Individual Events

I1	a	9	I2	a	15	I3	A	$\frac{\sqrt{7}}{4}$	I4	a	3
	b	125		b	999985		b	$\frac{\sqrt{3}}{3}$ or $\frac{1}{\sqrt{3}}$		b	2
	c	5		c	4		c	7		c	6
	d	$2\frac{1}{2}$ or 2.5		d	$\frac{509}{256}$		d	$\frac{100}{11}$		d	171

Group Events

G1	a	40	G2	a	7	G3	a	1	G4	a	1
	b	70		b	5		b	0.0625 or $\frac{1}{16}$		b	9
	C	4		C	35		c	0.5 or $\frac{1}{2}$		c	20
	d	20		d	6		d	6		d	6

Individual Event 1

I1.1 Suppose there are a numbers between 1 and 200 that can be divisible by 3 and 7, find the value of a.

The number which can be divisible be 3 and 7 are multiples of 21. 200÷21 = 9.52, $a = 9$

I1.2 Let p and q be prime numbers that are the two distinct roots of the equation $x^2 - 13x + R = 0$, where R is a real number. If $b = p^2 + q^2$, find the value of b.

$x^2 - 13x + R = 0$, roots p and q are prime numbers. $p + q = 13$, $pq = R$

The sum of two prime numbers is 13, so one is odd and the other is even, $p = 2$, $q = 11$

$b = p^2 + q^2 = 2^2 + 11^2 = 125$

I1.3 Given that $\tan\alpha = -\frac{1}{2}$. If $c = \frac{2\cos\alpha - \sin\alpha}{\sin\alpha + \cos\alpha}$, find the value of c.

$$\tan\alpha = -\frac{1}{2}.\quad c = \frac{2\cos\alpha - \sin\alpha}{\sin\alpha + \cos\alpha} = \frac{2 - \tan\alpha}{\tan\alpha + 1} = \frac{2 + \frac{1}{2}}{-\frac{1}{2} + 1} = 5$$

I1.4 Let r and s be the two distinct real roots of the equation $2\left(x^2 + \frac{1}{x^2}\right) - 3\left(x + \frac{1}{x}\right) = 1$. If $d = r + s$, find the value of d.

$2\left(x^2 + \frac{1}{x^2}\right) - 3\left(x + \frac{1}{x}\right) = 1$, real roots r, s. Let $t = x + \frac{1}{x}$, then $x^2 + \frac{1}{x^2} = t^2 - 2$.

$2(t^2 - 2) - 3t = 1$

$2t^2 - 3t - 5 = 0$

$(2t - 5)(t + 1) = 0$

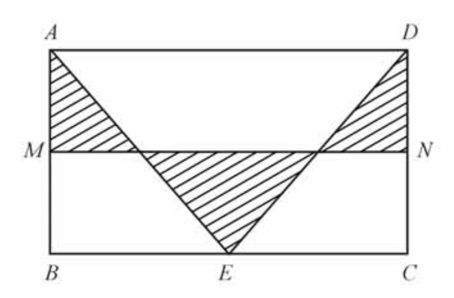
$t = \frac{5}{2}$ or -1

$x + \frac{1}{x} = \frac{5}{2}$ or $x + \frac{1}{x} = -1$

$x = 2$ or $\frac{1}{2}$ $\Rightarrow r = 2, s = \frac{1}{2} \Rightarrow d = r + s = \frac{5}{2}$

Individual Event 2

I2.1 In Figure 1, *ABCD* is a rectangle, $AB = 6$ cm and $BC = 10$ cm. *M* and *N* are the midpoints of *AB* and *DC* respectively. If the area of the shaded region is a cm^2, find the value of *a*.



$a = \frac{1}{4}$ area of rectangle $= \frac{1}{4} \times 6 \times 10 = 15$

I2.2 Let $b = 89 + 899 + 8999 + 89999 + 899999$, find the value of *b*.

$b = 89 + 899 + 8999 + 89999 + 899999$

= (90–1)+(900–1)+ (9000–1) + (90000–1) + (900000–1)

= 999990 – 5 = 999985

I2.3 Given that $2x + 5y = 3$. If $c = \sqrt{4^{x+\frac{1}{2}} \times 32^y}$, find the value of *c*.

$2x + 5y = 3$. $c = \sqrt{4^{x+\frac{1}{2}} \times 32^y} = \sqrt{2^{2x+1} \times 2^{5y}} = \sqrt{2^{2x+5y+1}} = \sqrt{2^{3+1}} = 4$

I2.4 Let $d = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \ldots + \frac{10}{2^{10}}$, find the value of *d*.

$d = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \ldots + \frac{10}{2^{10}}$, $2d = 1 + 1 + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \ldots + \frac{10}{2^9}$

$2d - d = 1 + 1 + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \ldots + \frac{10}{2^9} - \left(\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \ldots + \frac{10}{2^{10}}\right)$

$= 1 + (1 - \frac{1}{2}) + (\frac{3}{4} - \frac{2}{4}) + (\frac{4}{8} - \frac{3}{8}) + (\frac{5}{16} - \frac{4}{16}) + \ldots\ldots + (\frac{10}{2^9} - \frac{9}{2^9}) - \frac{10}{2^{10}}$

$d = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\ldots + \frac{1}{2^9} - \frac{10}{1024} = \frac{1 - \frac{1}{2^{10}}}{1 - \frac{1}{2}} - \frac{5}{512}$

$= \frac{1023}{512} - \frac{5}{512} = \frac{1018}{512} = \frac{509}{256}$

Individual Event 3

I3.1 Let 0° < α < 45°. If $\sin\alpha\cos\alpha = \frac{3\sqrt{7}}{16}$ and $A = \sin\alpha$, find the value of A.

Method 1 $2\sin\alpha\cos\alpha = \frac{3\sqrt{7}}{8}$

$\sin 2\alpha = \frac{3\sqrt{7}}{8}$

$\cos 2\alpha = \sqrt{1-\sin^2 2\alpha} = \sqrt{1-\left(\frac{3\sqrt{7}}{8}\right)^2}$

$= \frac{1}{8}\sqrt{64-63} = \frac{1}{8}$

$1 - 2\sin^2\alpha = \frac{1}{8} \Rightarrow \sin\alpha = \frac{\sqrt{7}}{4}$

Method 2 $2\sin\alpha\cos\alpha = \frac{3\sqrt{7}}{8}$,

$\sin 2\alpha = \frac{3\sqrt{7}}{8}$

$t = \tan\alpha$, $\sin 2\alpha = \frac{2t}{1+t^2} = \frac{3\sqrt{7}}{8}$

$16t = 3\sqrt{7} + 3\sqrt{7}\,t^2$

$3\sqrt{7}\,t^2 - 16t + 3\sqrt{7} = 0$

$(3t - \sqrt{7})(\sqrt{7}\,t - 3) = 0$

$t = \frac{\sqrt{7}}{3}$ or $\frac{3}{\sqrt{7}}$

$0° < \alpha < 45° \Rightarrow t = \tan\alpha < 1 \Rightarrow \tan\alpha = \frac{\sqrt{7}}{3}$

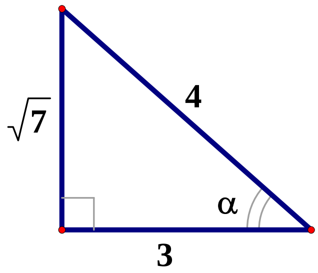
$A = \sin\alpha = \frac{\sqrt{7}}{4}$

Method 3

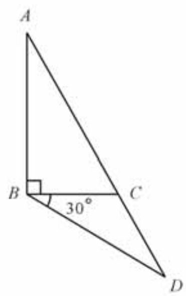
$\sin\alpha\cos\alpha = \frac{3\sqrt{7}}{16} = \frac{3}{4}\times\frac{\sqrt{7}}{4}$

$\sin\alpha = \frac{3}{4}$, $\cos\alpha = \frac{\sqrt{7}}{4}$ or $\sin\alpha = \frac{\sqrt{7}}{4}$, $\cos\alpha = \frac{3}{4}$

$\because 0° < \alpha < 45°$, $\therefore \sin\alpha < \cos\alpha$, $\sin\alpha = \frac{\sqrt{7}}{4}$



I3.2 In figure 1, C lies on AD, $AB = BD = 1$ cm, $\angle ABC = 90°$ and $\angle CBD = 30°$. If $CD = b$ cm, find the value of b.



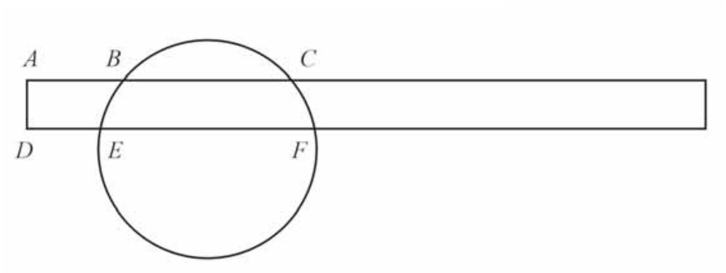
$AB = BD = 1$ cm, ΔABD is isosceles.

$\angle BAD = \angle BDA = (180° - 90° - 30°) \div 2 = 30°$ ($\angle$s sum of isosceles Δ)

ΔBCD is also isosceles.

$CD = b$ cm $= BC = AB\tan\angle BAD = 1\tan 30°$ cm $= \frac{1}{\sqrt{3}}$ cm

I3.3 In Figure 2, a rectangle intersects a circle at points B, C, E and F. Given that $AB = 4$ cm, $BC = 5$ cm and $DE = 3$ cm. If $EF = c$ cm, find the value of c.



$AB = 4$ cm, $BC = 5$ cm and $DE = 3$ cm. $EF = c$ cm

Draw $BG \perp DF$, $CH \perp DF$

$DG = AB = 4$ cm, $GH = BC = 5$ cm

$EG = DG - DE = 4$ cm $- 3$ cm $= 1$cm

Let O be the centre.

Let M be the foot of perpendicular of O on EF and produce OM to N on BC.

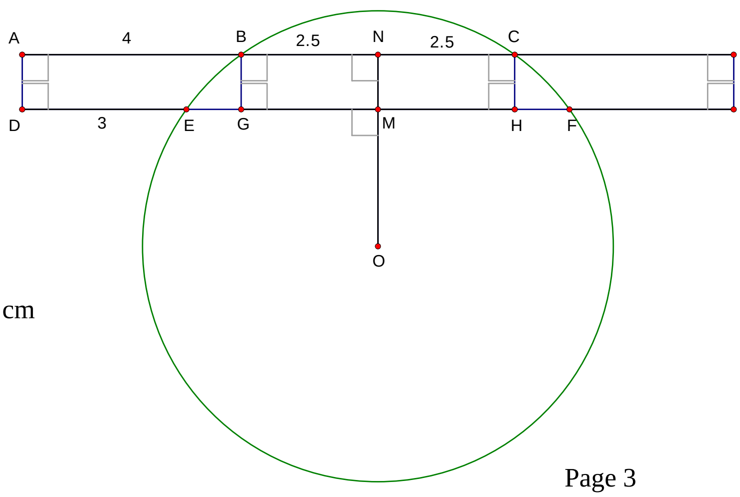
$ON \perp BC$ (corr. $\angle$s AC // DF)

$BN = NC = 2.5$ cm ($\perp$ from centre bisect chord)

$MF = EM$ ($\perp$ from centre bisect chords)

$= EG + GM = 1$ cm $+ BN = 1$ cm $+ 2.5$ cm $= 3.5$ cm

$EF = 2EM = 7$ cm



I3.4 Let x and y be two positive numbers that are inversely proportional to each other. If x is increased by 10 %, y will be decreased by d %, find the value of d.

$x y = k$, $x_1 = 1.1 x$

$x_1 y_1 = xy \Rightarrow 1.1xy_1 = xy$

$y_1 = \frac{10y}{11}$, Percentage decrease $= \frac{y - \frac{10y}{11}}{y} \times 100\% = \frac{100}{11}\%$, $d = \frac{100}{11}$

Individual Event 4

I4.1 If $a = \log_{\frac{1}{2}} 0.125$, find the value of a.

$$a = \log_{\frac{1}{2}} 0.125 = \frac{\log 0.125}{\log \frac{1}{2}} = \frac{\log \frac{1}{8}}{\log \frac{1}{2}} = \frac{\log 2^{-3}}{\log 2^{-1}} = \frac{-3\log 2}{-\log 2} = 3$$

I4.2 Suppose there are b distinct solutions of the equation $\big|\, x - |2x+1| \,\big| = 3$, find the value of b.

$\big|\, x - |2x+1| \,\big| = 3$

$x - |2x + 1| = 3$ or $x - |2x + 1| = -3$

$x - 3 = |2x + 1|$ or $x + 3 = |2x + 1|$

$x - 3 = 2x + 1$ or $3 - x = 2x + 1$ or $x + 3 = 2x + 1$ or $2x + 1 = -x - 3$

$x = -4$ or $\frac{2}{3}$ or -2 or $-\frac{4}{3}$

check: when $x = -4$ or $\frac{2}{3}$, $x - 3 = |2x + 1| \ge 0$, no solution

when $x = -2$ or $-\frac{4}{3}$, $x + 3 = |2x + 1| \ge 0$, accepted

There are 2 real solutions.

I4.3 If $c = 2\sqrt{3} \times \sqrt[3]{1.5} \times \sqrt[6]{12}$, find the value of c.

$$c = 2\sqrt{3} \times \sqrt[3]{1.5} \times \sqrt[6]{12} = 2 \times 3^{\frac{1}{2}} \times \left(\frac{3}{2}\right)^{1/3} \times \left(2^2 \times 3\right)^{1/6}$$

$$= 2^{1-\frac{1}{3}+\frac{2}{6}} \times 3^{\frac{1}{2}+\frac{1}{3}+\frac{1}{6}} = 2\times 3 = 6$$

I4.4 Given that $f_1 = 0$, $f_2 = 1$, and for any positive integer $n \ge 3$, $f_n = f_{n-1} + 2f_{n-2}$. If $d = f_{10}$, find the value of d.

The characteristic equation: $x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x + 1)(x - 2) = 0 \Rightarrow x = -1$ or 2

$f_n = A(-1)^n + B\times 2^n$, $n = 1, 2, 3, \ldots\ldots$

$f_1 = -A + 2B = 0$ ....(1)

$f_2 = A + 4B = 1$ ....(2)

(1) + (2) $6B = 1$, $B = \frac{1}{6}$

sub. into (1): $-A + \frac{1}{3} = 0$, $A = \frac{1}{3}$

$f_n = \frac{1}{3}(-1)^n + \frac{1}{6}\times 2^n$, $d = f_{10} = \frac{1}{3} + \frac{1}{6}\times 1024 = \frac{513}{3} = 171$

Method 2: $f_1 = 0$, $f_2 = 1$

$f_3 = f_2 + 2f_1 = 1 + 0 = 1$; $f_4 = f_3 + 2f_2 = 1 + 2 = 3$

$f_5 = f_4 + 2f_3 = 3 + 2\times 1 = 5$; $f_6 = f_5 + 2f_4 = 5 + 2\times 3 = 11$

$f_7 = f_6 + 2f_5 = 11 + 2\times 5 = 21$; $f_8 = f_7 + 2f_6 = 21 + 2\times 11 = 43$

$f_9 = f_8 + 2f_7 = 43 + 2\times 21 = 85$; $f_{10} = f_9 + 2f_8 = 85 + 2\times 43 = 171 = d$

Group Event 1

G1.1 There are a camels in a zoo. The number of one-hump camels exceeds that of two-hump camels by 10. If there have 55 humps altogether, find the value of a.

Suppose there are x one-hump camels, y two-hump camels.

$x - y = 10$ .... (1)

$x + 2y = 55$ .... (2)

(2) – (1) $3y = 45 \Rightarrow y = 15$

sub. $y = 15$ into (1): $x - 15 = 10 \Rightarrow x = 25$

$a = x + y = 25 + 15 = 40$

G1.2 If LCM(a, b) = 280 and HCF(a, b) = 10, find the value of b.

HCF × LCM = ab

$2800 = 40b$

$b = 70$

G1.3 Let C be a positive integer less than $\sqrt{b}$. If b is divided by C, the remainder is 2; when divided by $C + 2$, the remainder is C, find the value of C.

$C < \sqrt{70} \Rightarrow C \le 8$ ....(1)

$70 = mC + 2$ ....(2)

$70 = n(C + 2) + C$ ....(3)

from (2), $mC = 68$ $\because 2 < C \le 8$, $\therefore C = 4$ ($C \ne 1, 2$, otherwise remainder > divisor !!!)

G1.4 A regular 8-sided polygon has d diagonals, find the value of d.

The number of diagonals of a convex n-sided polygon is $\frac{n(n-3)}{2}$

$d = \frac{8\times 5}{2} = 20$

Group Event 2

G2.1 Mr. Chan has 8 sons and a daughters. Each of his sons has 8 sons and a daughters. Each of his daughters has a sons and 8 daughters. It is known that the number of his grand sons is one more than the number of his grand daughters and a is a prime number, find the value of a.

Grandsons = $8\times 8 + a\times a = a^2 + 64$

Grand daughters = $8\times a + a\times 8 = 16a$

$a^2 + 64 = 16a + 1$

$a^2 - 16a + 63 = 0$

$(a - 7)(a - 9) = 0$

$a = 7$ or $a = 9$

a is a prime number, $a = 7$

G2.2 Let $\frac{a}{7} = \sqrt[3]{2+\sqrt{b}} + \sqrt[3]{2-\sqrt{b}}$. Find the value of b.

$1 = \sqrt[3]{2+\sqrt{b}} + \sqrt[3]{2-\sqrt{b}}$

$1 = \left(\sqrt[3]{2+\sqrt{b}} + \sqrt[3]{2-\sqrt{b}}\right)^3$

$1 = 2+\sqrt{b} + 3\left(2+\sqrt{b}\right)^{2/3}\left(2-\sqrt{b}\right)^{1/3} + 3\left(2+\sqrt{b}\right)^{1/3}\left(2-\sqrt{b}\right)^{2/3} + 2-\sqrt{b}$

$1 = 4 + 3(4-b)^{1/3}\left(2+\sqrt{b}\right)^{1/3} + 3(4-b)^{1/3}\left(2-\sqrt{b}\right)^{1/3}$

$0 = 3 + 3(4-b)^{1/3}\left[\left(2+\sqrt{b}\right)^{1/3} + \left(2-\sqrt{b}\right)^{1/3}\right]$

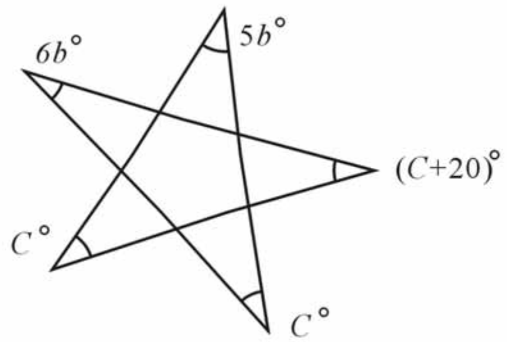
$0 = 1 + (4-b)^{1/3}$

$(4-b)^{1/3} = -1$

$4 - b = -1$

$b = 5$

G2.3 In Figure 1, find the value of C.



$6b° + 5b° + C° + C° + (C + 20)° = 180°$

$11\times 5 + 3C + 20 = 180$

$C = 35$

G2.4 Given that $P_1, P_2, \ldots, P_d$ are d consecutive prime numbers. If $P_1 + P_2 + \ldots + P_{d-2} = P_{d-1} + P_d = 36$, find the value of d.

By trial and error $5 + 7 + 11 + 13 = 17 + 19 = 36$, $d = 6$

Group Event 3

G3.1 Given that a is a positive real root of the equation $2^{x+1}=8^{\frac{1}{x}-\frac{1}{3}}$. Find the value of a.

$2^{x+1}=8^{\frac{1}{x}-\frac{1}{3}} \Rightarrow 2^{x+1}=2^{\frac{3}{x}-1} \Rightarrow x+1=\frac{3}{x}-1$

$x^2+2x-3=0$

$(x+3)(x-1)=0$

$x=-3$ or 1, $a=1$ is a real positive root.

G3.2 The area of the largest rectangle with perimeter a meter is b square meter, find the value of b.

The area of the largest rectangle with perimeter 1 m.

The rectangle is a square with side = 0.25m.

Area = 0.25 m × 0.25 m = 0.0625 m, $b=0.0625=\frac{1}{16}$

G3.3 If $c=(1234^3-1232\times(1234^2+2472))\times b$, find the value of c.

$c=(1234^3-1232\times(1234^2+2472))\times\frac{1}{16}$, let $x=1234$

$=\frac{1}{16}\left\{x^3-(x-2)\times\left[x^2+2(x+2)\right]\right\}=\frac{1}{16}\left\{x^3-(x-2)\times(x^2+2x+4)\right\}$

$=\frac{1}{16}\left\{x^3-(x^3-8)\right\}=\frac{8}{16}=\frac{1}{2}$

G3.4 If $\frac{1}{(c+1)(c+2)}+\frac{1}{(c+2)(c+3)}+\cdots+\frac{1}{(c+d)(c+d+1)}=\frac{8}{15}$, find the value of d.

$\frac{1}{(c+1)(c+2)}+\frac{1}{(c+2)(c+3)}+\cdots+\frac{1}{(c+d)(c+d+1)}=\frac{8}{15}$

$\left(\frac{1}{c+1}-\frac{1}{c+2}\right)+\left(\frac{1}{c+2}-\frac{1}{c+3}\right)+\cdots+\left(\frac{1}{c+d}-\frac{1}{c+d+1}\right)=\frac{8}{15}$

$\frac{1}{c+1}-\frac{1}{c+d+1}=\frac{8}{15} \Rightarrow \frac{1}{\frac{1}{2}+1}-\frac{1}{\frac{1}{2}+d+1}=\frac{8}{15}$

$\frac{2}{3}-\frac{8}{15}=\frac{2}{3+2d}$

$\frac{2}{15}=\frac{2}{3+2d}$

$3+2d=15$

$d=6$

Group Event 4

G4.1 If $A^2+B^2+C^2=AB+BC+CA=3$ and $a=A^2$, find the value of a.

$A^2+B^2+C^2=AB+BC+CA=3$ and $a=A^2$, find the value of a.

$2\left[A^2+B^2+C^2-(AB+BC+CA)\right]=6-6=0$

$A^2-2AB+B^2+B^2-2BC+C^2+C^2-2AC+A^2=0$

$(A-B)^2+(B-C)^2+(C-A)^2=0$ (sum of three non-negative numbers = 0)

$A-B=B-C=C-A=0$

$A=B=C$

$a=A^2=B^2=C^2=1$

G4.2 Given that n and b are integers satisfying the equation $29n+42b=1$. If $5<b<10$, find the value b.

$42=29+13 \Rightarrow 13=42-29$ ....(1)

$29=13\times2+3 \Rightarrow 3=29-13\times2$ ....(2)

$13=3\times4+1 \Rightarrow 1=13-3\times4$ ....(3)

Sub. (1) into (2): $3=29-(42-29)\times2=29\times3-42\times2$ ......(4)

Sub. (1), (4) into (3) $1=42-29-(29\times3-42\times2)\times4$

$1=29\times(-13)+42\times9$

$\therefore n=-13,\ b=9$

Method 2

$b=6, 7, 8, 9.$

By trial and error,

when $b=9$,

$29n+42\times9=1$

$n=-13$

$\therefore b=9$ satisfies the equation.

G4.3 If $\dfrac{\sqrt{3}-\sqrt{5}+\sqrt{7}}{\sqrt{3}+\sqrt{5}+\sqrt{7}}=\dfrac{c\sqrt{21}-18\sqrt{15}-2\sqrt{35}+b}{59}$, find the value of c.

$$\frac{\sqrt{3}-\sqrt{5}+\sqrt{7}}{\sqrt{3}+\sqrt{5}+\sqrt{7}}=\frac{c\sqrt{21}-18\sqrt{15}-2\sqrt{35}+9}{59}$$

$$\frac{\sqrt{3}+\sqrt{7}-\sqrt{5}}{\sqrt{3}+\sqrt{7}+\sqrt{5}}\cdot\frac{\sqrt{3}+\sqrt{7}-\sqrt{5}}{\sqrt{3}+\sqrt{7}-\sqrt{5}}=\frac{c\sqrt{21}-18\sqrt{15}-2\sqrt{35}+9}{59}$$

$$\frac{\left(\sqrt{3}+\sqrt{7}\right)^2-2\sqrt{5}\left(\sqrt{3}+\sqrt{7}\right)+5}{\left(\sqrt{3}+\sqrt{7}\right)^2-5}=\frac{c\sqrt{21}-18\sqrt{15}-2\sqrt{35}+9}{59}$$

$$\frac{3+7+2\sqrt{21}-2\sqrt{15}-2\sqrt{35}+5}{3+7+2\sqrt{21}-5}=\frac{c\sqrt{21}-18\sqrt{15}-2\sqrt{35}+9}{59}$$

$$\frac{15+2\sqrt{21}-2\sqrt{15}-2\sqrt{35}}{5+2\sqrt{21}}\cdot\frac{2\sqrt{21}-5}{2\sqrt{21}-5}=\frac{c\sqrt{21}-18\sqrt{15}-2\sqrt{35}+9}{59}$$

$$\frac{30\sqrt{21}+84-12\sqrt{35}-28\sqrt{15}-75-10\sqrt{21}+10\sqrt{15}+10\sqrt{35}}{4\times21-25}=\frac{c\sqrt{21}-18\sqrt{15}-2\sqrt{35}+9}{59}$$

$$\frac{20\sqrt{21}-18\sqrt{15}-2\sqrt{35}+9}{59}=\frac{c\sqrt{21}-18\sqrt{15}-2\sqrt{35}+9}{59}$$

$c=20$

Method 2 Cross multiplying $59\left(\sqrt{3}-\sqrt{5}+\sqrt{7}\right)=\left(\sqrt{3}+\sqrt{5}+\sqrt{7}\right)\cdot\left(c\sqrt{21}-18\sqrt{15}-2\sqrt{35}+9\right)$

Compare coefficient of $\sqrt{3}$: $9-90+7c=59 \Rightarrow c=20$

G4.4 If c has d positive factors, find the value of d.

The positive factors of 20 are 1, 2, 4, 5, 10 and 20.

$d=6$