

Individual Events

I1	a	1	I2	R	$\frac{9}{25}$	I3	S	10	I4	k	1
	b	$\frac{1}{2}$		S	1		R	30		r	2
	C	10		T	1		T	6		s	$\frac{1}{\sqrt{2}} (= \frac{\sqrt{2}}{2})$
	D	-2		W	$\sqrt{5}$		P	$\sqrt{7} + 2$		w	9

Group Events

G1	k	1	G2	w	45	G3	r	2006	G4	R	$12\sqrt{3}$
	B	$\frac{16}{15}$		z	-13		x	$\frac{7}{4} (=1.75)$		S	8
	C	$\frac{1}{4}$		s	$\frac{1}{4}$		z	30		T	$\frac{1}{2}$
	a	1		t	14		R	$\frac{15}{4} (=3.75)$		W	2013021

Individual Event 1

I1.1 If a is a real number satisfying $\log_2(x + 3) - \log_2(x + 1) = 1$, find the value of a .

$$\log_2 \frac{x+3}{x+1} = \log_2 2$$

$$x + 3 = 2x + 2$$

$$x = 1 \Rightarrow a = 1$$

I1.2 In Figure 1, O is the centre of the circle with radius 1 cm. If the length of the arc AB is equal to a cm and the area of the shaded sector OAB is equal to b cm², find the value of b . (Take $\pi = 3$)

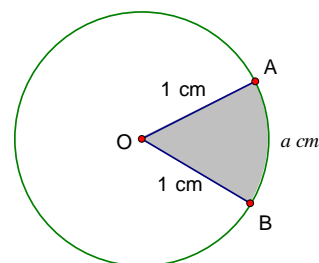


Figure 1

$$b = \frac{1}{2}rs = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$$

I1.3 An interior angle of a regular C -sided polygon is $288b^\circ$, find the value of C .

$$\text{Each interior angle} = 288b^\circ = 144^\circ$$

$$\text{Each exterior angle} = 36^\circ = \frac{360^\circ}{C}$$

$$C = 10$$

I1.4 Given that C is a root of the equation $kx^2 + 2x + 5 = 0$, where k is a constant. If D is another root, find the value of D .

$$100k + 20 + 5 = 0 \Rightarrow k = -\frac{1}{4}$$

$$C + D = \text{sum of roots} = -\frac{2}{k}$$

$$10 + D = 8 \Rightarrow D = -2$$

I2.1 Given that a, b and c are three numbers not equal to 0 and $a : b : c = 6 : 3 : 1$. If $R = \frac{3b^2}{2a^2 + bc}$, find the value of R .

$$\text{Let } a = 6k, b = 3k, c = k, \text{ then } R = \frac{3(3k)^2}{2(6k)^2 + (3k)k} = \frac{9}{25}$$

I2.2 Given that $\frac{|k + R|}{|R|} = 0$. If $S = \frac{|k + 2R|}{|2k + R|}$, find the value of S .

$$k + \frac{9}{25} = 0 \Rightarrow k = -\frac{9}{25}$$

$$S = \frac{|k + 2R|}{|2k + R|} = \frac{\left| -\frac{9}{25} + \frac{18}{25} \right|}{\left| -\frac{18}{25} + \frac{9}{25} \right|} = 1$$

I2.3 Given that $T = \sin 50^\circ \times (S + \sqrt{3} \times \tan 10^\circ)$, find the value of T .

$$\begin{aligned} T &= \sin 50^\circ \times \left(1 + \sqrt{3} \cdot \frac{\sin 10^\circ}{\cos 10^\circ} \right) \\ &= \frac{\sin 50^\circ}{\cos 10^\circ} \cdot (\cos 10^\circ + \sqrt{3} \sin 10^\circ) \\ &= \frac{2 \sin 50^\circ}{\cos 10^\circ} \left(\frac{1}{2} \cos 10^\circ + \frac{\sqrt{3}}{2} \sin 10^\circ \right) \\ &= \frac{2 \sin 50^\circ}{\cos 10^\circ} (\cos 60^\circ \cdot \cos 10^\circ + \sin 60^\circ \cdot \sin 10^\circ) \\ &= \frac{2 \sin 50^\circ \cos 50^\circ}{\cos 10^\circ} = \frac{\sin 100^\circ}{\cos 10^\circ} = 1 \end{aligned}$$

I2.4 Given that x_0 and y_0 are real numbers satisfying the system of equations $\begin{cases} y = \frac{T}{x} \\ y = |x| + T \end{cases}$.

If $W = x_0 + y_0$, find the value of W .

$$\begin{cases} y = \frac{1}{x} \\ y = |x| + 1 \end{cases}$$

$$\frac{1}{x} = |x| + 1 \dots\dots\dots (*)$$

$$\frac{1}{x} = x + 1 \text{ or } \frac{1}{x} = -x + 1$$

$$1 = x^2 + x \text{ or } 1 = -x^2 + x$$

$$x^2 + x - 1 = 0 \text{ or } x^2 - x + 1 = 0$$

$$x = \frac{-1 \pm \sqrt{5}}{2} \text{ or no solution}$$

$$\text{check: sub. } x = \frac{-1 - \sqrt{5}}{2} \text{ into } (*)$$

$$\text{LHS} = \frac{2}{-1 - \sqrt{5}} < 0, \text{ RHS} > 0 \text{ (rejected)}$$

$$\text{When } x = \frac{-1 + \sqrt{5}}{2}; \text{ LHS} = \frac{2}{-1 + \sqrt{5}} = \frac{\sqrt{5} + 1}{2}; \text{ RHS} = \frac{-1 + \sqrt{5}}{2} + 1 = \frac{1 + \sqrt{5}}{2} \text{ (accepted)}$$

$$y = \frac{1 + \sqrt{5}}{2} \Rightarrow W = x_0 + y_0 = \frac{-1 + \sqrt{5}}{2} + \frac{1 + \sqrt{5}}{2} = \sqrt{5}$$

Individual Event 3

I3.1 Given that $\frac{2x-3}{x^2-x} = \frac{A}{x-1} + \frac{B}{x}$, where A and B are constants. If $S = A^2 + B^2$, find the value of S

$$\frac{2x-3}{x^2-x} = \frac{Ax+B(x-1)}{(x-1)x}$$

$$A+B=2, -B=-3$$

$$A=-1, B=3$$

$$S = (-1)^2 + 3^2 = 10$$

I3.2 In Figure 1, $ABCD$ is an inscribed rectangle, $AB = (S-2)$ cm and $AD = (S-4)$ cm. If the circumference of the circle is R cm, find the value of R . (Take $\pi = 3$)

$$AB = 8, CD = 6$$

$$AC = 10 \text{ cm (Pythagoras' theorem)}$$

$$R = 10\pi = 30$$

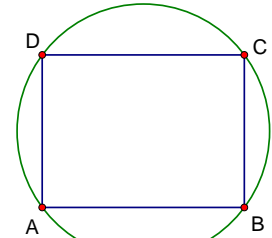


Figure 1

I3.3 Given that x and y are integers satisfying the equation $\frac{R}{2}xy = 21x + 20y - 13$. If $T = xy$, find the value of T .

$$15xy = 21x + 20y - 13$$

$$(3x-4)(5y-7) = 15$$

$$\begin{cases} 3x-4=1 \\ 5y-7=15 \end{cases} \text{ or } \begin{cases} 3x-4=3 \\ 5y-7=5 \end{cases} \text{ or } \begin{cases} 3x-4=5 \\ 5y-7=3 \end{cases} \text{ or } \begin{cases} 3x-4=15 \\ 5y-7=1 \end{cases} \text{ or } \begin{cases} 3x-4=-1 \\ 5y-7=-15 \end{cases} \text{ or } \\ \begin{cases} 3x-4=-3 \\ 5y-7=-5 \end{cases} \text{ or } \begin{cases} 3x-4=-5 \\ 5y-7=-3 \end{cases} \text{ or } \begin{cases} 3x-4=-15 \\ 5y-7=-1 \end{cases}$$

For integral solution: $3x-4=5, 5y-7=3$

$$x=3, y=2 \Rightarrow T=3 \times 2 = 6$$

I3.4 Let a be the positive root of the equation $x^2-2x-T=0$. If $P = 3 + \frac{T}{2 + \frac{T}{2 + \frac{T}{2 + \frac{T}{a}}}}$, find the value

of P .

$$x^2 - 2x - 6 = 0$$

$$a = 1 + \sqrt{7}$$

$$a^2 - 2a - 6 = 0 \Rightarrow a^2 = 2a + 6 \Rightarrow a = 2 + \frac{6}{a}$$

$$2 + \frac{T}{a} = 2 + \frac{6}{a} = a \Rightarrow 2 + \frac{T}{2 + \frac{T}{a}} = 2 + \frac{6}{a} = a \Rightarrow 2 + \frac{T}{2 + \frac{T}{2 + \frac{T}{a}}} = 2 + \frac{6}{a} = a$$

$$P = 3 + \frac{6}{a} = 1 + 2 + \frac{6}{a} = 1 + a = 1 + 1 + \sqrt{7} = 2 + \sqrt{7}$$

Individual Event 4

I4.1 Let $\frac{k}{4} = \left(1 + \frac{1}{2} + \frac{1}{3}\right) \times \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) \times \left(\frac{1}{2} + \frac{1}{3}\right)$, find the value of k .

Let $x = 1 + \frac{1}{2} + \frac{1}{3}$, $y = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$, then $\frac{k}{4} = x(y - 1) - y(x - 1) = -x + y = \frac{1}{4} \Rightarrow k = 1$

I4.2 Let x and y be real numbers satisfying the equation $y^2 + 4y + 4 + \sqrt{x + y + k} = 0$.

If $r = |xy|$, find the value of r .

The equation is equivalent to $(y + 2)^2 + \sqrt{x + y + k} = 0$

which is a sum of two non-negative numbers.

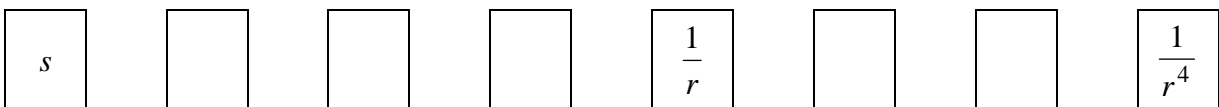
$\Rightarrow y + 2 = 0$ and $x + y + 1 = 0$

$y = -2$ and $x - 2 + 1 = 0$, $x = 1$

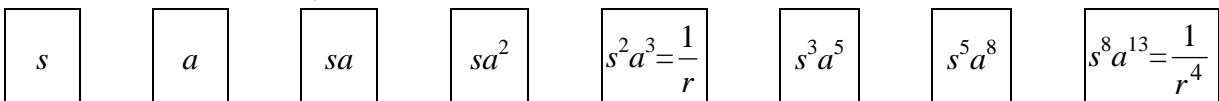
$r = |-2 \times 1| = 2$

I4.3 In Figure 1, there are eight positive numbers in series. Starting from the 3rd number, each number is the product of the previous two numbers. Given that the 5th number is $\frac{1}{r}$ and the

8th number is $\frac{1}{r^4}$. If the first number is s , find the value of s .



Let the 2nd number be a , then other numbers are as follows:



$s^2a^3 = \frac{1}{r}$ (1)

$s^8a^{13} = \frac{1}{r^4}$ (2)

$(2) \div (1)^4: a = 1$

sub. $a = 1$ into (1): $s^2 = \frac{1}{r} = \frac{1}{2}; s > 0 \Rightarrow s = \frac{1}{\sqrt{2}}$

I4.4 Let $[x]$ be the largest integer not greater than x . For example, $[2.5] = 2$.

Let $w = 1 + [10 \times s^2] + [10 \times s^4] + [10 \times s^6] + \dots + [10 \times s^{2n}] + \dots$, find the value of w .

$w = 1 + [10 \times \frac{1}{2}] + [10 \times \frac{1}{4}] + [10 \times \frac{1}{8}] + \dots + [10 \times \frac{1}{2^n}] + \dots$

$= 1 + 5 + 2 + 1 + 0 + 0 + \dots = 9$

Group Event 1

G1.1 Given that k is a real number. If $x^2 + 2kx - 3k^2$ can be divisible by $x - 1$, find the greatest value of k .

By factor theorem, $1^2 + 2k - 3k^2 = 0$

$$3k^2 - 2k - 1 = 0$$

$$(3k + 1)(k - 1) = 0$$

$$k = -\frac{1}{3} \text{ or } 1$$

Greatest value of $k = 1$

G1.2 Given that $x = x_0$ and $y = y_0$ satisfy the system of equations $\begin{cases} \frac{x}{3} + \frac{y}{5} = 1 \\ \frac{x}{5} + \frac{y}{3} = 1 \end{cases}$. If $B = \frac{1}{x_0} + \frac{1}{y_0}$, find the value of B .

$$\frac{x}{3} + \frac{y}{5} = \frac{x}{5} + \frac{y}{3} = 1 \Rightarrow \frac{x}{3} - \frac{x}{5} = \frac{y}{3} - \frac{y}{5} \Rightarrow x = y$$

$$\text{Sub. } x = y \text{ into the first equation: } \frac{x}{3} + \frac{x}{5} = 1 \Rightarrow x = y = \frac{15}{8} \Rightarrow B = \frac{16}{15}$$

G1.3 Given that $x = 2 + \sqrt{3}$ is a root of the equation $x^2 - (\tan \alpha + \cot \alpha)x + 1 = 0$.

If $C = \sin \alpha \times \cos \alpha$, find the value of C .

Let the other root be y , $(2 + \sqrt{3})y = \text{product of roots} = 1 \Rightarrow y = 2 - \sqrt{3}$

$$\tan \alpha + \cot \alpha = \text{sum of roots} = 2 + \sqrt{3} + 2 - \sqrt{3} = 4$$

$$\frac{\sin \alpha}{\cos \alpha} + \frac{\cos \alpha}{\sin \alpha} = 4 \Rightarrow \frac{1}{\sin \alpha \cos \alpha} = 4 \Rightarrow C = \sin \alpha \times \cos \alpha = \frac{1}{4}$$

G1.4 Let a be an integer. If the inequality $|x + 1| < a - 1.5$ has no integral solution, find the greatest value of a .

$\because |x + 1| \geq 0$, In order that the equation has no integral solution, it is sufficient that $a - 1.5 < 0$
 $a < 1.5$

Greatest integral value of $a = 1$

Group Event 2

G2.1 In Figure 1, PRS is a straight line, $PQ = PR = QS$ and

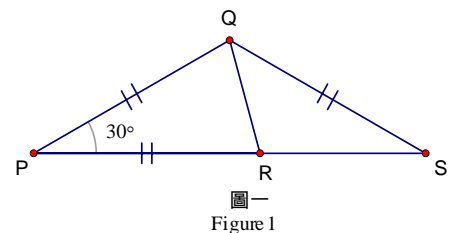
$\angle QPR = 30^\circ$. If $\angle RQS = w^\circ$, find the value of w .

$\angle QPR = \angle QSP = 30^\circ$ (base \angle s isos. Δ)

$\angle PQS = 120^\circ$ (\angle s sum of Δ)

$\angle PQR = \angle PRQ = (180^\circ - 30^\circ) \div 2 = 75^\circ$ (\angle s sum of isos. Δ)

$\angle RQS = 120^\circ - 75^\circ = 45^\circ$



G2.2 Let $f(x) = px^7 + qx^3 + rx - 5$, where p, q and r are real numbers.

If $f(-6) = 3$ and $z = f(6)$ Find the value of z .

$$f(-6) = 3 \Rightarrow -p \times 6^7 - q \times 6^3 - 6r - 5 = 3$$

$$f(6) = p \times 6^7 + q \times 6^3 + 6r - 5 = -(-p \times 6^7 - q \times 6^3 - 6r - 5) - 10 = -3 - 10 = -13$$

G2.3 If $n \neq 0$ and $s = \left(\frac{20}{2^{2n+4} + 2^{2n+2}} \right)^{\frac{1}{n}}$, find the value of s .

$$s = \left(\frac{20}{16 \cdot 2^{2n} + 4 \cdot 2^{2n}} \right)^{\frac{1}{n}} = \left(\frac{20}{20 \cdot 2^{2n}} \right)^{\frac{1}{n}} = \frac{1}{4}$$

G2.4 Given that x and y are positive integers and $x + y + xy = 54$. If $t = x + y$, find the value of t .

$$1 + x + y + xy = 55$$

$$(1 + x)(1 + y) = 55$$

$$1 + x = 5, 1 + y = 11 \text{ or } 1 + x = 11, 1 + y = 5$$

$$x = 4, y = 10 \text{ or } x = 10, y = 4$$

$$t = 14$$

G3.1 Given that $r = 2006 \times \frac{\sqrt{8} - \sqrt{2}}{\sqrt{2}}$, find the value of r . It is easy to show that $r = 2006$.

G3.2 Given that $6^{x+y} = 36$ and $6^{x+5y} = 216$, find the value of x .

$$x + y = 2 \dots\dots\dots (1)$$

$$x + 5y = 3 \dots\dots\dots (2)$$

$$5(1) - (2): 4x = 7 \Rightarrow x = \frac{7}{4}$$

G3.3 Given that $\tan x + \tan y + 1 = \cot x + \cot y = 6$. If $z = \tan(x + y)$, find the value of z .

$$\tan x + \tan y + 1 = \frac{\tan y + \tan x}{\tan x \tan y} = 6$$

$$\tan x + \tan y = 5; \tan x \tan y = \frac{5}{6}$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{5}{1 - \frac{5}{6}} = 30$$

G3.4 In Figure 1, $ABCD$ is a rectangle, F is the midpoint of CD and $BE : EC = 1 : 3$. If the area of the rectangle $ABCD$ is 12 cm^2 and the area of $BEFD$ is $R \text{ cm}^2$, find the value of R .

$$\text{Area of } \triangle BCD = 6 \text{ cm}^2$$

$$\text{Area of } \triangle CEF = \frac{3}{4} \cdot \frac{1}{2} \cdot 6 \text{ cm}^2 = \frac{9}{4} \text{ cm}^2$$

$$\text{Area of } BEFD = (6 - \frac{9}{4}) \text{ cm}^2 = \frac{15}{4} \text{ cm}^2$$

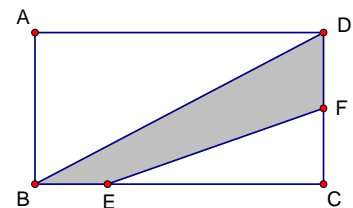


Figure 1

G4.1 In Figure 1, $ABCD$ is a parallelogram, $BE \perp CD$, $BF \perp AD$, $CE = 2$ cm, $DF = 1$ cm and $\angle EBF = 60^\circ$. If the area of the parallelogram $ABCD$ is R cm², find the value of R .

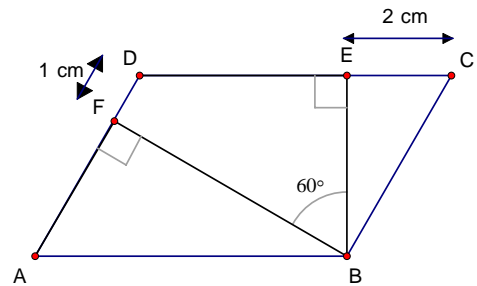


Figure 1

$\angle EDF = 360^\circ - 90^\circ - 90^\circ - 60^\circ = 120^\circ$ (\angle s sum of polygon)

$\angle BAD = \angle BCD = 180^\circ - 120^\circ = 60^\circ$ (int. \angle s // -lines)

$$BC = \frac{2}{\cos 60^\circ} \text{ cm} = 4 \text{ cm} = AD$$

$$BE = 2 \tan 60^\circ = 2\sqrt{3} \text{ cm}$$

$$AF = (4 - 1) \text{ cm} = 3 \text{ cm}$$

$$AB = \frac{3}{\cos 60^\circ} \text{ cm} = 6 \text{ cm}$$

$$\text{Area of } ABCD = AB \times BE = 6 \times 2\sqrt{3} \text{ cm}^2 = 12\sqrt{3} \text{ cm}^2$$

G4.2 Given that a and b are positive real numbers and $a + b = 2$. If $S = \left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2$, find

the minimum value S .

$$\left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0 \Rightarrow a + b - 2\sqrt{ab} \geq 0 \Rightarrow 1 \geq \sqrt{ab} \Rightarrow 1 \geq ab \dots\dots\dots (1)$$

$$a^2 + b^2 = (a + b)^2 - 2ab = 4 - 2ab \geq 4 - 2 = 2 \dots\dots\dots (2)$$

$$\frac{1}{ab} \geq 1 \Rightarrow \frac{1}{a^2 b^2} \geq 1 \dots\dots\dots (3)$$

$$S = \left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 = a^2 + b^2 + \frac{1}{a^2} + \frac{1}{b^2} + 4$$

$$= a^2 + b^2 + \frac{a^2 + b^2}{a^2 b^2} + 4$$

$$= (a^2 + b^2) \left(1 + \frac{1}{a^2 b^2}\right) + 4$$

$$\geq 2 \times (1 + 1) + 4 \text{ (by (2) and (3))}$$

$$= 8$$

G4.3 Let $2^x = 7^y = 196$. If $T = \frac{1}{x} + \frac{1}{y}$, find the value of T .

$$x \log 2 = y \log 7 = \log 196$$

$$x = \frac{\log 196}{\log 2}, y = \frac{\log 196}{\log 7}$$

$$T = \frac{1}{x} + \frac{1}{y} = \frac{\log 2 + \log 7}{\log 196} = \frac{\log 14}{\log 14^2} = \frac{1}{2}$$

G4.4 If $W = 2006^2 - 2005^2 + 2004^2 - 2003^2 + \dots + 4^2 - 3^2 + 2^2 - 1^2$, find the value of W .

$$W = (2006 + 2005)(2006 - 2005) + (2004 + 2003)(2004 - 2003) + \dots + (4 + 3)(4 - 3) + (2 + 1)(2 - 1)$$

$$= 2006 + 2005 + 2004 + \dots + 4 + 3 + 2 + 1$$

$$= \frac{2006}{2}(2006 + 1) = 1003 \times 2007 = 2013021$$