

Individual Events

I1	$P$	5	I2	$P$	23	I3	$P$	4	I4	$P$	12
	$Q$	4		$Q$	4		$Q$	33		$Q$	$\frac{2}{3}$
	$R$	1		$R$	8		$R$	3		$R$	4
	$S$	62		$S$	8		$S$	$3\sqrt{2}$		$S$	144

Group Events

G1	$a$	29	G2	$a$	12	G3	$a$	334501	G4	$\alpha$	$\frac{180}{7}$
	$b$	7		$b$	6		$b$	$\frac{1}{3}$		$b$	$\frac{1}{5}$
	$c$	100		$c$	16		$c$	$1 + \sqrt{2}$		$c$	10
	$d$	206		$d$	$\frac{44}{125}$		$d$	3		$d$	$\frac{1 + \sqrt{5}}{2}$

Individual Event 1

I1.1 Let  $P$  be the unit digit of  $3^{2003} \times 5^{2002} \times 7^{2001}$ . Find the value of  $P$ .

$3^{2003} \times 7^{2001}$  is an odd number, and the unit digit of  $5^{2002}$  is 5;  $P = 5$

I1.2 If the equation  $(x^2 - x - 1)^{x+P-1} = 1$  has  $Q$  integral solutions, find the value of  $Q$ .

The equation is  $(x^2 - x - 1)^{x+4} = 1$

Either  $x^2 - x - 1 = 1$  .....(1) or  $x + 4 = 0$  .....(2) or  $(x^2 - x - 1 = -1$  and  $x + 4$  is even) .....(3)

(1):  $x = 2$  or  $-1$ ; (2):  $x = -4$ ; (3):  $x = 0$  or  $1$  and  $x$  is even  $\Rightarrow x = 0$  only

Conclusion:  $x = -4, -1, 0, 2$

$Q = 4$

I1.3 Let  $x, y$  be real numbers and  $xy = 1$ . If the minimum value of  $\frac{1}{x^4} + \frac{1}{Qy^4}$  is  $R$ , find the value of  $R$ .

$$\frac{1}{x^4} + \frac{1}{Qy^4} = \frac{1}{x^4} + \frac{1}{4y^4} \geq 2\sqrt{\frac{1}{x^4} \cdot \frac{1}{4y^4}} = 1 = R \text{ (AM} \geq \text{GM)}$$

I1.4 Let  $x_R, x_{R+1}, \dots, x_K$  ( $K > R$ ) be  $K - R + 1$  distinct positive integers and  $x_R + x_{R+1} + \dots + x_K = 2003$ . If  $S$  is the maximum possible value of  $K$ , find the value of  $S$ .

$$x_1 + x_2 + \dots + x_K = 2003$$

For maximum possible value of  $K$ ,  $x_1 = 1, x_2 = 2, \dots, x_{K-1} = K - 1$

$$1 + 2 + \dots + K - 1 + x_K = 2003$$

$$\frac{(K-1)K}{2} + x_K = 2003, x_K \geq K$$

$$2003 \geq \frac{(K-1)K}{2} + K$$

$$4006 \geq K^2 + K$$

$$K^2 + K - 4006 \leq 0$$

$$\left( K - \frac{-1 - \sqrt{1 + 4 \times 4006}}{2} \right) \left( K - \frac{-1 + \sqrt{1 + 4 \times 4006}}{2} \right) \leq 0$$

$$0 \leq K \leq \frac{-1 + \sqrt{1 + 4 \times 4006}}{2}$$

$$\frac{-1 + \sqrt{1 + 4 \times 4006}}{2} \approx \frac{-1 + \sqrt{4 \times 4006}}{2} = \sqrt{4006} - 0.5 \geq \sqrt{3969} - 0.5 = \sqrt{63^2} - 0.5 = 62.5$$

Maximum possible  $K = 62 = S$

$$1 + 2 + \dots + 62 = 1953 = 2003 - 50; 1 + 2 + \dots + 61 + 112 = 2003$$

Individual Event 2

- 12.1 If the 50<sup>th</sup> power of a two-digit number  $P$  is a 69-digit number, find the value of  $P$ .  
 (given that  $\log 2 = 0.3010$ ,  $\log 3 = 0.4771$ ,  $\log 11 = 1.0414$ )

$$P^{50} = y, 10 < P \leq 99, 10^{68} \leq y < 10^{69}$$

$$P = y^{\frac{1}{50}}; 10^{68 \div 50} < P < 10^{69 \div 50}$$

$$1.34 < \log P < 1.38$$

$$\log 22 = \log 2 + \log 11 = 1.3424; \log 24 = 3\log 2 + \log 3 = 1.3801$$

$$\log 22 < \log P < \log 24, P = 23$$

- 12.2 The roots of the equation  $x^2 + ax - P + 7 = 0$  are  $\alpha$  and  $\beta$ , whereas the roots of the equation  $x^2 + bx - r = 0$  are  $-\alpha$  and  $-\beta$ . If the positive root of the equation  $(x^2 + ax - P + 7) + (x^2 + bx - r) = 0$  is  $Q$ , find the value of  $Q$ .

$$\alpha + \beta = -a, \alpha\beta = -16; -\alpha - \beta = -b, (-\alpha)(-\beta) = -r$$

$$\therefore b = -a, r = 16$$

$$(x^2 + ax - P + 7) + (x^2 + bx - r) = 0 \text{ is equivalent to } (x^2 + ax - 16) + (x^2 - ax - 16) = 0$$

$$2x^2 - 32 = 0$$

$$x = 4 \text{ or } -4$$

$$Q = \text{positive root} = 4$$

- 12.3 Given that  $\triangle ABC$  is an isosceles triangle,  $AB = AC = \sqrt{2}$ , and  $D_1, D_2, \dots, D_Q$  are  $Q$  points on  $BC$ . Let  $m_i = AD_i^2 + BD_i \cdot D_iC$ . If  $m_1 + m_2 + m_3 + \dots + m_Q = R$ , find the value of  $R$ .

As shown in the figure,  $AB = AC = \sqrt{2}$

$BD = x, CD = y, AD = t, \angle ADC = \theta$

Apply cosine formula on  $\triangle ABD$  and  $\triangle ACD$

$$\cos \theta = \frac{t^2 + y^2 - 2}{2ty}$$

$$\cos(180^\circ - \theta) = \frac{t^2 + x^2 - 2}{2tx}$$

since  $\cos(180^\circ - \theta) = -\cos \theta$

Add these equations and multiply by  $2txy$ :

$$x(t^2 + y^2 - 2) + y(t^2 + x^2 - 2) = 0$$

$$(x + y)t^2 + (x + y)xy - 2(x + y) = 0$$

$$(x + y)(t^2 + xy - 2) = 0$$

$$t^2 + xy - 2 = 0$$

$$AD^2 + BD \cdot DC = 2$$

$$R = m_1 + m_2 + m_3 + m_4 = 2 + 2 + 2 + 2 = 8$$

- 12.4 There are 2003 bags arranged from left to right. It is given that the leftmost bag contains  $R$  balls, and every 7 consecutive bags contains 19 balls altogether. If the rightmost bag contains  $S$  balls, find the value of  $S$ .

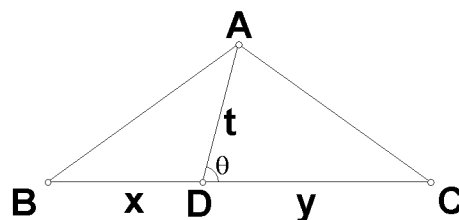
The leftmost bag contains 8 balls.

Starting from left to right, the total number of balls from 2<sup>nd</sup> bag to the 7<sup>th</sup> bag is 11.

The number of balls in the 8<sup>th</sup> bag is therefore 8.

Similarly, the number of balls in the 15<sup>th</sup> bag, 22<sup>th</sup> bag, 29<sup>th</sup> bag, ... are all 8.

$2003 = 7 \times 286 + 1$ , the rightmost bag should have the same number of balls as the leftmost bag.



Individual Event 3

I3.1 Given that  $\begin{cases} wxyz = 4 \\ w - xyz = 3 \end{cases}$  and  $w > 0$ . If the solution of  $w$  is  $P$ , find the value of  $P$ .

From (2),  $xyz = w - 3$ .....(3), sub. into (1)

$$w(w - 3) = 4$$

$$w^2 - 3w - 4 = 0$$

$$w = 4 \text{ or } w = -1 \text{ (rejected)}$$

$$P = 4$$

I3.2 Let  $[y]$  represents the integral part of the decimal number  $y$ . For example,  $[3.14] = 3$ . If  $\left[ (\sqrt{2} + 1)^p \right] = Q$ , find the value of  $Q$ .

Note that  $0 < \sqrt{2} - 1 < 1$  and  $0 < (\sqrt{2} - 1)^4 < 1$

$$(\sqrt{2} + 1)^4 + (\sqrt{2} - 1)^4 = 2(\sqrt{2}^4 + 6\sqrt{2}^2 + 1) = 2(4 + 12 + 1) = 34$$

$$33 < (\sqrt{2} + 1)^4 < 34$$

$$Q = \left[ (\sqrt{2} + 1)^4 \right] = 33$$

I3.3 Given that  $x_0y_0 \neq 0$  and  $Qx_0^2 - 22\sqrt{3}x_0y_0 + 11y_0^2 = 0$ . If  $\frac{6x_0^2 + y_0^2}{6x_0^2 - y_0^2} = R$ , find the value of  $R$ .

$$33x_0^2 - 22\sqrt{3}x_0y_0 + 11y_0^2 = 0$$

$$3x_0^2 - 2\sqrt{3}x_0y_0 + y_0^2 = 0$$

$$(\sqrt{3}x_0 - y_0)^2 = 0$$

$$y_0 = \sqrt{3}x_0$$

$$R = \frac{6x_0^2 + y_0^2}{6x_0^2 - y_0^2} = \frac{6x_0^2 + 3x_0^2}{6x_0^2 - 3x_0^2} = 3$$

I3.4 The diagonals  $AC$  and  $BD$  of a quadrilateral  $ABCD$  are perpendicular to each other. Given that  $AB = 5$ ,  $BC = 4$ ,  $CD = R$ . If  $DA = S$ , find the value of  $S$ .

Suppose  $AC$  and  $BD$  intersect at  $O$ .

Let  $OA = a$ ,  $OB = b$ ,  $OC = c$ ,  $OD = d$ .

$$a^2 + b^2 = 5^2 \text{ .....(1)}$$

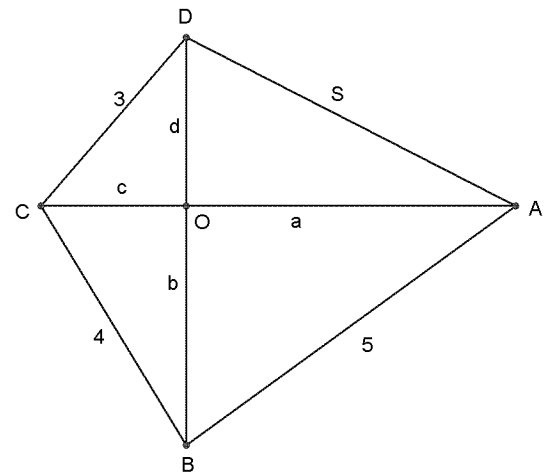
$$b^2 + c^2 = 4^2 \text{ .....(2)}$$

$$c^2 + d^2 = 3^2 \text{ .....(3)}$$

$$d^2 + a^2 = S^2 \text{ .....(4)}$$

$$(1) + (3) - (2): S^2 = d^2 + a^2 = 5^2 + 3^2 - 4^2 = 18$$

$$S = 3\sqrt{2}$$



Individual Event 4

14.1 Suppose the 9-digit number  $\overline{32x35717y}$  is a multiple of 72, and  $P = xy$ , find the value of  $P$ .

$72 = 8 \times 9$ , the number is divisible by 8 and 9.

$\overline{17y}$  is divisible by 8, i.e.  $y = 6$ .

$3 + 2 + x + 3 + 5 + 7 + 1 + 7 + 6 = 9m$ , where  $m$  is an integer.

$34 + x = 9m, x = 2$

$P = xy = 2 \times 6 = 12$

14.2 Given that the lines  $4x + y = \frac{P}{3}$ ,  $mx + y = 0$  and  $2x - 3my = 4$  cannot form a triangle. Suppose that  $m > 0$  and  $Q$  is the minimum possible value of  $m$ , find  $Q$ .

Slope of  $L_1 = -4$ , slope of  $L_2 = -m$ , slope of  $L_3 = \frac{2}{3m}$

If  $L_1 \parallel L_2$ :  $m = 4$ ; if  $L_2 \parallel L_3$ :  $m^2 = -\frac{2}{3}$  (no solution); if  $L_1 \parallel L_3$ :  $m = -\frac{1}{6}$  (rejected,  $\because m > 0$ )

If they are concurrent:  $\begin{cases} 4x + y = 4 \dots\dots(1) \\ mx + y = 0 \dots\dots(2) \\ 2x - 3my = 4 \dots\dots(3) \end{cases}$

Solve (1), (2) gives:  $x = \frac{4}{4-m}; y = \frac{-4m}{4-m}$

Sub. into (3):  $\frac{2 \times 4}{4-m} - \frac{3m(-4m)}{4-m} = 4$

$3m^2 + m - 2 = 0$

$(m + 1)(3m - 2) = 0$

$m = \frac{2}{3}$  (rejected  $-1, \because m > 0$ )

Minimum positive  $m = \frac{2}{3}$

14.3 Given that  $R, x, y, z$  are integers and  $R > x > y > z$ . If  $R, x, y, z$  satisfy the equation

$2^R + 2^x + 2^y + 2^z = \frac{495Q}{16}$ , find the value of  $R$ .

$2^R + 2^x + 2^y + 2^z = \frac{495 \cdot \frac{2}{3}}{16} = \frac{165}{8} = 21 + \frac{5}{8} = 16 + 1 + \frac{1}{2} + \frac{1}{16}$

$R = 4$

14.4 In Figure 1,  $Q$  is the interior point of  $\triangle ABC$ . Three straight lines passing through  $Q$  are parallel to the sides of the triangle such that  $FE \parallel AB, GK \parallel AC$  and  $HJ \parallel BC$ . Given that the areas of  $\triangle KQE, \triangle JFQ$  and  $\triangle QGH$  are  $R, 9$  and  $49$  respectively. If the area of  $\triangle ABC$  is  $S$ , find the value of  $S$ .

It is easy to show that all triangles are similar.

By the ratio of areas of similar triangles,

$S_{\triangle KQE} : S_{\triangle JFQ} : S_{\triangle QGH} = (QE)^2 : (FQ)^2 : (GH)^2$

$4 : 9 : 49 = (QE)^2 : (FQ)^2 : (GH)^2$

$QE : FQ : GH = 2 : 3 : 7$

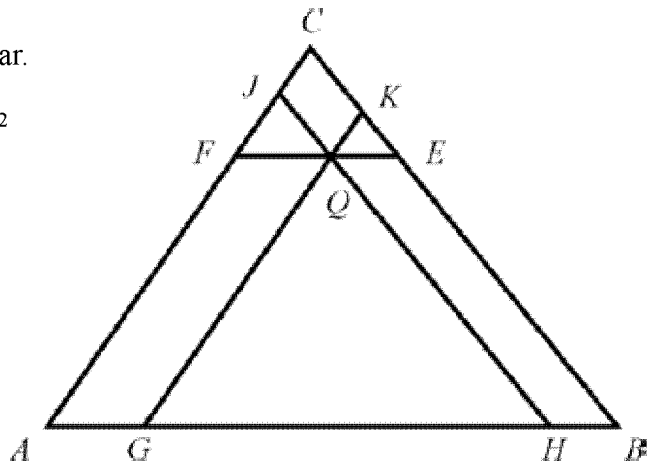
Let  $QE = 2t, FQ = 3t, GH = 7t$

$AFQG$  and  $BEQH$  are parallelograms.

$AG = 3t, BH = 2t$  (opp. sides of // - gram)

$AB = 3t + 7t + 2t = 12t$

$S_{\triangle ABC} = 4 \times \left(\frac{12}{2}\right)^2 = 144 = S$



Group Event 1

G1.1 Given that  $n$  and  $k$  are natural numbers and  $1 < k < n$ . If  $\frac{(1+2+3+\dots+n)-k}{n-1} = 10$  and

$n + k = a$ , find the value of  $a$ .

$$\frac{n(n+1)}{2} - k = 10n - 10 \Rightarrow n^2 - 19n + 2(10 - k) = 0$$

$\Delta = 281 + 8k$ ,  $n$  is an integer  $\Rightarrow \Delta$  is a perfect square.

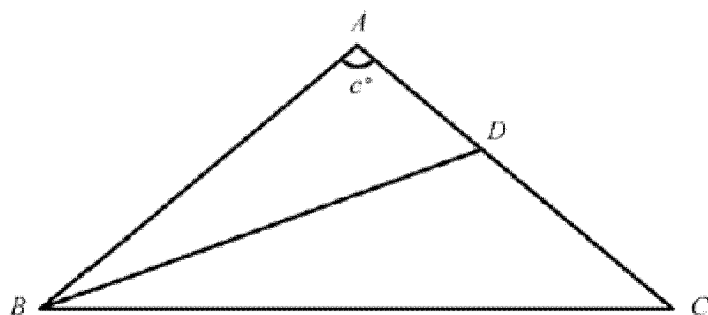
$281 + 8k = 289, 361, 441, \dots \Rightarrow k = 1, 10, 20, \dots$  Given  $1 < k < n$ ,  $\therefore k = 10, 20, \dots$

when  $k = 10$ ,  $n = 19$ ;  $a = n + k = 29$ ; when  $k = 20$ ,  $n = 20$  rejected.

G1.2 Given that  $(x - 1)^2 + y^2 = 4$ , where  $x$  and  $y$  are real numbers. If the maximum value of  $2x + y^2$  is  $b$ , find the value of  $b$ .

$$\begin{aligned} 2x + y^2 &= 2x + 4 - (x - 1)^2 \\ &= -x^2 + 2x - 1 + 2x + 4 \\ &= -x^2 + 4x + 3 \\ &= -(x^2 - 4x + 4) + 7 \\ &= -(x - 2)^2 + 7 \leq 7 = b \end{aligned}$$

G1.3 In Figure 1,  $\triangle ABC$  is an isosceles triangle and  $AB = AC$ . Suppose the angle bisector of  $\angle B$  meets  $AC$  at  $D$  and  $BC = BD + AD$ . Let  $\angle A = c^\circ$ , find the value of  $c$ .



Let  $AB = n = AC$ ;  $AD = q$ ,  $BD = q$ ,  $CD = n - q$

$\angle ABD = x = \angle CBD$ ;  $\angle ACB = 2x$ .

Let  $E$  be a point on  $BC$  such that  $BE = p$ ,  $EC = q$

Apply sine formula on  $\triangle ABD$  and  $\triangle BCD$

$$\frac{n}{\sin \angle ADB} = \frac{q}{\sin x}; \frac{p+q}{\sin \angle BDC} = \frac{n-q}{\sin x}$$

$\therefore \sin \angle ADB = \sin \angle BDC$

Dividing the above two equations

$$\frac{n}{p+q} = \frac{q}{n-q}$$

$$\frac{AB}{BC} = \frac{EC}{CD} \text{ and } \angle ABC = \angle ECD = 2x$$

$\triangle ABC \sim \triangle ECD$  (2 sides proportional, included angle)

$\therefore \angle CDE = 2x$  (corr.  $\angle$ s,  $\sim \Delta$ 's)

$\angle BED = 4x$  (ext.  $\angle$  of  $\triangle CDE$ )

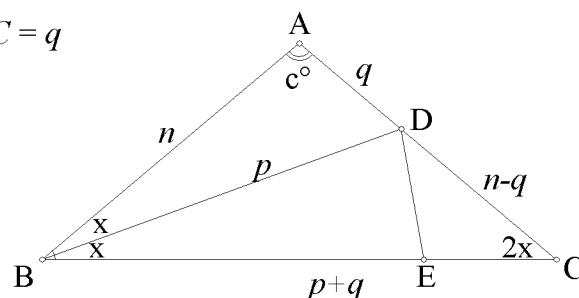
$\angle BDE = 4x$  ( $BD = BE = p$ , base  $\angle$ s, isos.  $\Delta$ )

$\angle ADB = 3x$  (ext.  $\angle$  of  $\triangle BCD$ )

$2x + 3x + 4x = 180^\circ$  (adj.  $\angle$ s on st. line  $ADC$ )

$x = 20^\circ$

$c^\circ = 180^\circ - 4x = 100^\circ$  ( $\angle$ s sum of  $\triangle ABC$ )



G1.4 Given that the sum of two prime numbers is 105. If the product of these prime numbers is  $d$ , find the value of  $d$ .

"2" is the only prime number which is an even integer.

The sum of two prime number is 105, which is odd

$\Rightarrow$  One prime is odd and the other prime is even

$\Rightarrow$  One prime is odd and the other prime is 2

$\Rightarrow$  One prime is 103 and the other prime is 2

$d = 2 \times 103 = 206$

Group Event 2

G2.1. Given that the equation  $ax(x + 1) + bx(x + 2) + c(x + 1)(x + 2) = 0$  has roots 1 and 2.

If  $a + b + c = 2$ , find the value of  $a$ .

Expand and rearrange the terms in descending orders of  $x$ :

$$(a + b + c)x^2 + (a + 2b + 3c)x + 2c = 0$$

$$2x^2 + (a + b + c + b + 2c)x + 2c = 0$$

$$2x^2 + (b + 2c + 2)x + 2c = 0$$

It is identical to  $2(x - 1)(x - 2) = 0$

$$\therefore b + 2c + 2 = -6; 2c = 4$$

Solving these equations give  $c = 2, b = -12, a = 12$

G2.2 Given that  $48^x = 2$  and  $48^y = 3$ . If  $8^{\frac{x+y}{1-x-y}} = b$ , find the value of  $b$ .

Take logarithms on the two given equations:

$$x \log 48 = \log 2, y \log 48 = \log 3$$

$$\therefore x = \frac{\log 2}{\log 48}; y = \frac{\log 3}{\log 48}$$

$$\begin{aligned} \frac{x+y}{1-x-y} &= \frac{\frac{\log 2}{\log 48} + \frac{\log 3}{\log 48}}{1 - \frac{\log 2}{\log 48} - \frac{\log 3}{\log 48}} \\ &= \frac{\log 2 + \log 3}{\log 48 - \log 2 - \log 3} \\ &= \frac{\log 6}{\log 8} \end{aligned}$$

$$b = 8^{\frac{x+y}{1-x-y}}$$

$$\begin{aligned} \log b &= \log \left( 8^{\frac{x+y}{1-x-y}} \right) = \frac{x+y}{1-x-y} \log 8 \\ &= \frac{\log 6}{\log 8} \cdot \log 8 = \log 6 \Rightarrow b = 6 \end{aligned}$$

G2.3 In Figure 1, the square  $PQRS$  is inscribed in  $\triangle ABC$ . The areas of  $\triangle APQ$ ,  $\triangle PBS$  and  $\triangle QRC$  are 4, 4 and 12 respectively. If the area of the square is  $c$ , find the value of  $c$ .

Let  $BC = a, PS = x$ , the altitude from  $A$  onto  $BC = h$ .

$$\text{Area of } \triangle PBS = \frac{1}{2}x \cdot BS = 4 \Rightarrow BS = \frac{8}{x}$$

$$\text{Area of } \triangle CQR = \frac{1}{2}x \cdot CR = 12 \Rightarrow CR = \frac{24}{x}$$

$$BC = BS + SR + RC = \frac{8}{x} + x + \frac{24}{x} = x + \frac{32}{x} \dots\dots\dots(1)$$

$$\text{Area of } \triangle APQ = \frac{1}{2}x(h-x) = 4 \Rightarrow h = \frac{8}{x} + x \dots\dots\dots(2)$$

$$\text{Area of } \triangle ABC = \frac{1}{2}h \cdot BC = 4 + 4 + 12 + x^2 = 20 + x^2 \dots\dots\dots(3)$$

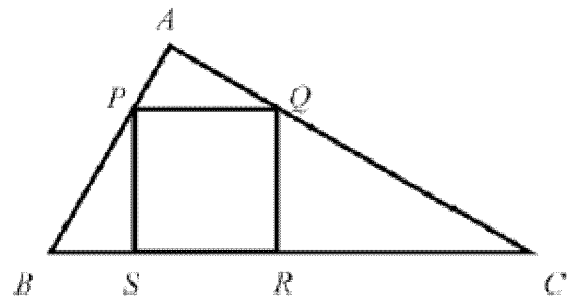
$$\text{Sub. (1) and (2) into (3): } \frac{1}{2} \left( \frac{8}{x} + x \right) \cdot \left( x + \frac{32}{x} \right) = 20 + x^2$$

$$(8 + x^2)(x^2 + 32) = 40x^2 + 2x^4$$

$$x^4 + 40x^2 + 256 = 40x^2 + 2x^4$$

$$x^4 = 256$$

$$c = \text{area of the square} = x^2 = 16$$



G2.4 In  $\triangle ABC$ ,  $\cos A = \frac{4}{5}$  and  $\cos B = \frac{7}{25}$ . If  $\cos C = d$ , find the value of  $d$ .

$$\sin A = \frac{3}{5}, \sin B = \frac{24}{25}$$

$$\begin{aligned}\cos C &= \cos(180^\circ - A - B) = -\cos(A + B) = -\cos A \cos B + \sin A \sin B \\ &= -\frac{4}{5} \frac{7}{25} + \frac{3}{5} \frac{24}{25} = \frac{44}{125}\end{aligned}$$

### Group Event 3

G3.1 Let  $f$  be a function such that  $f(1) = 1$  and for any integers  $m$  and  $n$ ,  $f(m + n) = f(m) + f(n) + mn$ .

If  $a = \frac{f(2003)}{6}$ , find the value of  $a$ .

$$f(n+1) = f(n) + n + 1 = f(n-1) + n + n + 1 = f(n-2) + n-1 + n + n+1 = \dots = 1 + 2 + \dots + n + n+1$$

$$\therefore f(n) = 1 + 2 + \dots + n = \frac{n(n+1)}{2}$$

$$\frac{f(2003)}{6} = \frac{2004 \times 2003}{12} = 334501$$

G3.2 Suppose  $x^{\frac{1}{2}} + x^{-\frac{1}{2}} = 3$ ,  $b = \frac{x^{\frac{3}{2}} + x^{-\frac{3}{2}} - 3}{x^2 + x^{-2} - 2}$ , find the value of  $b$ .

$$\left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)^2 = 9 \Rightarrow x + x^{-1} = 7 \Rightarrow \left(x + x^{-1}\right)^2 = 49 \Rightarrow x^2 + x^{-2} = 47$$

$$\left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)\left(x + x^{-1}\right) = 3 \times 7 \Rightarrow x^{\frac{3}{2}} + x^{-\frac{3}{2}} + x^{\frac{1}{2}} + x^{-\frac{1}{2}} = 21 \Rightarrow x^{\frac{3}{2}} + x^{-\frac{3}{2}} = 18$$

$$b = \frac{x^{\frac{3}{2}} + x^{-\frac{3}{2}} - 3}{x^2 + x^{-2} - 2} = \frac{18 - 3}{47 - 2} = \frac{1}{3}$$

G3.3 Given that  $f(n) = \sin \frac{n\pi}{4}$ , where  $n$  is an integer. If  $c = f(1) + f(2) + \dots + f(2003)$ , find the value of  $c$ .

$$\begin{aligned}& f(1) + f(2) + f(3) + f(4) + f(5) + f(6) + f(7) + f(8) \\ &= \frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}} + 0 - \frac{1}{\sqrt{2}} - 1 - \frac{1}{\sqrt{2}} + 0 = 0\end{aligned}$$

and the function repeats for every multiples of 8.

$$c = f(2001) + f(2002) + f(2003) = \frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}} = 1 + \sqrt{2}$$

G3.4 Given that  $f(x) = \begin{cases} -2x+1, & \text{when } x < 1 \\ x^2 - 2x, & \text{when } x \geq 1 \end{cases}$ . If  $d$  is the maximum integral solution of  $f(x) = 3$ , find the value of  $d$ .

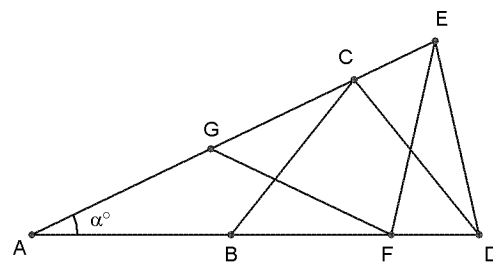
When  $x \geq 1$ ,  $f(x) = 3 \Rightarrow x^2 - 2x = 3 \Rightarrow x^2 - 2x - 3 = 0 \Rightarrow x = 3$  or  $-1$  (rejected)

When  $x < 1$ , the solution cannot be greater than 3.

$$\therefore d = 3$$

Group Event 4

G4.1 In Figure 1,  $AE$  and  $AD$  are two straight lines and  $AB = BC = CD = DE = EF = FG = GA$ . If  $\angle DAE = \alpha^\circ$ , find the value of  $\alpha$ .



$$\begin{aligned} \angle AFG &= \alpha^\circ = \angle ACB \text{ (base } \angle \text{s isos. } \Delta) \\ \angle CBD &= 2\alpha^\circ = \angle FGE \text{ (ext. } \angle \text{ of } \Delta) \\ \angle FEG &= 2\alpha^\circ = \angle BDC \text{ (base } \angle \text{s isos. } \Delta) \\ \angle DFE &= 3\alpha^\circ = \angle DCE \text{ (ext. } \angle \text{ of } \Delta) \\ \angle ADE &= 3\alpha^\circ = \angle AED \text{ (base } \angle \text{s isos. } \Delta) \\ \alpha^\circ + 3\alpha^\circ + 3\alpha^\circ &= 180^\circ \text{ (} \angle \text{s sum of } \Delta) \\ \alpha &= \frac{180}{7} \end{aligned}$$

G4.2 Suppose  $P(x) = a_0 + a_1x + a_2x^2 + \dots + a_8x^8$  is a polynomial of degree 8 with real coefficients  $a_0, a_1, \dots, a_8$ . If  $P(k) = \frac{1}{k}$  when  $k = 1, 2, \dots, 9$ , and  $b = P(10)$ , find the value of  $b$ .

$$P(k) = \frac{1}{k}, \text{ for } k = 1, 2, \dots, 9.$$

Let  $F(x) = xP(x) - 1$ , then  $F(k) = kP(k) - 1 = 0$ , for  $k = 1, 2, \dots, 9$ .

$F(x)$  is a polynomial of degree 9 and the roots are  $1, 2, \dots, 9$ .

$$F(x) = xP(x) - 1 = c(x-1)(x-2)\dots(x-9)$$

$$P(x) = \frac{c(x-1)(x-2)\dots(x-9)+1}{x}, \text{ which is a polynomial of degree 8.}$$

constant term of  $c(x-1)(x-2)\dots(x-9) + 1 = 0$

$$-c \cdot 9! + 1 = 0$$

$$c = \frac{1}{9!} \Rightarrow P(x) = \frac{(x-1)(x-2)\dots(x-9)+9!}{9!x}$$

$$P(10) = \frac{9!+9!}{9! \cdot 10} = \frac{1}{5}$$

G4.3 Given two positive integers  $x$  and  $y$ ,  $xy - (x + y) = \text{HCF}(x, y) + \text{LCM}(x, y)$ , where  $\text{HCF}(x, y)$  and  $\text{LCM}(x, y)$  are respectively the greatest common divisor and the least common multiple of  $x$  and  $y$ . If  $c$  is the maximum possible value of  $x + y$ , find  $c$ .

Without loss of generality assume  $x \geq y$ .

Let the H.C.F. of  $x$  and  $y$  be  $m$  and  $x = ma, y = mb$  where the H.C.F. of  $a, b$  is 1.

L.C.M. of  $x$  and  $y = mab$ .  $a \geq b$ .

$$xy - (x + y) = \text{HCF} + \text{LCM} \Rightarrow m^2ab - m(a + b) = m + mab$$

$$ab(m-1) = a + b + 1$$

$$m-1 = \frac{1}{a} + \frac{1}{b} + \frac{1}{ab}$$

$$1 \leq m-1 = \frac{1}{a} + \frac{1}{b} + \frac{1}{ab} \leq 3$$

$$m = 2, 3 \text{ or } 4$$

$$\text{when } m = 2, \frac{1}{a} + \frac{1}{b} + \frac{1}{ab} = 1 \Rightarrow a + b + 1 = ab \Rightarrow ab - a - b - 1 = 0$$

$$ab - a - b + 1 = 2$$

$$(a-1)(b-1) = 2$$

$$a = 3, b = 2, m = 2, x = 6, y = 4, c = x + y = 10$$

$$\text{when } m = 3, \frac{1}{a} + \frac{1}{b} + \frac{1}{ab} = 2 \Rightarrow a + b + 1 = 2ab \Rightarrow 2ab - a - b - 1 = 0$$

$$4ab - 2a - 2b + 1 = 3$$

$$(2a - 1)(2b - 1) = 3$$

$$a = 2, b = 1, m = 3, x = 6, y = 3, c = x + y = 9$$

$$\text{When } m = 4, \frac{1}{a} + \frac{1}{b} + \frac{1}{ab} = 3 \Rightarrow a + b + 1 = 3ab \Rightarrow 3ab - a - b - 1 = 0$$

$$9ab - 3a - 3b + 1 = 4$$

$$(3a - 1)(3b - 1) = 4$$

$$a = 1, b = 1, m = 4, x = 4, y = 4, c = x + y = 8$$

$$\text{Maximum } c = 10$$

G4.4 In Figure 2,  $\triangle ABC$  is an equilateral triangle, points  $M$  and  $N$  are the midpoints of sides  $AB$  and  $AC$  respectively, and  $F$  is the intersection of the line  $MN$  with the circle  $ABC$ . If  $d = \frac{MF}{MN}$ , find

the value of  $d$ .

Let  $O$  be the centre, join  $AO$ .

Suppose  $MN$  intersects  $AO$  at  $H$ .

Produce  $FNM$  to meet the circle at  $E$ .

Then it is easy to show that:

$MN \parallel BC$  (mid-point theorem)

$\triangle AMO \cong \triangle ANO$  (SSS)

$\triangle AMH \cong \triangle ANH$  (SAS)

$AO \perp MN$  and  $MH = HN$  ( $\cong \Delta$ 's)

$EH = HF$  ( $\perp$  from centre bisect chords)

Let  $EM = r$ ,  $MN = a$ ,  $NF = p$ .

$$r = EH - MH = HF - HN = p$$

By intersecting chords theorem,

$$AN \times NC = FN \times NE$$

$$a^2 = p(p + a)$$

$$p^2 + ap - a^2 = 0$$

$$\left(\frac{p}{a}\right)^2 + \frac{p}{a} - 1 = 0$$

$$\frac{p}{a} = \frac{-1 + \sqrt{5}}{2} \text{ or } \frac{-1 - \sqrt{5}}{2} \text{ (rejected)}$$

$$d = \frac{MF}{MN} = \frac{a + p}{a} = 1 + \frac{-1 + \sqrt{5}}{2} = \frac{1 + \sqrt{5}}{2}$$

