

Individual Events

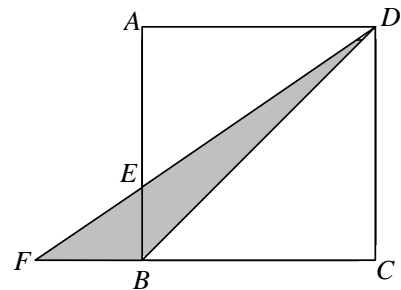
I1	P	40	I2	P	$\frac{99}{100}$	I3	P	12	I4	P	4
	Q	72		Q	1		Q	1		Q	8
	R	648		R	3		R	615		R	4
	S	40.5		S	$\frac{1}{12}$		S	60		S	10

Group Events

G1	a	21	G2	a	24	G3	a	2005	G4	a	4032
	b	2.5		b	52		b	2		b	2
	c	19		c	2005003		c	649		c	1
	d	300°		d	3		d	8		d	2

Individual Event 1

11.1 In the following figure,  $ABCD$  is a square of length 10cm.  $AEB$ ,  $FED$  and  $FBC$  are straight lines. The area of  $\triangle AED$  is larger than that of  $\triangle FEB$  by  $10\text{cm}^2$ . If the area of  $\triangle DFB$  is  $P\text{cm}^2$ , find the value of  $P$ .



Let the area of  $\triangle BDE$  be  $x$ .

Then area of  $\triangle AED + x - (\text{area of } \triangle BEF + x) = 10$

area of  $\triangle ABD - \text{area of } \triangle BDF = 10$

$$\frac{1}{2} 10 \times 10 - \text{area of } \triangle BDF = 10$$

area of  $\triangle BDF = 40$

method 2

Area of  $\triangle ADE - \text{area of } \triangle BFE = 10$  (given)

$\Rightarrow$  Area of  $\triangle ADE + \text{area of } \triangle AEF - \text{area of } \triangle BFE - \text{area of } \triangle AEF = 10$

$\Rightarrow$  Area of  $\triangle ADF - \text{area of } \triangle AFB = 10$

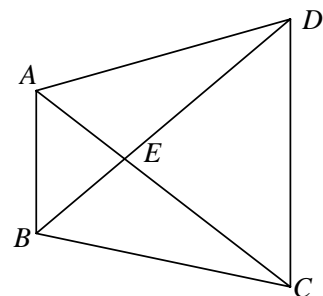
$$\Rightarrow \frac{1}{2} \cdot 10 \times 10 - \text{area of } \triangle DFB = 10$$

Area of  $\triangle DFB = 50 - 10 = 40$

11.2 Workman A needs 90 days to finish a task independently while workman B needs  $Q$  days for the same task. If they only need  $P$  days to finish the task when working together, find the value of  $Q$ .

$$\frac{1}{90} + \frac{1}{Q} = \frac{1}{40}; Q = 72$$

11.3 In the following figure,  $AB \parallel CD$ , the area of trapezium  $ABCD$  is  $R\text{cm}^2$ . Given that the areas of  $\triangle ABE$  and  $\triangle CDE$  are  $Q\text{cm}^2$  and  $4Q\text{cm}^2$  respectively, find the value of  $R$ .



It is easy to show that  $\triangle ABE \sim \triangle CDE$  (equiangular)

$$Q : 4Q = (AB)^2 : (CD)^2 \Rightarrow AB : CD = 1 : 2$$

$AE : EC = BE : ED = 1 : 2$  (ratio of sides,  $\sim \Delta$ 's)

$S_{\triangle AEB} : S_{\triangle AED} = BE : ED = 1 : 2$  (the 2  $\Delta$ s have the same heights)

$$S_{\triangle AED} = 2Q$$

$S_{\triangle AEB} : S_{\triangle BEC} = AE : EC = 1 : 2$  (the 2  $\Delta$ s have the same heights)

$$S_{\triangle BEC} = 2Q$$

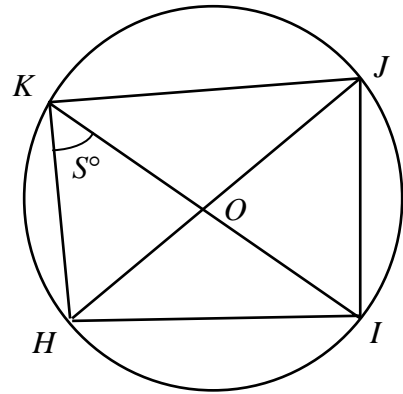
$$S_{ABCD} = Q + 4Q + 2Q + 2Q = 9Q = 648; R = 648$$

11.4 In the following figure,  $O$  is the centre of the circle,  $HJ$  and  $IK$  are diameters and  $\angle HKI = S^\circ$ .

Given that  $\angle HKI + \angle HOI + \angle HJI = \frac{1}{4} R^\circ$ , find the value of  $S$ .

$$S^\circ + 2S^\circ + S^\circ = \frac{1}{4} \times 648^\circ \Rightarrow 4S^\circ = 162^\circ \quad (\angle \text{ at centre} = 2\angle \text{ at } ^\circ\text{e})$$

$$S = 40.5$$



Individual Event 2

12.1 Given that  $P = \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{99 \times 100}$ , find the value of  $P$ .

$$P = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{99} - \frac{1}{100} = 1 - \frac{1}{100} = \frac{99}{100}$$

12.2 Given that  $99Q = P \times (1 + \frac{99}{100} + \frac{99^2}{100^2} + \frac{99^3}{100^3} + \dots)$ , find the value of  $Q$ .

$$99Q = \frac{99}{100} \times (1 + \frac{99}{100} + \frac{99^2}{100^2} + \frac{99^3}{100^3} + \dots) = \frac{99}{100} \times \frac{1}{1 - \frac{99}{100}} = 99; Q = 1$$

12.3 Given that  $x$  and  $R$  are real numbers and  $\frac{2x^2 + 2Rx + R}{4x^2 + 6x + 3} \leq Q$  for all  $x$ , find the maximum value of  $R$ .

$$2x^2 + 2Rx + R \leq 4x^2 + 6x + 3$$

$$2x^2 + 2(3 - R)x + 3 - R \geq 0$$

$$\Delta \leq 0$$

$$(3 - R)^2 - 2(3 - R) \leq 0$$

$$(3 - R)(1 - R) \leq 0$$

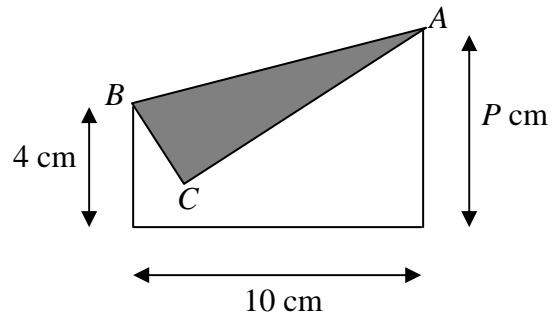
$$1 \leq R \leq 3; \text{ the maximum value of } R = 3$$

12.4 Given that  $S = \log_{144} \sqrt[3]{2} + \log_{144} \sqrt[2]{R}$ , find the value of  $S$ .

$$S = \frac{\frac{1}{3} \log 2}{\log 144} + \frac{\frac{1}{2} \log 3}{\log 144} = \frac{2 \log 2 + \log 3}{6 \log 144} = \frac{\log 12}{6 \log 12^2} = \frac{\log 12}{12 \log 12} = \frac{1}{12}$$

Individual Event 3

I3.1 A rectangular piece of paper is folded into the following figure. If the area of  $\triangle ABC$  is  $\frac{1}{3}$  of the area of the original rectangular piece of paper, find the value of  $P$ .



$BC = P - 4, AC = 10, \angle ACB = 10$

$$\frac{(P-4) \cdot 10}{2} = \frac{1}{3} \times P \times 10 \Rightarrow P = 12$$

I3.2 If  $Q$  is the positive integral solution of the equation  $\frac{P}{2}(4^x + 4^{-x}) - 35(2^x + 2^{-x}) + 62 = 0$ , find the value of  $Q$ .

Let  $t = 2^x + 2^{-x}$ , then  $t^2 = 4^x + 4^{-x} + 2 \Rightarrow 4^x + 4^{-x} = t^2 - 2$

The equation becomes  $6(t^2 - 2) - 35t + 62 = 0$

$6t^2 - 35t + 50 = 0$

$(2t - 5)(3t - 10) = 0$

$t = \frac{5}{2}$  or  $\frac{10}{3}$

$2^x + 2^{-x} = \frac{5}{2}$  or  $2^x + 2^{-x} = \frac{10}{3}$

$2^x + \frac{1}{2^x} = \frac{5}{2}$  or  $2^x + \frac{1}{2^x} = \frac{10}{3}$

$2(2^x)^2 + 2 = 5(2^x)$  or  $3(2^x)^2 + 3 = 10(2^x)$

$2(2^x)^2 - 5(2^x) + 2 = 0$  or  $3(2^x)^2 - 10(2^x) + 3 = 0$

$(2 \cdot 2^x - 1)(2^x - 2) = 0$  or  $(3 \cdot 2^x - 1)(2^x - 3) = 0$

$2^x = \frac{1}{2}, 2, \frac{1}{3}$  or  $3$

For positive integral solution  $x = 1; Q = 1$

I3.3 Let  $[a]$  be the largest integer not greater than  $a$ . For example,  $[2.5] = 2$ .

If  $R = [\sqrt{1}] + [\sqrt{2}] + \dots + [\sqrt{99Q}]$ , find the value of  $R$ .

$$R = [\sqrt{1}] + [\sqrt{2}] + \dots + [\sqrt{99}] = 1 + 1 + 1 + \underbrace{2 + \dots + 2}_{5 \text{ times}} + \underbrace{3 + \dots + 3}_{7 \text{ times}} + \dots + \underbrace{9 + \dots + 9}_{19 \text{ times}}$$

$R = 3 \times 1 + 5 \times 2 + 7 \times 3 + \dots + 19 \times 9$

$R = (2 \times 1 + 1) \times 1 + (2 \times 2 + 1) \times 2 + (2 \times 3 + 1) \times 3 + \dots + (2 \times 9 + 1) \times 9$

$R = 2 \times 1^2 + 1 + 2 \times 2^2 + 2 + 2 \times 3^2 + 3 + \dots + 2 \times 9^2 + 9$

$R = 2 \times (1^2 + 2^2 + 3^2 + \dots + 9^2) + (1 + 2 + 3 + \dots + 9)$

$$R = 2 \times \frac{1}{6} \cdot 9(9+1)(2 \times 9 + 1) + \frac{(1+9)9}{2} = 3 \times 10 \times 19 + 45 = 570 + 45 = 615$$

I3.4 In a convex polygon, other than the interior angle  $A$ , the sum of all the remaining interior angles is equal to  $4R^\circ$ . If  $A = S^\circ$ , find the value of  $S$ .

$4 \times 615 + S = 180 \times (n - 2)$

$S = 180(n - 2) - 2460$

$\therefore$  the polygon is convex,  $\therefore S < 180. S = 180(14) - 2460 = 60$

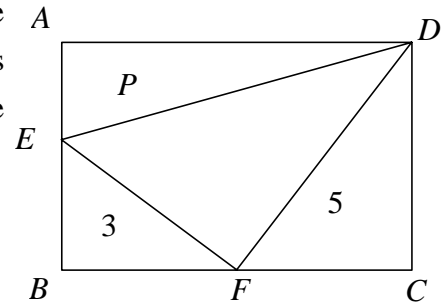
Individual Event 4

14.1 Given that  $f(x) = (x^2 + x - 2)^{2002} + 3$  and  $f\left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right) = P$ , find the value of  $P$ .

$$x = \frac{\sqrt{5}}{2} - \frac{1}{2} \Rightarrow 2x = \sqrt{5} - 1 \Rightarrow (2x + 1) = \sqrt{5} \Rightarrow (2x + 1)^2 = 5 \Rightarrow 4x^2 + 4x - 4 = 0 \Rightarrow x^2 + x = 1$$

$$f(x) = (x^2 + x - 2)^{2002} + 3 = (1 - 2)^{2002} + 3 = 1 + 3 = 4$$

14.2 In the following figure,  $ABCD$  is a rectangle.  $E$  and  $F$  are points on  $AB$  and  $BC$  respectively. The areas of triangles  $AED$ ,  $EBF$  and  $FCD$  are  $P$ , 3 and 5 respectively. If the area of  $\triangle EFD$  is  $Q$ , find the value of  $Q$ .



Let  $AE = x$ ,  $CF = y$ ,  $AD = b$ ,  $CD = a$ .

Then  $BE = a - x$ ,  $BF = b - y$

Given the area of  $\triangle ADE = 4 \Rightarrow bx = 8 \dots\dots\dots(1)$

the area of  $\triangle CDF = 5 \Rightarrow ay = 10 \dots\dots\dots(2)$

the area of  $\triangle BEF = 3 \Rightarrow (a - x)(b - y) = 6 \Rightarrow ab - bx - ay + xy = 6 \dots\dots\dots(3)$

sub (1), (2) into (3)  $ab - 8 - 10 + xy = 6$

sub (1), (2) into the equation again:  $ab - 18 + \frac{80}{ab} = 6$

Solving for  $ab$ ,  $ab = 20$  or  $4$  (rejected)

The area of  $\triangle DEF = 20 - 3 - 4 - 5 = 8$ ;  $Q = 8$

14.3 It is given that  $x$  and  $y$  are positive integers. If the number of solutions  $(x, y)$  of the inequality  $x^2 + y^2 \leq Q$  is  $R$ , find the value of  $R$ .

$$x^2 + y^2 \leq 8 \Rightarrow (x, y) = (1, 1), (1, 2), (2, 1), (2, 2); R = 4$$

14.4 It is given that  $\alpha$  and  $\beta$  are roots of the equation  $x^2 - ax + a - R = 0$ , where  $a$  is real. If the minimum value of  $(\alpha+1)^2 + (\beta+1)^2$  is  $S$ , find the value of  $S$ .

$$x^2 - ax + a - 4 = 0; \alpha + \beta = a, \alpha\beta = a - 4$$

$$(\alpha+1)^2 + (\beta+1)^2 = \alpha^2 + 2\alpha + 1 + \beta^2 + 2\beta + 1 = (\alpha + \beta)^2 - 2\alpha\beta + 2(\alpha + \beta) + 2$$

$$= a^2 - 2(a - 4) + 2a + 2 = a^2 + 10 \geq 10; \text{ the minimum value is } S = 10.$$

Group Event 1

G1.1 Assume that the curve  $x^2 + 3y^2 = 12$  and the straight line  $mx + y = 16$  intersect at only one point. If  $a = m^2$ , find the value of  $a$ .

Sub.  $y = 16 - mx$  into  $x^2 + 3y^2 = 12 \Rightarrow x^2 + 3(16 - mx)^2 = 12$

$x^2 + 3(256 - 32mx + m^2x^2) = 12 \Rightarrow (1 + 3m^2)x^2 - 96mx + 756 = 0$

The straight line is a tangent  $\Rightarrow \Delta = (-96m)^2 - 4(1 + 3m^2)756 = 0$

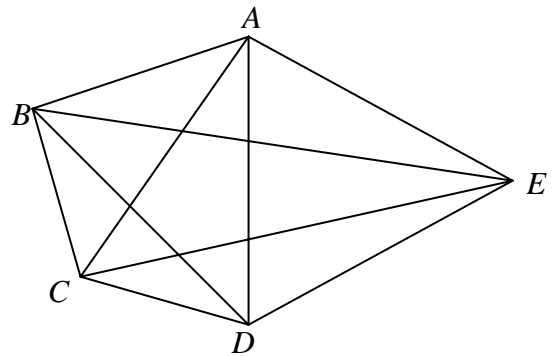
$576m^2 - 189(1 + 3m^2) = 0 \Rightarrow 64m^2 - 21(1 + 3m^2) = 0 \Rightarrow m^2 = 21 \Rightarrow a = 21$

G1.2 It is given that  $x + y = 1$  and  $x^2 + y^2 = 2$ . If  $x^3 + y^3 = b$ , find the value of  $b$ .

$(x + y)^2 = 1 \Rightarrow x^2 + y^2 + 2xy = 1 \Rightarrow 2 + 2xy = 1 \Rightarrow xy = -\frac{1}{2}$

$b = x^3 + y^3 = (x + y)(x^2 + y^2 - xy) = 1(2 + \frac{1}{2}) = \frac{5}{2}$

G1.3 In the following figure,  $AC = AD = AE = ED = DB$  and  $\angle BEC = c^\circ$ . Given that  $\angle BDC = 26^\circ$  and  $\angle ADB = 46^\circ$ , find the value of  $c$ .



$\triangle ADE$  is an equilateral triangle.

$\angle DAE = \angle ADE = \angle AED = 60^\circ$

$\therefore BD = DE$  and  $\angle BDE = 46^\circ + 60^\circ = 106^\circ$

$\therefore \angle BED = (180^\circ - 106^\circ) \div 2 = 37^\circ$  ( $\angle$ s sum of  $\triangle$ )

$\angle AEB = 60^\circ - 37^\circ = 23^\circ$

$\angle ADC = 26^\circ + 46^\circ = 72^\circ$

$\therefore AC = AD$  and  $\angle ADC = 72^\circ = \angle ACD$  (base  $\angle$ , isos.  $\triangle$ )

$\therefore \angle CAD = 180^\circ - 72^\circ \times 2 = 36^\circ$  ( $\angle$ s sum of  $\triangle$ )

$\therefore AC = AE$  and  $\angle CAE = 36^\circ + 60^\circ = 96^\circ$

$\therefore \angle AEC = (180^\circ - 96^\circ) \div 2 = 42^\circ$  ( $\angle$ s sum of  $\triangle$ )

$\angle CED = 60^\circ - 42^\circ = 18^\circ$

$\angle BCE = 60^\circ - 18^\circ - 23^\circ = 19^\circ$ ;  $c = 19$

G1.4 It is given that  $4 \cos^4 \theta + 5 \sin^2 \theta - 4 = 0$ , where  $0^\circ < \theta < 360^\circ$ . If the maximum value of  $\theta$  is  $d$ , find the value of  $d$ .

$4 \cos^4 \theta + 5 \sin^2 \theta - 4 = 0 \Rightarrow 4 \cos^4 \theta + 5(1 - \cos^2 \theta) - 4 = 0 \Rightarrow 4 \cos^4 \theta - 5 \cos^2 \theta + 1 = 0$

$(4 \cos^2 \theta - 1)(\cos^2 \theta - 1) = 0$ ;  $\cos^2 \theta = \frac{1}{4}$  or  $1 \Rightarrow \cos \theta = \frac{1}{2}, -\frac{1}{2}, 1$  or  $-1$ .

$\theta = 60^\circ, 300^\circ, 120^\circ, 180^\circ, 240^\circ$ . The maximum value of  $\theta = 300^\circ$ ;  $d = 300^\circ$

Group Event 2

G2.1 It is given that the lengths of the sides of a triangle are 6, 8, and 10. If the area of the triangle is  $a$ , find the value of  $a$ .

$$6^2 + 8^2 = 36 + 64 = 100 = 10^2; \text{ It is a right angled triangle.}$$

$$\text{The area of the triangle} = 6 \times 8 \div 2 = 24; a = 24$$

G2.2 Given that  $f\left(x + \frac{1}{x}\right) = x^3 + \frac{1}{x^3}$  and  $f(4) = b$ , find the value of  $b$ .

$$\text{Let } y = x + \frac{1}{x}; y^2 - 2 = x^2 + \frac{1}{x^2}, x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)\left(x^2 - 1 + \frac{1}{x^2}\right) = y(y^2 - 3) = y^3 - 3y$$

$$f(y) = y^3 - 3y; b = f(4) = 4^3 - 3(4) = 52$$

G2.3 Given that  $2002^2 - 2001^2 + 2000^2 - 1999^2 + \dots + 4^2 - 3^2 + 2^2 - 1^2 = c$ , find the value of  $c$ .

$$c = (2002 + 2001)(2002 - 2001) + (2000 + 1999)(2000 - 1999) + \dots + (4 + 3)(4 - 3) + (2 + 1)(2 - 1)$$

$$c = 4003 + 3999 + \dots + 7 + 3 = \frac{4003 + 3}{2} \times 1001 = 2005003$$

G2.4  $PQRS$  is a square,  $PTU$  is an isosceles triangle, and  $\angle TPU = 30^\circ$ . Points  $T$  and  $U$  lie on  $QR$  and  $RS$  respectively. The area of  $\triangle PTU$  is 1. If the area of  $PQRS$  is  $d$ , find the value of  $d$ .

$$\text{Let } PT = a = PU$$

$$\frac{1}{2}a^2 \sin 30^\circ = 1 \Rightarrow a = 2$$

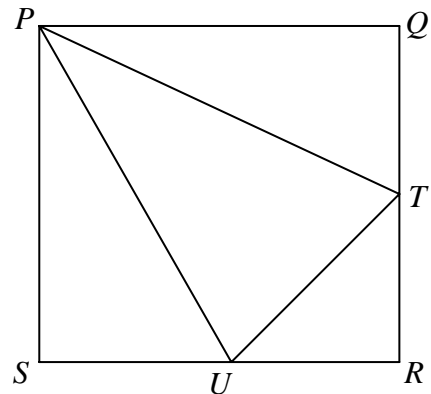
$$\triangle PSU \cong \triangle PQT \text{ (RHS)}$$

$$\text{Let } PS = x = PQ; SU = y = QT$$

$$\angle SPU = \angle QPT = 30^\circ \text{ (corr. } \angle\text{s } \cong \Delta)$$

$$x = PU \cos 30^\circ = 2 \frac{\sqrt{3}}{2} = \sqrt{3}$$

$$d = \text{area of } PQRS = \sqrt{3}^2 = 3$$



Group Event 3

G3.1 If  $\frac{2002^3 + 4 \times 2002^2 + 6006}{2002^2 + 2002} = a$ , find the value of  $a$ .

$$a = \frac{2002(2002^2 + 4 \times 2002 + 3)}{2002(2002 + 1)} = \frac{(2002 + 1)(2002 + 3)}{2002 + 1} = 2005$$

G3.2 It is given that the real numbers  $x$  and  $y$  satisfy the relation  $y = \frac{x}{2x - 1}$ . If the minimum value

of  $\frac{1}{x^2} + \frac{1}{y^2}$  is  $b$ , find the value of  $b$ .

$$\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{x^2} + \frac{(2x - 1)^2}{x^2} = \frac{4x^2 - 4x + 2}{x^2}; \text{ Let } T = \frac{4x^2 - 4x + 2}{x^2}; Tx^2 = 4x^2 - 4x + 2$$

$$(T - 4)x^2 + 4x - 2 = 0$$

$$\Delta = 4^2 + 4 \times 2(T - 4) \geq 0$$

$$2 + T - 4 \geq 0 \Rightarrow T \geq 2; \text{ the minimum value is } 2; b = 2$$

G3.3 Suppose two different numbers are chosen randomly from the 50 positive integers 1, 2, 3, ..., 50, and the sum of these two numbers is not less than 50. If the number of ways of choosing these two numbers is  $c$ , find the value of  $c$ .

Possible combinations may be: (1, 49), (1, 50),

(2, 48), (2, 49), (2, 50),

(3, 47), (3, 48), (3, 49), (3, 50),

.....

(24, 26), (24, 27), ..., (24, 50),

(25, 26), (25, 27), ..., (25, 50),

(26, 27), ..., (26, 50)

.....

(49, 50)

$$\text{Total number of combinations} = (2 + 3 + \dots + 25) + 25 + (24 + 23 + \dots + 1)$$

$$= (1 + 2 + \dots + 24) \times 2 + 24 + 25 = 25 \times 24 + 49 = 649$$

G3.4 Given that  $x - y = 1 + \sqrt{5}$ ,  $y - z = 1 - \sqrt{5}$ . If  $x^2 + y^2 + z^2 - xy - yz - zx = d$ , find the value of  $d$ .

$$2d = (x - y)^2 + (y - z)^2 + (z - x)^2 = (1 + \sqrt{5})^2 + (1 - \sqrt{5})^2 + [(z - y) - (x - y)]^2$$

$$2d = 1 + 2\sqrt{5} + 5 + 1 - 2\sqrt{5} + 5 + [-1 + \sqrt{5} - (1 + \sqrt{5})]^2 = 12 + 4 = 16; d = 8$$

Group Event 4

G4.1 If  $a$  is the sum of all the positive factors of 2002, find the value of  $a$ .

$$2002 = 2 \times 7 \times 11 \times 13$$

The positive factors may be  $2^a 7^b 11^c 13^d$ , where  $0 \leq a, b, c, d \leq 1$  are integers.

The sum of all positive factors are  $(1 + 2)(1 + 7)(1 + 11)(1 + 13) = 3 \times 8 \times 12 \times 14 = 4032 = a$

G4.2 It is given that  $x > 0, y > 0$  and  $\sqrt{x}(\sqrt{x} + \sqrt{y}) = 3\sqrt{y}(\sqrt{x} + 5\sqrt{y})$ . If  $b = \frac{2x + \sqrt{xy} + 3y}{x + \sqrt{xy} - y}$ , find

the value of  $b$ .

$$\sqrt{x}(\sqrt{x} + \sqrt{y}) = 3\sqrt{y}(\sqrt{x} + 5\sqrt{y}) \Rightarrow x + \sqrt{xy} = 3\sqrt{xy} + 15y \Rightarrow x - 2\sqrt{xy} - 15y = 0$$

$$(\sqrt{x} + 3\sqrt{y})(\sqrt{x} - 5\sqrt{y}) = 0 \Rightarrow \sqrt{x} = 5\sqrt{y} \quad \text{and} \quad x = 25y$$

$$b = \frac{2x + \sqrt{xy} + 3y}{x + \sqrt{xy} - y} = \frac{50y + \sqrt{25y^2} + 3y}{25y + \sqrt{25y^2} - y} = \frac{58y}{29y} = 2; \quad b = 2$$

G4.3 Given that the equation  $||x - 2| - 1| = c$  has only 3 integral solutions, find the value of  $c$ .

$|x - 2| - 1 = \pm c \Rightarrow |x - 2| = 1 \pm c$ , in order that it has only 3 integral solutions,  $c = 1$

G4.4 If  $d$  is the positive real root of the equation  $\frac{1}{2} \left\{ \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} x^2 + 2 \right) + 2 \right] + 2 \right\} = 2$ , find the value

of  $d$ .

$$\frac{1}{2} \left\{ \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} x^2 + 2 \right) + 2 \right] + 2 \right\} = 2 \Rightarrow \left\{ \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} x^2 + 2 \right) + 2 \right] + 2 \right\} = 4 \Rightarrow \frac{1}{2} \left[ \frac{1}{2} \left( \frac{1}{2} x^2 + 2 \right) + 2 \right] = 2$$

$$\left[ \frac{1}{2} \left( \frac{1}{2} x^2 + 2 \right) + 2 \right] = 4 \Rightarrow \frac{1}{2} \left( \frac{1}{2} x^2 + 2 \right) = 2 \Rightarrow \frac{1}{2} x^2 + 2 = 4 \Rightarrow \frac{1}{2} x^2 = 2 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

$d =$  the positive real root  $= 2$