Individual Events

I1	P	1	I2	P	12	I3	P	4	I4	P	35
	Q	4		Q	14		Q	7		Q	10
	R	2		R	1		R	14		R	10
	S	32		S	2		S	34		S	222

Group Events

G1	a	4	G2	a	2	G3	a	3	G4	a	3840
	b	1001		b	3		b	20		b	1
	c	8		c	333333		c	14		c	3
	d	3		d	46		d	15		d	1853

Individual Event 1

I1.1 a, b and c are the lengths of the opposite sides $\angle A$, $\angle B$ and $\angle C$ of the ΔABC respectively. If $\angle C = 60°$ and $\frac{a}{b+c}+\frac{b}{a+c}=P$, find the value of P.

$c^2 = a^2 + b^2 - 2ab\cos 60° = a^2 + b^2 - ab \Rightarrow a^2 + b^2 = c^2 + ab$

$$P=\frac{a}{b+c}+\frac{b}{a+c}=\frac{a(a+c)+b(b+c)}{(b+c)(a+c)}$$

$$P=\frac{a^2+ac+b^2+bc}{ab+ac+bc+c^2}=\frac{ab+ac+bc+c^2}{ab+ac+bc+c^2}=1$$

I1.2 Given that $f(x) = x^2 + ax + b$ is the common factor of $x^3 + 4x^2 + 5x + 6$ and $2x^3 + 7x^2 + 9x + 10$ If $f(P) = Q$, find the value of Q.

Let $g(x) = x^3 + 4x^2 + 5x + 6$; $h(x) = 2x^3 + 7x^2 + 9x + 10$

$g(-3) = -27 + 36 - 15 + 6 = 0$, $(x + 3)$ is a factor of $g(x)$; by division, $g(x) = (x + 3)(x^2 + x + 2)$

$h(-2.5)= -31.25+43.75-22.5+10 = 0$, $(2x+5)$ is a factor of $h(x)$; by division, $h(x) = (2x+5)(x^2+x+2)$

$f(x)$ = common factor $= (x^2 + x + 2)$; $Q = f(P) = f(1) = 1 + 1 + 2 = 4$

I1.3 Given that $\frac{1}{a}+\frac{1}{b}=\frac{Q}{a+b}$ and $\frac{a}{b}+\frac{b}{a}=R$, find the value of R.

$\frac{1}{a}+\frac{1}{b}=\frac{4}{a+b} \Rightarrow (a + b)^2 = 4ab \Rightarrow a^2 + 2ab + b^2 = 4ab \Rightarrow a^2 - 2ab + b^2 = 0 \Rightarrow (a \Rightarrow b)^2 = 0$

$a = b \Rightarrow R=\frac{a}{b}+\frac{b}{a} = 2$

I1.4 Given that $\begin{cases} a+b=R \\ a^2+b^2=12 \end{cases}$ and $a^3 + b^3 = S$, find the value of S.

$$\begin{cases} a+b=2 \cdots\cdots(1) \\ a^2+b^2=12\cdots\cdots(2) \end{cases}$$

$(1)^2 - (2)$: $2ab = -8 \Rightarrow \begin{cases} ab=-4 \\ a+b=2 \end{cases}$

$S = a^3 + b^3 = (a + b)(a^2 - ab + b^2) = 2(12 + 4) = 32$

Individual Event 2

I2.1 Suppose P is an integer and $5 < P < 20$. If the roots of the equation

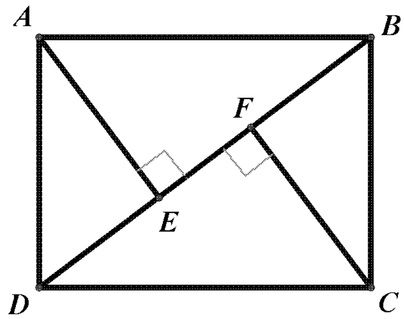
$x^2 - 2(2P - 3)x + 4P^2 - 14P + 8 = 0$ are integers, find the value of P.

$\Delta = 4(2P - 3)^2 - 4(4P^2 - 14P + 8) = m^2$

$\left(\frac{m}{2}\right)^2 = 4P^2 - 12P + 9 - 4P^2 + 14P - 8 = 2P + 1$

$\because 5 < P < 20$ $\therefore P = 12$ so that $2P + 1 = 25 = 5^2$

I2.2 $ABCD$ is a rectangle. $AB = 3P + 4$, $AD = 2P + 6$. AE and CF are perpendiculars to the diagonal BD. If $EF = Q$, find the value of Q.



$AB = 40$, $AD = 30$, $BD = 50$, let $\angle ADB = \theta$, $\cos\theta = \frac{3}{5}$

$DE = AD\cos\theta = 30\times\frac{3}{5} = 18 = BF$

$EF = 50 - 18 - 18 = 14$

I2.3 There are less than $4Q$ students in a class. In a mathematics test, $\frac{1}{3}$ of the students got grade A, $\frac{1}{7}$ of the students got grade B, half of the students got grade C, and the rest failed. Given that R students failed in the mathematics test, find the value of R.

$4Q = 56$, let the number of students be x, then x is divisible by 2, 3 and 7.

i.e. x is divisible by 42, as $x < 56$, so $x = 42$

R = number of students failed in Mathematics $= 42\times\left(1-\frac{1}{3}-\frac{1}{7}-\frac{1}{2}\right) = 1$; $R = 1$

I2.4 $[a]$ represents the largest integer not greater than a. For example, $\left[2\frac{1}{3}\right] = 2$. Given that the sum of the roots of the equation $[3x + R] = 2x + \frac{3}{2}$ is S, find the value of S.

$[3x + 1] = 2x + \frac{3}{2} \Rightarrow 3x + 1 = 2x + \frac{3}{2} + a$, where $0 \le a < 1$

$a = x - \frac{1}{2} \Rightarrow 0 \le x - \frac{1}{2} < 1 \Rightarrow 2.5 \le 2x + \frac{3}{2} < 4.5$

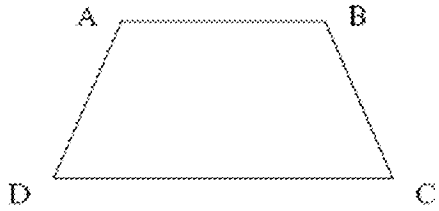
$\because 2x + \frac{3}{2}$ is an integer $\therefore$ $2x + \frac{3}{2} = 4$ or 3

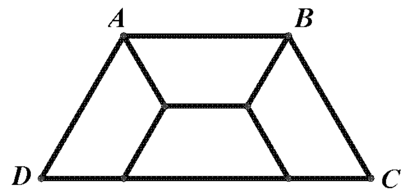
$x = 0.75$ or 1.25

$S = 0.75 + 1.25 = 2$

Individual Event 3

I3.1 $ABCD$ is a trapezium such that $\angle ADC = \angle BCD = 60°$ and $AB = BC = AD = \frac{1}{2}CD$. If this trapezium is divided into P equal portions ($P > 1$) and each portion is similar to trapezium $ABCD$ itself. Find the minimum value of P.





From the graph, $P = 4$

I3.2 The sum of tens and unit digits of $(P + 1)^{2001}$ is Q. Find the value of Q.

$5^{2001} = 100a + 25$, where a is a positive integer. $Q = 2 + 5 = 7$.

I3.3 If $\sin 30° + \sin^2 30° + \ldots + \sin^Q 30° = 1 - \cos^R 45°$, find the value of R.

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^7}=1-\frac{1}{\sqrt{2}^R}$$

$$1-\frac{1}{2^7}=1-\frac{1}{2^{\frac{R}{2}}};\ R = 14$$

I3.4 Let α and β be the roots of the equation $x^2 - 8x + (R + 1) = 0$. If $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$ are the roots of the equation $225x^2 - Sx + 1 = 0$, find the value of S.

$x^2 - 8x + 15 = 0$, $\alpha = 3$, $\beta = 5$

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{1}{9}+\frac{1}{25}=\frac{34}{225}=\frac{S}{225};\ S = 34$$

Individual Event 4

I4.1 Let $a^{\frac{2}{3}}+b^{\frac{2}{3}}=17\frac{1}{2}$, $x=a+3a^{\frac{1}{3}}b^{\frac{2}{3}}$ and $y=b+3a^{\frac{2}{3}}b^{\frac{1}{3}}$. If $P=(x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}$, find the value of P.

$$P=\left(a+3a^{\frac{2}{3}}b^{\frac{1}{3}}+3a^{\frac{1}{3}}b^{\frac{2}{3}}+b\right)^{\frac{2}{3}}+\left(a-3a^{\frac{2}{3}}b^{\frac{1}{3}}+3a^{\frac{1}{3}}b^{\frac{2}{3}}-b\right)^{\frac{2}{3}}=\left(a^{\frac{1}{3}}+b^{\frac{1}{3}}\right)^{3\times\frac{2}{3}}+\left(a^{\frac{1}{3}}-b^{\frac{1}{3}}\right)^{3\times\frac{2}{3}}$$

$$P=\left(a^{\frac{1}{3}}+b^{\frac{1}{3}}\right)^2+\left(a^{\frac{1}{3}}-b^{\frac{1}{3}}\right)^2=a^{\frac{2}{3}}+2a^{\frac{1}{3}}b^{\frac{1}{3}}+a^{\frac{2}{3}}+a^{\frac{2}{3}}-2a^{\frac{1}{3}}b^{\frac{1}{3}}+a^{\frac{2}{3}}=2\left(a^{\frac{2}{3}}+a^{\frac{2}{3}}\right)=2\times17.5=35$$

I4.2 If a regular Q-sided polygon has P diagonals, find the value of Q.

The number of diagonals $= {}_QC_2 - Q = 35$; $\frac{Q(Q-1)}{2}-Q=35$; $Q^2 - 3Q - 70 = 0$; $Q = 10$

I4.3 Let $x=\sqrt{\frac{Q}{2}+\sqrt{\frac{Q}{2}}}$ and $y=\sqrt{\frac{Q}{2}-\sqrt{\frac{Q}{2}}}$. If $R=\frac{x^6+y^6}{40}$, find the value of R.

$$R=\frac{(x^2+y^2)(x^4+y^4-x^2y^2)}{40}=\frac{\left(\frac{Q}{2}+\sqrt{\frac{Q}{2}}+\frac{Q}{2}-\sqrt{\frac{Q}{2}}\right)\left[\left(\frac{Q}{2}+\sqrt{\frac{Q}{2}}\right)^2+\left(\frac{Q}{2}-\sqrt{\frac{Q}{2}}\right)^2-\left(\frac{Q}{2}+\sqrt{\frac{Q}{2}}\right)\left(\frac{Q}{2}-\sqrt{\frac{Q}{2}}\right)\right]}{40}$$

$$R=\frac{Q\left[2\left(\frac{Q}{2}\right)^2+2\left(\sqrt{\frac{Q}{2}}\right)^2-\left(\frac{Q}{2}\right)^2+\left(\sqrt{\frac{Q}{2}}\right)^2\right]}{40}=\frac{10(5^2+3\times5)}{40}=10$$

I4.4 $[a]$ represents the largest integer not greater than a. For example, $[2.5] = 2$.

If $S=\left[\frac{2001}{R}\right]+\left[\frac{2001}{R^2}\right]+\left[\frac{2001}{R^3}\right]+\cdots$, find the value of S.

$$S=\left[\frac{2001}{10}\right]+\left[\frac{2001}{100}\right]+\left[\frac{2001}{1000}\right]+\cdots=200+20+2+0+\ldots=222$$

Group Event 1

G1.1 Given that $(a + b + c)^2 = 3(a^2 + b^2 + c^2)$ and $a + b + c = 12$, find the value of a.

Sub. (2) into (1), $12^2 = 3(a^2 + b^2 + c^2) \Rightarrow a^2 + b^2 + c^2 = 48$ ……(3)

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca) \Rightarrow 12^2 = 48 + 2(ab + bc + ca) \Rightarrow ab + bc + ca = 48$

$2[a^2 + b^2 + c^2 - (ab + bc + ca)] = (a - b)^2 + (b - c)^2 + (c - a)^2$

$2[48 - 48] = 0 = (a - b)^2 + (b - c)^2 + (c - a)^2 \Rightarrow a = b = c$

$a + b + c = 3a = 12 \Rightarrow a = 4$

G1.2 Given that $b\left[\frac{1}{1\times 3}+\frac{1}{3\times 5}+\cdots+\frac{1}{1999\times 2001}\right]=2\times\left[\frac{1^2}{1\times 3}+\frac{2^2}{3\times 5}+\cdots+\frac{1000^2}{1999\times 2001}\right]$, find the value of b

Note that $\frac{1}{(2r-1)\times(2r+1)}=\frac{1}{2}\left(\frac{1}{2r-1}-\frac{1}{2r+1}\right)$ and $\frac{r^2}{(2r-1)\times(2r+1)}=\frac{1}{4}+\frac{1}{8}\left(\frac{1}{2r-1}-\frac{1}{2r+1}\right)$

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\cdots+\frac{1}{1999\times 2001}=\frac{1}{2}\left[1-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\cdots+\frac{1}{1999}-\frac{1}{2001}\right]=\frac{1}{2}\left(1-\frac{1}{2001}\right)=\frac{1000}{2001}$$

$$\frac{1^2}{1\times 3}+\frac{2^2}{3\times 5}+\cdots+\frac{1000^2}{1999\times 2001}=\frac{1}{4}+\frac{1}{8}\left(1-\frac{1}{3}\right)+\frac{1}{4}+\frac{1}{8}\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots+\frac{1}{4}+\frac{1}{8}\left(\frac{1}{1999}-\frac{1}{2001}\right) \text{ (1000 terms)}$$

$$=\frac{1000}{4}+\frac{1}{8}\left(1-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\cdots+\frac{1}{1999}-\frac{1}{2001}\right)=\frac{1000}{4}+\frac{1}{8}\left(1-\frac{1}{2001}\right)$$

$$=\frac{1000}{4}+\frac{1}{8}\cdot\frac{2000}{2001}=250+\frac{250}{2001}=250\left(1+\frac{1}{2001}\right)=\frac{250\cdot 2002}{2001}$$

The given equation becomes: $b\cdot\frac{1000}{2001}=2\cdot\frac{250\cdot 2002}{2001}\Rightarrow b=1001$

G1.3 A six-digit number $1234xy$ is divisible by both 8 and 9. Given that $x + y = c$, find the value of c

The number formed by last 3 digits must be divisible by 8 and the sum of digits must be divisible by 9. i.e. $400 + 10x + y$ is divisible by 8 and $1 + 2 + 3 + 4 + x + y = 9m$

$10x + y = 8n$ ……(1); $x + y = 9m - 10$ ……(2)

(1) – (2): $9x = 8n - 9m + 9 + 1 \Rightarrow n = 1$ or 10

When $n = 1$, (1) has no solution; when $n = 10$, $x = 8$, $y = 0$; $c = x + y = 8$

G1.4 Suppose $\log_x t = 6$, $\log_y t = 10$ and $\log_z t = 15$. If $\log_{xyz} t = d$, find the value of d.

$\frac{\log t}{\log x}=6,\ \frac{\log t}{\log y}=10,\ \frac{\log t}{\log z}=15 \Rightarrow \frac{\log x}{\log t}=\frac{1}{6},\ \frac{\log y}{\log t}=\frac{1}{10},\ \frac{\log t}{\log z}=\frac{1}{15}$

$\frac{\log x}{\log t}+\frac{\log y}{\log t}+\frac{\log z}{\log t}=\frac{1}{6}+\frac{1}{10}+\frac{1}{15}=\frac{10}{30}=\frac{1}{3}$

$\frac{\log x+\log y+\log z}{\log t}=\frac{1}{3}$

$\frac{\log xyz}{\log t}=\frac{1}{3}$

$\frac{\log t}{\log xyz}=3=d$

Group Event 2

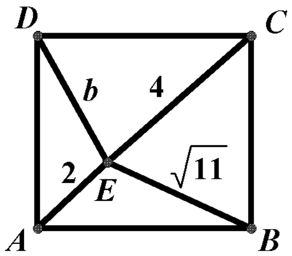
G2.1 Given that $x=\sqrt{7-4\sqrt{3}}$ and $\frac{x^2-4x+5}{x^2-4x+3}=a$, find the value of a.

$x=\sqrt{7-4\sqrt{3}}=\sqrt{4-2\sqrt{12}+3}=\sqrt{\sqrt{4}^2-2\sqrt{4}\sqrt{3}+\sqrt{3}^2}=\sqrt{\left(\sqrt{4}-\sqrt{3}\right)^2}=\sqrt{4}-\sqrt{3}=2-\sqrt{3}$

$\sqrt{3}=2-x \Rightarrow 3=(2-x)^2 \Rightarrow x^2-4x+1=0$

$a=\frac{x^2-4x+5}{x^2-4x+3}=\frac{x^2-4x+1+4}{x^2-4x+1+2}=2$

G2.2 E is an interior point of the rectangle $ABCD$. Given that the lengths of EA, EB, EC and ED are 2, $\sqrt{11}$, 4 and b respectively, find the value of b.



P, Q R, S be the foot of perpendiculars drawn from E onto AB, BC, CD, DA respectively.

Using Pythagoras Theorem, it can be proved that

$p^2+s^2=4$ ....(1)

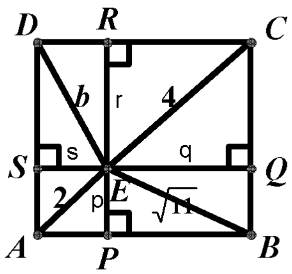
$p^2+q^2=11$ ....(2)

$q^2+r^2=16$ ....(3)

$r^2+s^2=b^2$ ....(4)

(1) + (3) – (2) – (4): $0=4+16-11-b^2$

$b=3$



G2.3 Given that 111111222222 = $c\times(c+1)$, find the value of c.

111111222222 = 111111000000 + 222222 = 111111×1000000 + 2×111111 = 111111×1000002

111111222222 = 111111×3×333334 = 333333 × 333334; c = 333333

Reference 1996 Final group event 7 (ii)

A six-digit figure 111*aaa* is the product of two consecutive positive integers b and $b+1$, find the value of b.

G2.4 Given that cos 16° = sin 14° + sin d° and $0<d<90$, find the value of d.

sin d° = cos 16° – sin 14°

sin d° = sin 74° – sin 14°

sin d° = 2 cos88° sin 30°

sin d° = cos 44° = sin 46°

d = 46

Group Event 3

G3.1 Given that the solution of the equation $\sqrt{3x+1}+\sqrt{3x+6}=\sqrt{4x-2}+\sqrt{4x+3}$ is a, find the value of a.

$\sqrt{3x+6}-\sqrt{4x-2}=\sqrt{4x+3}-\sqrt{3x+1}$

$\left(\sqrt{3x+6}-\sqrt{4x-2}\right)^2=\left(\sqrt{4x+3}-\sqrt{3x+1}\right)^2$

$3x+6+4x-2-2\sqrt{12x^2+18x-12}=4x+3+3x+1-2\sqrt{12x^2+13x+3}$

$\sqrt{12x^2+18x-12}=\sqrt{12x^2+13x+3}$

$12x^2+18x-12=12x^2+13x+3$

$x=3$

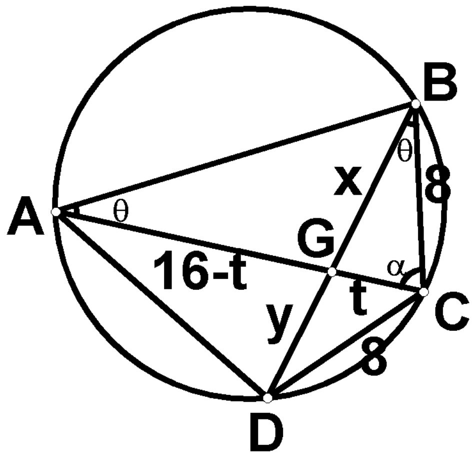
G3.2 Suppose the equation $x^2y-x^2-3y-14=0$ has only one positive integral solution (x_0, y_0). If $x_0+y_0=b$, find the value of b.

$(y-1)x^2=3y+14$

$x^2=\frac{3y+14}{y-1}=\frac{3y-3+17}{y-1}=3+\frac{17}{y-1}=3+1$

$y=18$, $x=2$, $c=20$

G3.3 $ABCD$ is a cyclic quadrilateral. AC and BD intersect at G. Suppose $AC=16$ cm, $BC=CD=8$ cm, $BG=x$ cm and $GD=y$ cm. If x and y are integers and $x+y=c$, find the value of c.



As shown in the figure, let $CG=t$, $AG=16-t$.

Let $\angle CBG=\theta$, $\angle ACB=\alpha$.

Then $\angle CAB=\theta$ (eq. chords eq. $\angle$s)

Then $\Delta BCG\sim\Delta ACB$ (equiangular)

$t:8=8:16$ (ratio of sides, ~Δs)

$t=4$

It is easy to see that $\Delta ADG\sim\Delta BCG$ (equiangular)

$(16-t):y=x:t$ (ratio of sides, ~Δs)

$(16-4)\times4=xy$

$xy=48$

Assume that x and y are integers, then possible pairs of (x, y) are (1,48), (2, 24), ..., (6, 8), ..., (48, 1).

Using triangle inequality $x+t>8$ and $8+t>x$ in ΔBCG, the only possible combinations are:

$(x, y)=(6, 8)$ or $(8, 6)$

$c=x+y=14$

G3.4 Given that $5^{\log 30}\times\left(\frac{1}{3}\right)^{\log 0.5}=d$, find the value of d.

$\log 30\log 5+\log 0.5\log\frac{1}{3}=\log d$

$\log(3\times10)\log\frac{10}{2}+(-\log 2)(-\log 3)=\log d$

$(\log 3+1)(1-\log 2)+\log 2\log 3=\log d$

$\log 3+1-\log 3\log 2-\log 2+\log 2\log 3=\log d$

$\log d=\log 3+1-\log 2=\log\frac{3\times10}{2}$

$d=15$

Group Event 4

G4.1 $x_1 = 2001$. When $n > 1$, $x_n = \frac{n}{x_{n-1}}$. Given that $x_1x_2x_3\ldots x_{10} = a$, find the value of a.

$x_2 = \frac{2}{x_1} \Rightarrow x_1x_2 = 2$

$x_4 = \frac{4}{x_3} \Rightarrow x_3x_4 = 4$

$x_6 = \frac{6}{x_5} \Rightarrow x_5x_6 = 6$

$x_8 = \frac{8}{x_7} \Rightarrow x_7x_8 = 8$

$x_{10} = \frac{10}{x_9} \Rightarrow x_9x_{10} = 10$

Multiply these equations gives $a = x_1x_2x_3\ldots x_{10} = 2\times4\times6\times8\times10 = 32\times120 = 3840$

G4.2 Given that the unit digit of $1^3 + 2^3 + 3^3 + \ldots + 2001^3$ is b, find the value of b.
Arrange the numbers in groups of 10 in ascending order, the unit digit of sum each group is the same (except the last number, 2001^3). So $b = 1$

G4.3 A and B ran around a circular path with constant speeds. They started from the same place and at the same time in opposite directions. After their first meeting, B took 1 minute to go back to the starting place. If A and B need 6 minutes and c minutes respectively to complete one round of the path, find the value of c.

In one minute, A and B ran $\frac{1}{6}+\frac{1}{c}=\frac{c+6}{6c}$ of the total distance.

They will meet after $\frac{6c}{c+6}$ minutes at the first time.

After 1 more minute, (i.e. total time elapsed = $\frac{6c}{c+6}+1$ minutes), B retuned to the starting point. So $\left(\frac{6c}{c+6}+1\right)\times\frac{1}{c}=1$

$6c + c + 6 = c^2 + 6c$

$c^2 - c - 6 = 0$

$(c - 3)(c + 2) = 0$

$c = 3$

G4.4 The roots of the equation $x^2 - 45x + m = 0$ are prime numbers. Given that the sum of the squares of the roots is d, find the value of d.

Let the roots be α, β. $\alpha + \beta = 45$, $\alpha\beta = m$

The sum of two prime numbers $\alpha + \beta = 45$

$\alpha = 2$, $\beta = 43$ (2 is the only even prime number)

$d = \alpha^2 + \beta^2 = 4 + 43^2 = 1853$