03-04 Individual	1	–2009010	2	7	3	45	4	700	5	6
	6	12.5	7	6	8	$-\frac{3}{16}$	9	12	10	$\frac{19}{4}$

03-04 Group	1	2475	2	1	3	6	4	32	5	5
	6	500	7	34.56	8	$\frac{1}{6}$	9	10	10	$\frac{5}{3}$

Individual Events

I1 Let $A = 1^2 - 2^2 + 3^2 - 4^2 + \dots + 2003^2 - 2004^2$, find the value of A.

$A = (1^2 - 2^2) + (3^2 - 4^2) + \dots + (2003^2 - 2004^2)$

$= -3-7-11 - \dots - 4007$, this is an Arithmetic series, $a = -3$, $l = -4007 = a + (n-1)(-4)$, $n = 1002$

$= -\frac{3+4007}{2} \times 1002 = -2009010$

I2 If $\sqrt[2003]{B} = 2003$, C is the unit digit of B, find the value of C.

$B = 2003^{2003}$; $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, $3^5 = 243$; the unit digit repeats for every multiples of 4. $2003^{2003} = 2003^{4\times500+3}$; the unit digit is 7; $C = 7$.

I3 If $x + y + z = 10$, $x^2 + y^2 + z^2 = 10$ and $xy + yz + zx = m$, find the value of m.

$(x + y + z)^2 = 10^2 \Rightarrow x^2 + y^2 + z^2 + 2(xy + yz + zx) = 100 \Rightarrow 10 + 2m = 100 \Rightarrow m = 45$

I4 Arrange the natural numbers in the following order. In this arrangement, 9 is in the row 3 and the column 2. If the number 2003 is in the row x and the column y, find the value of xy.

1	2	4	7	11	16	...
3	5	8	12	17	...	
6	9	13	18	...		
10	14	19	...			
15	20	...				
21	...					

Consider the integers in the first column of each row: 1, 3, 6, 10, ...

The first integer in the first column of the n$^{\text{th}}$ row $= \frac{n(n+1)}{2}$

$\frac{n(n+1)}{2} < 2003 \Rightarrow n(n + 1) < 4006$

$\because 62\times63 = 3906$, $63\times64 = 4032$; $\therefore$ greatest possible $n = 62$

$3906 \div 2 = 1953$

The 63$^{\text{rd}}$ element of the first row = 1954

The 62$^{\text{nd}}$ element of the second row = 1955, ... and so on.

$2003 = 1953 + 50$; $63 - 50 + 1 = 14$

The 14$^{\text{th}}$ element of the 50$^{\text{th}}$ row is 2003; $x = 50$, $y = 14$

$xy = 50\times14 = 700$

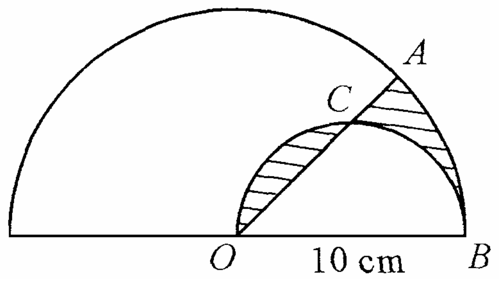
I5 Let $E = \sqrt{12+6\sqrt{3}} + \sqrt{12-6\sqrt{3}}$, find the value of E.

$\sqrt{12+6\sqrt{3}} = \sqrt{9+3+2\sqrt{9\times3}} = \sqrt{a+b+2\sqrt{ab}} = \sqrt{a}+\sqrt{b} = 3+\sqrt{3}$

$\sqrt{12-6\sqrt{3}} = \sqrt{9+3-2\sqrt{9\times3}} = \sqrt{a+b-2\sqrt{ab}} = \sqrt{a}-\sqrt{b} = 3-\sqrt{3}$

$\sqrt{12+6\sqrt{3}} + \sqrt{12-6\sqrt{3}} = 3+\sqrt{3}+3-\sqrt{3} = 6$

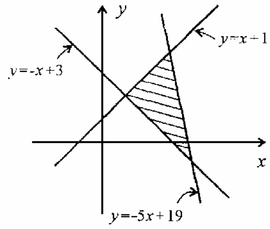
I6 In the figure, O is the centre of the bigger semicircle with radius 10 cm, OB is the diameter of the smaller semicircle and C is the midpoint of arc OB and it lies on the segment OA. Let the area of the shaded region be K cm^2, find the value of K. (Take $\pi = 3$)



Shaded area = area of sector OAB – area of ΔOCB

$= \frac{1}{2}10^2 \cdot \frac{\pi}{4} - \frac{1}{2}10 \cdot 5 = 12.5$

I7 In the figure, let the shaded area formed by the three straight lines $y = -x + 3$, $y = x + 1$ and $y = -5x + 19$ be R, find the value of R.



Intersection points are A(1, 2), B(3,4), C(4, –1).

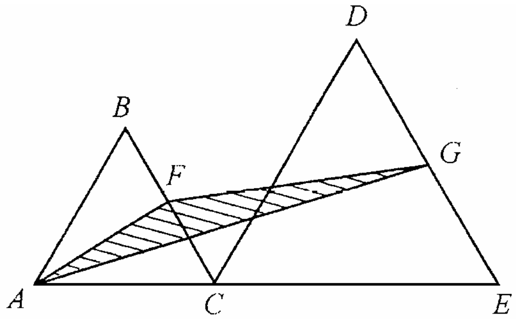
$\angle CAB = 90°$

Area = $\frac{1}{2}\sqrt{8}\sqrt{18} = 6$ sq.unit

I8 If $t = \sin^4\frac{\pi}{6} - \cos^2\frac{2\pi}{6}$, find the value of t.

$t = \sin^4\frac{\pi}{6} - \cos^2\frac{2\pi}{6} = \left(\frac{1}{2}\right)^4 - \left(\frac{1}{2}\right)^2 = \frac{1}{16} - \frac{1}{4} = -\frac{3}{16}$

I9 In the figure, C lies on AE, ΔABC and ΔCDE are equilateral triangles, F and G are the mid-points of BC and DE respectively. If the area of ΔABC is 24 cm^2, the area of ΔCDE is 60 cm^2, and the area of ΔAFG is Q cm^2, find the value of Q.



$\angle FAC = \angle GCE = 30°$

AF // CG (corr. $\angle$s eq.)

Area of ΔAFG = Area of ΔACF = 12 cm^2

(They have the same bases *AF* and the same height)

I10 If α and β are the roots of the quadratic equation $4x^2 - 10x + 3 = 0$ and $k = \alpha^2 + \beta^2$, find the value of k.

$k = \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \left(\frac{5}{2}\right)^2 - 2\cdot\left(\frac{3}{4}\right) = \frac{19}{4}$

Group Events

G1 If $x = \frac{1}{2} + \left(\frac{1}{3} + \frac{2}{3}\right) + \left(\frac{1}{4} + \frac{2}{4} + \frac{3}{4}\right) + \left(\frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{4}{5}\right) + \cdots + \left(\frac{1}{100} + \frac{2}{100} + \cdots + \frac{99}{100}\right)$, find the value of x.

$x = \frac{1}{2} + \frac{2}{2} + \frac{3}{2} + \frac{4}{2} + \cdots + \frac{99}{2} = \frac{100}{4} \times 99 = 2475$

G2 If z is the positive root of the equation $6\times4^x - 13\times6^x + 6\times9^x = 0$, find the value of z.

$6\times(4^x + 9^x) = 13\times6^x$

$x = 1$

G3 If there are at most k mutually non-congruent isosceles triangles whose perimeter is 25cm and the lengths of the three sides are positive integers when expressed in cm, find the value of k.

Possible triangles are {7,7,11}, {8,8,9}, {9,9,7}, {10,10,5}, {11,11,3}, {12,12,1}

G4 Given that a, b are positive real numbers satisfying $a^3 = 2004$ and $b^2 = 2004$. If the number of integers x that satisfy the inequality $a < x < b$ is h, find the value of h.

$12^3 = 1728$, $44^2 = 1976$

$a^3 = 2004 \Rightarrow 12 < a < 13$; $b^2 = 2004 \Rightarrow 44 < b < 45$

$a < x < b \Rightarrow 12 < x < 45 \Rightarrow$ number of integral values of $x = 32$

G5 If the sum of R consecutive positive integers is 1000 (where $R > 1$), find the least value of R.

Let the smallest positive integer be x.

$x + (x + 1) + ... + (x + R - 1) = 1000$

$\frac{R}{2}\times[2x + R - 1] = 1000$

$R(2x + R - 1) = 2000$

Factors of 2000 are 1,2,4,5,8,10,16,20,25,40,50,80,100,125,250,400,500,1000,2000.

After testing the possible values of R one by one, smallest R > 1 is 5, x = 198.

198 + 199 + 200 + 201 + 202 = 2000

G6 If a, b and c are positive integers such that $abc + ab + bc + ac + a + b + c = 2003$, find the least value of abc.

$(a + 1)(b + 1)(c + 1) = 2004 = 2^2 \times 3 \times 167$

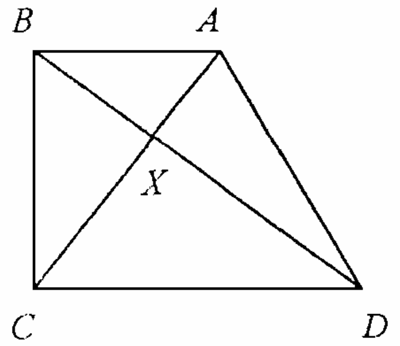
abc is the least when the difference between a, b and c are the greatest.

$a + 1 = 2$, $b + 1 = 2$, $c + 1 = 501$

$a = 1$, $b = 1$, $c = 500$

$abc = 500$

G7 In the figure, ABCD is a trapezium, the segments AB and CD are both perpendicular to BC and the diagonals AC and BD intersect at X. If AB = 9 cm, BC = 12 cm and CD = 16 cm, and the area of ΔBXC is W cm^2, find the value of W.



$\Delta ABX \sim \Delta CDX$

$AX : CX = AB : CD = 9 : 16$

$S_{\Delta ABX} : S_{\Delta CDX} = 9^2 : 16^2 = 81 : 256$

Let $S_{\Delta ABX} = 81y$, $S_{\Delta CDX} = 256y$

Let AX = 9t, CX = 16t ($\because \Delta ABX \sim \Delta CDX$)

ΔABX and ΔBCX have the same height.

$S_{\Delta BCX} = S_{\Delta ABX} \times \frac{16t}{9t} = 81y \times \frac{16}{9} = 144y$

$S_{\Delta ABC} = S_{\Delta ABX} + S_{\Delta BCX}$

$\frac{9\times 12}{2} = 81y + 144y$

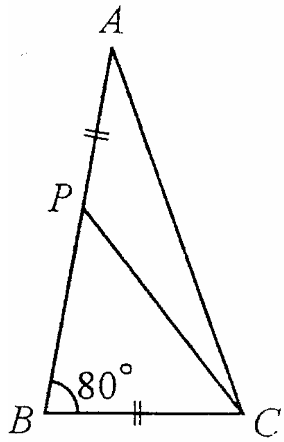
$y = \frac{6}{25} \Rightarrow S_{\Delta BCX} = 144y = 144\times\frac{6}{25} = 34.56$

G8 Let $y = \log_{1400}\sqrt{2} + \log_{1400}\sqrt[3]{5} + \log_{1400}\sqrt[6]{7}$, find the value of y.

$$y = \frac{\log\sqrt{2}\times\sqrt[3]{5}\times\sqrt[6]{7}}{\log 1400} = \frac{\frac{1}{2}\log 2 + \frac{1}{3}\log 5 + \frac{1}{6}\log 7}{\log 1400} = \frac{3\log 2 + 2\log 5 + \log 7}{6\log 1400}$$

$$y = \frac{\log 8 + \log 25 + \log 7}{6\log 1400} = \frac{\log(8\times 25\times 7)}{6\log 1400} = \frac{\log 1400}{6\log 1400} = \frac{1}{6}$$

G9 In the figure, ΔABC is an isosceles triangle with AB = AC and $\angle$ABC = 80°. If P is a point on the AB such that AP = BC, $\angle$ACP = k°, find the value of k.



$\angle ACB = 80°$, $\angle BAC = 20°$

Let $\angle ACP = k°$, AP = BC = $2x$

Then AC = $\frac{x}{\sin 10^\circ}$

$\angle APC = 180° - (20° + k°)$

Apply sine formula on ΔACP:

$$\frac{\frac{x}{\sin 10^\circ}}{\sin(20^\circ + k^\circ)} = \frac{2x}{\sin k^\circ}$$

$\sin k° = 2 \sin 10° \sin(20° + k°)$

$\sin k° = \cos(10° + k°) - \cos(30° + k°)$

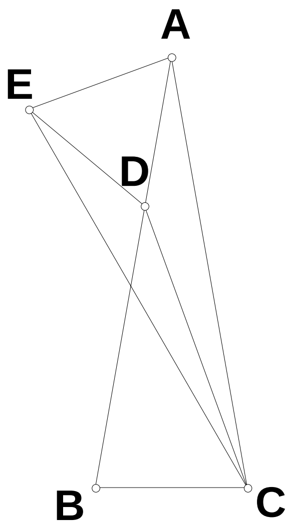
$\sin k° = \cos 10° \cos k° - \sin 10° \sin k° - \cos 30° \cos k° + \sin 30° \sin k°$

$\sin k° (1 - \frac{1}{2} + \sin 10°) = (\cos 10° - \cos 30°)\cos k°$

$\tan k° = \frac{\cos 10^\circ - \cos 30^\circ}{\sin 10^\circ + \sin 30^\circ}$

$= \frac{2\sin 10^\circ \sin 20^\circ}{2\sin 20^\circ \cos 10^\circ} = \tan 10°$; $k = 10$

method 2 Construct an equilateral triangle ΔAED as shown. Join EC.



$\angle BAC = 180° - 2\times 80° = 20°$ ∠s of ΔABC, $AB = AC$.

$\angle EAD = 60°$ equilateral Δ

$\angle EAC = 60° + 20° = 80° = \angle ABC$

$\because EA = BC = a$

$AC = AC$ common

$\therefore \Delta ABC \cong \Delta CAE$ SAS

$\therefore \Delta AEC = 80° = \angle ACB$ corr. ∠s ≅Δ's

$\angle CED = 80° - 60° = 20° = \angle CAD$

$CD = CD$ common

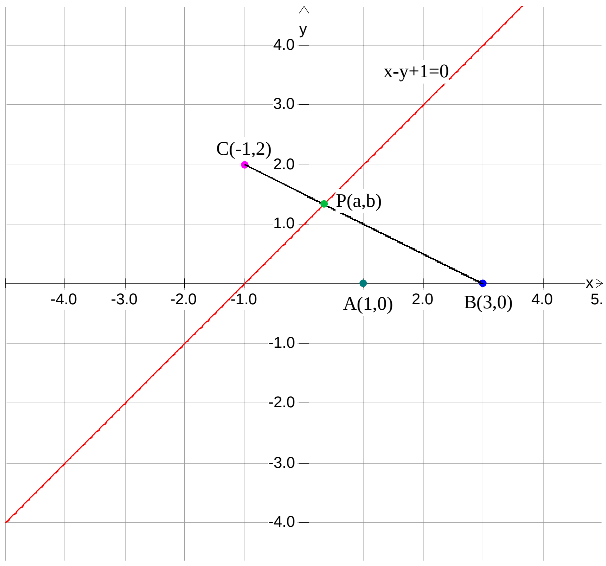
$AD = a = ED$

$\Delta ADC \cong \Delta EDC$ SSS

$\therefore \angle ACE = \angle BAC = 20°$ corr. ∠s $\Delta ABC \cong \Delta CAE$

$\angle ACD = \angle ECD = 10°$ corr. ∠s $\Delta ACD \cong \Delta ECD$

G10 Suppose P(a, b) is a point on the straight line $x - y + 1 = 0$ such that the sum of the distance between P and the point A(1,0) and the distance between P and the point B(3,0) is the least, find the value of $a + b$.



Regard $x - y + 1 = 0$ as mirror.

C(–1,2) is the mirror image of A(1,0).

Sum of distance is the least

⇒P(a, b) lies on BC.

$P(a, b)$ lies on $x - y + 1 = 0$

$\Rightarrow b = a + 1$

$m_{PB} = m_{BC}$

$$\frac{a+1}{a-3} = \frac{2}{-4}$$

$-2a - 2 = a - 3$

$$a = \frac{1}{3},\quad b = \frac{4}{3}$$

$$a + b = \frac{5}{3}$$