta° - 7\sec\theta° + 1 = 0$ and $0° \le \theta° \le 180°$, find the value of θ.

$4\sec^2\theta° - (\sec^2\theta° - 1) - 7\sec\theta° + 1 = 0$

$3\sec^2\theta° - 7\sec\theta° + 2 = 0$

$(3\sec\theta° - 1)(\sec\theta° - 2) = 0$

$\sec\theta° = \frac{1}{3}$ or $2 \Rightarrow \cos\theta° = 3$ (rejected) or $\frac{1}{2}\Rightarrow \theta = 60$

I6 Given that w, x, y and z are positive integers which satisfy the equation $w + x + y + z = 12$. If there are W sets of different positive integral solutions of the equation, find the value of W.

Reference: 2001 Heat Group Q2 ..... how many ways can 10 balls be put into 3 different boxes...?

$1 \le w \le 9$, keep w fixed, we shall find the number of solutions to $x + y + z = 12 - w$ ......(1)

$1 \le x \le 10 - w$, keep x fixed, we shall find the number of solutions to $y + z = 12 - w - x$ ......(2)

$1 \le y \le 11 - w - x$, keep y fixed, the number of solution to $z = 12 - w - x - y$ is 1

Total number of solutions $=\sum_{w=1}^{9}\sum_{x=1}^{10-w}\sum_{y=1}^{11-w-x}1=\sum_{w=1}^{9}\sum_{x=1}^{10-w}(11\text{-}w\text{-}x)$

$$=\sum_{w=1}^{9}\left[(11-w)(10-w)-(1+2+\cdots 10-w)\right]$$

$$=\sum_{w=1}^{9}\left[(11-w)(10-w)-\frac{1}{2}(11-w)(10-w)\right]$$

$$=\sum_{w=1}^{9}\left[\frac{1}{2}(11-w)(10-w)\right]=\frac{1}{2}\sum_{w=1}^{9}\left(110-21w+w^2\right)$$

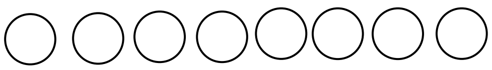
$$=\frac{1}{2}\left(990-21\times 45+\frac{9}{6}\cdot 10\cdot 19\right)$$

$$=\frac{45}{2}\left(22-21+\frac{19}{3}\right)=\frac{45}{2}\cdot\frac{22}{3}=165$$

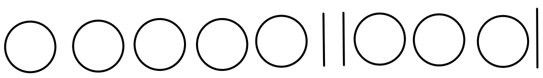
Method 2

The problem is equivalent to: put 12 identical balls into 4 labeled containers w, x, y and z. Each container should have at least one ball to ensure positive integral solution.

First we put one ball into each container, then the remaining 8 balls can be put into the containers in any order. The number of ways of putting the 8 balls is the number of positive integral solution to the equation. We shall align the 8 balls



Next, we shall divide these 8 balls into 4 groups by putting 3 sticks between the gaps of the balls.



For instance, 2 sticks are put between the fifth and the sixth balls, the third stick is put on the rightmost position. Then the 4 groups of balls are 5, 0, 3, 0 (i.e. the number of balls between two sticks.) then $w = 1 + 4 = 5$, $x = 1 + 0 = 1$, $y = 1 + 3 = 4$, $z = 1 + 0 = 1$ is a solution of the equation.

Thirdly, regard these 8 balls and the 3 sticks as 11 objects in one line.

					X	X				X

The number of ways is equivalent the number of choosing 3 objects as sticks from 11 objects, and regard the remaining 8 objects as balls.

The number of ways is ${}_{11}C_3=\frac{11}{1}\cdot\frac{10}{2}\cdot\frac{9}{3}=165$

I7 Given that the number of prime numbers in the sequence 1001, 1001001, 1001001001, ..., $1\underbrace{00}_{2}1\underbrace{00}_{2}1\cdots 1\underbrace{00}_{2}1$, ... is R, find the value of R.

$1001 = 7\times 11\times 13$ which is not a prime

Suppose there are n '1's in $1\underbrace{00}_{2}1\underbrace{00}_{2}1\cdots 1\underbrace{00}_{2}1$.

If n is divisible by 3, then the number itself is divisible by 3.

Otherwise, $1\underbrace{00}_{2}1\underbrace{00}_{2}1\cdots 1\underbrace{00}_{2}1=1+10^3+10^6+\ldots\ldots+10^{3(n-1)}$

$$=\frac{10^{3n}-1}{10^3-1}=\frac{10^{3n}-1}{999}$$

$$=\frac{(10^n-1)(10^{2n}+10^n+1)}{999}$$

$$=\frac{\underbrace{99\cdots 9}_{n}(10^{2n}+10^n+1)}{999}$$

$$= \frac{\underbrace{11\cdots1}_{n}\left(10^{2n}+10^{n}+1\right)}{111}$$

$\because n$ is not divisible by 3, $\underbrace{11\cdots1}_{n}$ and 111 are relatively prime.

LHS = $1\underbrace{00}_{2}1\underbrace{00}_{2}1\cdots1\underbrace{00}_{2}1$ is an integer $\Rightarrow$ RHS is an integer $\Rightarrow \dfrac{10^{2n}+10^{n}+1}{111}$ is an integer $\neq 1$

$\therefore 1\underbrace{00}_{2}1\underbrace{00}_{2}1\cdots1\underbrace{00}_{2}1$ = product of two integers

$\therefore 1\underbrace{00}_{2}1\underbrace{00}_{2}1\cdots1\underbrace{00}_{2}1$ is not a prime number $\Rightarrow$ there are no primes, $R = 0$

I8 Let $\lfloor x \rfloor$ be the largest integer not greater than x, for example, $\lfloor 2.5 \rfloor = 2$. If $B = \left\lfloor \log_7\left(462 + \log_2\lfloor \tan 60^\circ \rfloor + \sqrt{9872}\right)\right\rfloor$, find the value of B.

$B = \left\lfloor \log_7\left(462 + \log_2\lfloor \sqrt{3} \rfloor + \sqrt{9872}\right)\right\rfloor = \left\lfloor \log_7\left(462 + \log_2 1 + \sqrt{9872}\right)\right\rfloor = \left\lfloor \log_7\left(462 + \sqrt{9872}\right)\right\rfloor$

$\sqrt{99^2} = \sqrt{9801} < \sqrt{9872} < \sqrt{10000}$, $\sqrt{9872} = 99 + a$, $0 < a < 1$

$462 + \sqrt{9872} = 462 + 99 + a = 561 + a = 7\times80 + 1 + a = 7\times(7\times11 + 3) + 1 + a$

$= 7^2\times11 + 7\times3 + 1 + a = 7^3 + 7^2\times4 + 7\times3 + 1 + a$

$7^3 < 462 + \sqrt{9872} < 7^4$

$\log_7 7^3 < \log_7\left(462 + \sqrt{9872}\right) < \log_7 7^4$

$3 < \log_7\left(462 + \sqrt{9872}\right) < 4$

$\left\lfloor \log_7\left(462 + \sqrt{9872}\right)\right\rfloor = 3$

I9 Given that the units digit of 7^{2006} is C, find the value of C.

$7^1 = 7$, $7^2 \equiv 9 \pmod{10}$, $7^3 \equiv 3 \pmod{10}$, $7^4 \equiv 1$ (mode 10)

$7^{2006} = (7^4)^{501}\times7^2 \equiv 9 \bmod 10$

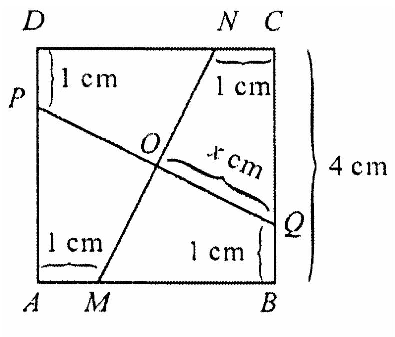
I10 In Figure 2, $ABCD$ is a square with side length equal to 4 cm. The line segments PQ and MN intersect at the point O. If the lengths of PD, NC, BQ and AM are 1 cm and the length of OQ is x cm, find the value of x.

$AP = BM = CQ = DN = 3$ cm

$PMQN$ is a rhombus (4 sides equal)

$PQ \perp MN$, $PO = OQ$, $MO = NO$ (property of rhombus)

$MN = PQ = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$ cm $\Rightarrow x = \sqrt{5}$



圖二
Figure 2

Group Events

G1 Let a, b and c are three prime numbers. If $a < b < c$ and $c = a^2 + b^2$, find the value of a.

Note that 2 is the only prime number which is even.

If $a \neq 2$ and $b \neq 2$, the a and b must be odd prime number, then c = odd + odd = even prime number > 2, which is a contradiction.

$\therefore a = 2$, by trail and error, $b = 3$, $c = 13 = 2^2 + 3^2$

G2 If $\log\left(\log\left(\log\left(\overbrace{100\cdots0}^{n\text{ zeros}}\right)\right)\right) = 1$, find the value of n.

$$\log\left(\log\left(\overbrace{100\cdots0}^{n\text{ zeros}}\right)\right) = 10 \Rightarrow \log\left(10^n\right) = 10^{10} \Rightarrow 10^n = 10^{\left(10^{10}\right)} \Rightarrow n = 10^{10}$$

G3 Given that $0° < \theta < 90°$ and $1 + \sin\theta + \sin^2\theta + ... = \frac{3}{2}$. If $y = \tan\theta$, find the value of y.

$$\frac{1}{1-\sin\theta} = \frac{3}{2} \Rightarrow 2 = 3 - 3\sin\theta \Rightarrow \sin\theta = \frac{1}{3} \Rightarrow \tan\theta = y = \frac{1}{\sqrt{8}} = \frac{\sqrt{2}}{4}$$

G4 Consider the quadratic equation $x^2 - (a - 2)x - a - 1 = 0$, where a is a real number. Let α and β be the roots of the equation. Find the value of a such that the value of $\alpha^2 + \beta^2$ will be the least.

$\alpha + \beta = a - 2$, $\alpha\beta = -a - 1$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (a - 2)^2 - 2(-a - 1) = a^2 - 4a + 4 + 2a + 2 = a^2 - 2a + 6 = (a-1)^2+5$

$\alpha^2 + \beta^2$ will be the least when $(a - 1)^2 = 0 \Rightarrow a = 1$

G5 Given that the sum of k consecutive positive integers is 2006, find the maximum possible value of k. (reference: HKMO 2004 heat group Q5)

Let the smallest positive integer be x: $x + x + 1 + ... + x + k - 1 = 2006$, $x > 0$

$$\frac{k}{2}\left[2x + (k-1)\right] = 2006$$

$k(2x + k - 1) = 4\times17\times59$, factors of 4012 are 1, 2, 4, 34, 59, 68, 118, 236, 1003, 2006, 4012

if $k = 4012$, then $2x + k - 1 = 2x + 4011 = 1$, no solution for $x > 0$

Similarly, if $k = 2006, 1003, 236, 118$ or 68, then there is no solution for $x > 0$

if $k = 59$, then $2x + k - 1 = 2x + 58 = 68$, $x = 5 \Rightarrow$ maximum possible $k = 59$

G6 Let a, b, c and d be real numbers such that $a^2 + b^2 = c^2 + d^2 = 1$ and $ac + bd = 0$. If $R = ab + cd$, find the value of R.

Let $a = \sin A$, $b = \cos A$, $c = \cos B$, $d = \sin B$

$ac + bd = 0 \Rightarrow \sin A\cos B + \cos A\sin B = 0 \Rightarrow \sin(A + B) = 0$

$$R = ab + cd = \sin A\cos A + \cos B\sin B = \frac{1}{2}(\sin 2A + \sin 2B) = \frac{1}{2}[2\sin(A + B)\cos(A - B)] = 0$$

Method 2

There is no need to let the sum = 1.

$R^2 = R^2 - 0^2 = (ab + cd)^2 - (ac + bd)^2$

$\quad = a^2b^2 + c^2d^2 - a^2c^2 - b^2d^2$

$\quad = a^2(b^2 - c^2) - d^2(b^2 - c^2)$

$R^2 = (b^2 - c^2)(a^2 - d^2)$ ....(1)

$a^2 + b^2 = c^2 + d^2 \Rightarrow b^2 - c^2 = d^2 - a^2$ ....(2)

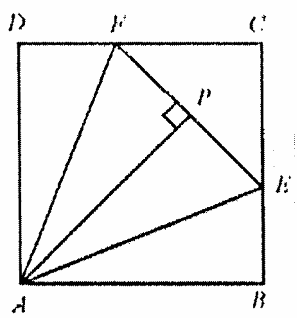
sub. (2) into (1): $R^2 = (d^2 - a^2)(a^2 - d^2) = -(a^2 - d^2)^2$

LHS ≥ 0, whereas RHS $\leq 0 \Rightarrow$ LHS = RHS = 0

$\therefore R = ab + cd = 0$

G7 In Figure 1, $ABCD$ is a square with perimeter equal to 16 cm, $\angle EAF = 45°$ and $AP \perp EF$. If the length of AP is equal to R cm, find the value of R.

(reference:http://twg.hkcampus.net/~twg-htw/Geometry/others/transform/Q5.pdf)



$AB = BC = CD = DA = 4$ cm

Let $BE = x$ cm, $DF = y$ cm.

Rotate ΔABE about A in anti-clockwise direction by 90°

Then $\Delta ABE \cong \Delta ADG$; $GD = x$ cm, $AG = AE$ (corr. sides $\cong$ Δ's)

$AE = AE$ (common side)

$\angle EAG = 90°$ ($\angle$ of rotation)

$\angle GAF = 90° - 45° = 45° = \angle EAF$

$\therefore \Delta AEF \cong \Delta AGF$ (SAS)

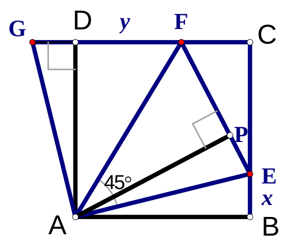
$\angle AGD = \angle AEP$ (corr. $\angle$s. $\cong\Delta$'s)

$\angle ADG = 90° = \angle APE$ (by rotation)

$AD = AB$ (sides of a square)

$\therefore \Delta ADG \cong \Delta APE$ (AAS)

$AP = AD = 4$ cm (corr. sides $\cong$ Δ's)



G8 Given that x and y are real numbers and satisfy the system of the equations

$$\begin{cases} \dfrac{100}{x+y}+\dfrac{64}{x-y}=9 \\ \dfrac{80}{x+y}+\dfrac{80}{x-y}=9 \end{cases}$$

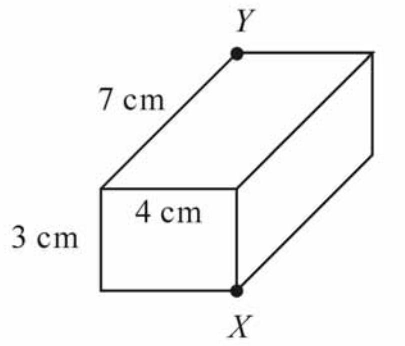
. If $V = x^2 + y^2$, find the value of V.

$$\frac{100}{x+y}+\frac{64}{x-y}=9=\frac{80}{x+y}+\frac{80}{x-y}\Rightarrow\frac{20}{x+y}=\frac{16}{x-y}\Rightarrow\frac{x+y}{5}=\frac{x-y}{4}=k$$

$x + y = 5k$, $x - y = 4k \Rightarrow \dfrac{100}{5k}+\dfrac{64}{4k}=9 \Rightarrow 20 + 16 = 9k \Rightarrow k = 4$

$x + y = 20$, $x - y = 16 \Rightarrow x = 18$, $y = 2$, $V = 18^2 + 2^2 = 328$

G9 In Figure 2, given a rectangular box with dimensions 3 cm, 4 cm, and 7 cm respectively. If the length of the shortest path on the surface of the box from point X to point Y is K cm, find the value of K.



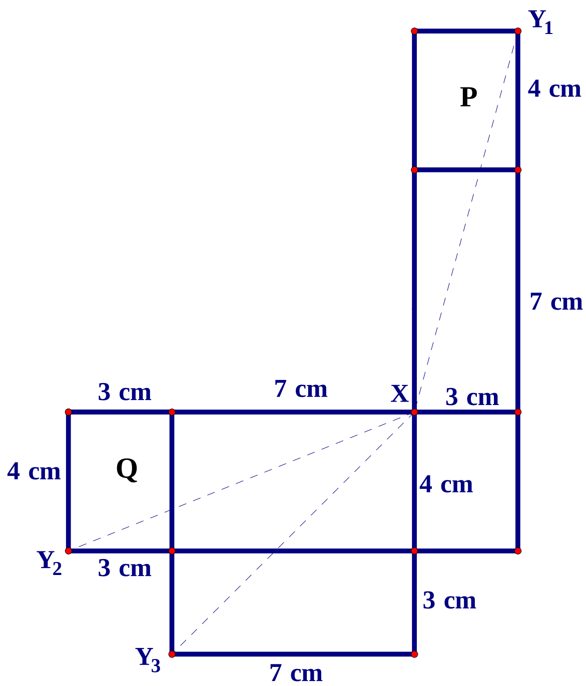
Unfold the rectangular box as follow. Note that the face P is identical to the face Q. There are 3 possible routes to go from X to Y:

$XY_1=\sqrt{3^2+(4+7)^2}\text{ cm}=\sqrt{130}\text{cm}$

$XY_2=\sqrt{4^2+(3+7)^2}\text{ cm}=\sqrt{116}\text{cm}$

$XY_3=\sqrt{7^2+(3+4)^2}\text{ cm}=7\sqrt{2}\text{cm}$

The length of the shortest path is $7\sqrt{2}$ cm.



G10 Given that x is a positive real number which satisfy the inequality $|x - 5| - |2x + 3| \le 1$, find the least value of x.

Reference: 2000-01 Heat Group Q9 ...... $|x - 3| + |x - 5| = 2$.......

Case 1: $x \le -1.5$, $5 - x + 2x + 3 \le 1$, $x \le -7$

Case 2: $-1.5 < x \le 5$, $5 - x - 2x - 3 \le 1$, $\dfrac{1}{3} \le x \Rightarrow \dfrac{1}{3} \le x \le 5$

Case 3: $5 < x$, $x - 5 - 2x - 3 \le 1$, $-9 \le x \Rightarrow 5 < x$

Combined solution: $x \le -7$ or $\dfrac{1}{3} \le x$

The least positive value of $x = \dfrac{1}{3}$