Definition

$$X[K] = X(e^{j\omega})\big|_{\omega=2\pi k/N} = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} \quad ,0 \le k \le N-1$$

$X[K]$ เป็น finite-length sequence (N) เรียกว่า DFT ของ x[n]

แทนค่า $w_N = e^{-j2\pi/N}$

DFT $$X[K] = \sum_{n=0}^{N-1} x[n]w_N^{kn} \qquad 0 \le k \le N-1$$

IDFT $$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[K]w_N^{-kn} \qquad 0 \le n \le N-1$$

<u>Note</u> จากเดิมต้องหา ∞ จุด เพราะ ω ต่อเนื่อง แต่DFT จะหาเพียง N จุด

$(0 \le k \le N-1)$

Ex 3.9 Determine the DFT of the sequence $x[n]=\begin{cases}1, & n=0\\ 0, & otherwise\end{cases}$

Ex 3.10. Consider the length-12 sequence, defined for 0<=n<=11,

$$\{x[n]\} = \{3 \;\; -1 \;\; 2 \;\; 4 \;\; -3 \;\; -2 \;\; 0 \;\; 1 \;\; -4 \;\; 6 \;\; 2 \;\; 5\}$$

with a 12-point DFT given by X[k], 0<=k<=11. Evaluate the following functions:

a) X[0]

b) X[6]

c) $\sum_{k=0}^{11} X[k]$

Matrix Relations

The DFT samples defined in Eq. (3.25) can be expressed in matrix form as

$$\mathbf{X} = \mathbf{D}_N \mathbf{x},$$

where $\mathbf{X}$ is the vector composed of the N DFT samples,

$$\mathbf{X} = [\, X[0] \quad X[1] \quad \cdots \quad X[N-1]\,]^T ,$$

$\mathbf{x}$ is the vector of N input samples,

$$\mathbf{x} = [\, x[0] \quad x[1] \quad \cdots \quad x[N-1]\,]^T ,$$

and $\mathbf{D}_N$ is the $N \times N$ *DFT matrix* given by

$$\mathbf{D}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)\times(N-1)} \end{bmatrix}.$$

Matrix Relations

Likewise, the IDFT relations can be expressed in matrix form as

$$\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} = \mathbf{D}_N^{-1} \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix},$$

where $\mathbf{D}_N^{-1}$ is the $N \times N$ *IDFT matrix* given by

$$\mathbf{D}_N^{-1} = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)\times(N-1)} \end{bmatrix}.$$

It follows from Eqs. (3.40) and (3.42) that

$$\mathbf{D}_N^{-1} = \frac{1}{N}\mathbf{D}_N^{*}.$$

Ex 3.11. General representation of 4-point DFT G[k] of the length-4 sequence g[n]

is given by $G[k] = g[0] + g[1]e^{-j2\pi k/4} + g[2]e^{-j4\pi k/4} + g[3]e^{-j6\pi k/4}$

Or by Matrix Relations approach

3.3 Z-Transform

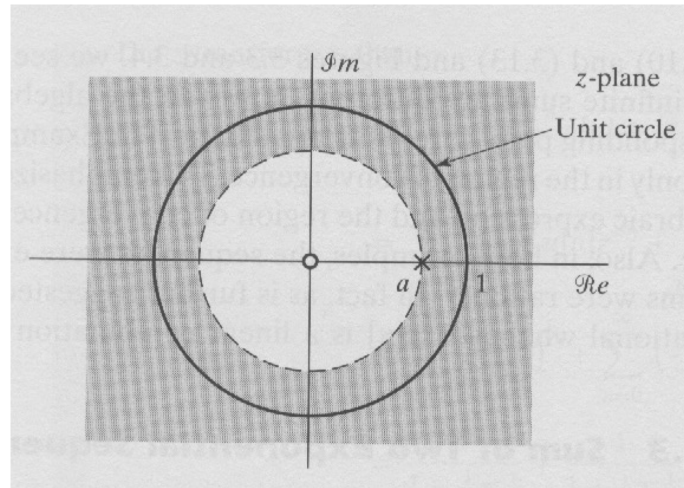
-Generalization of DTFT
-DTFT ของบาง sequence หาไม่ได้ แต่ Z-transform หาได้
-เป็นอุปกรณ์สำคัญในการวิเคราะห์และออกแบบ digital filter

$$X(Z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \quad \text{[two-side Z-transform]}$$

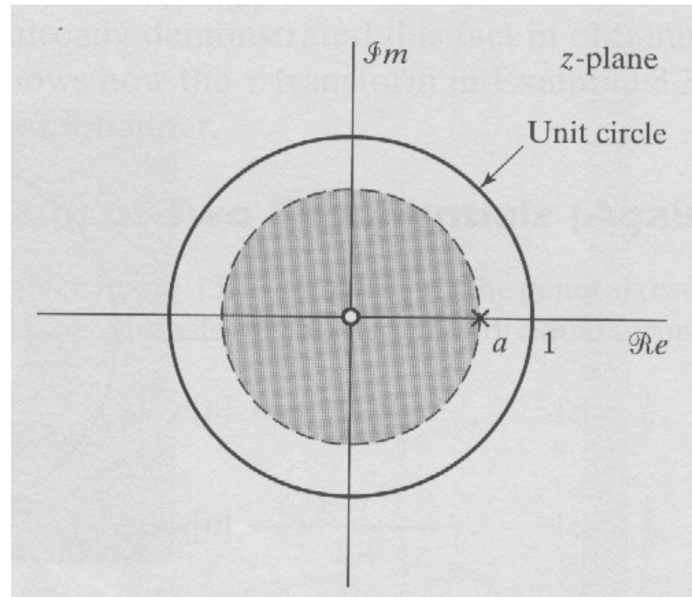
$$X(Z) = \sum_{n=0}^{\infty} x[n] z^{-n} \quad \text{[one-sided z-transform]}$$

$$Z = re^{jw}$$

Ex 3.12 Right-Sided Exponential Sequence $x[n] = a^n u[n]$

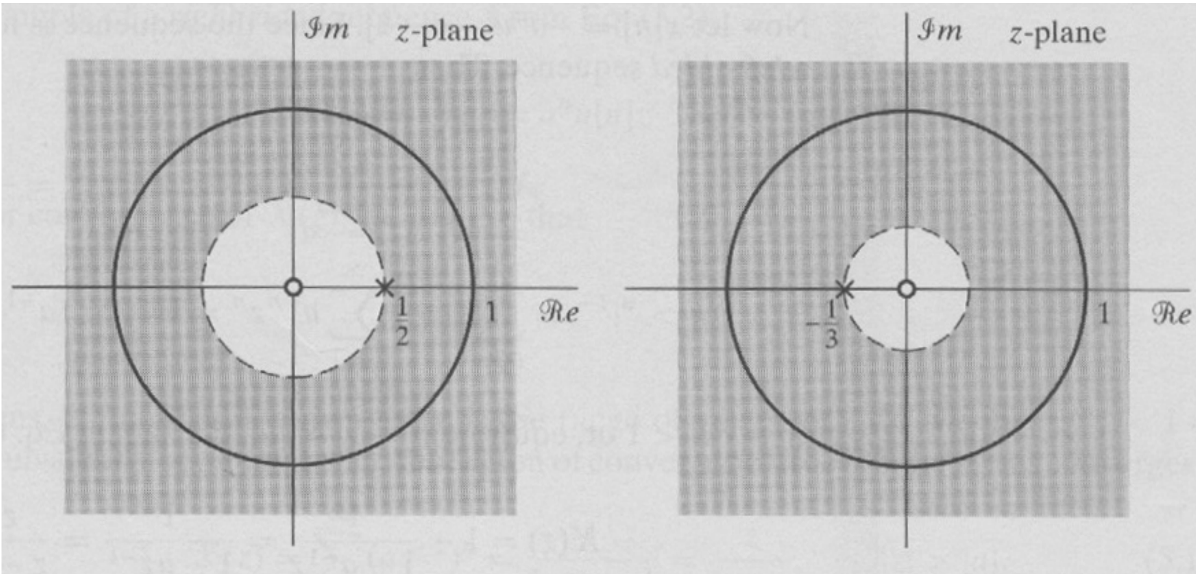


Ex 3.13 Left-Sided Exponential Sequence $x[n] = -a^n u[-n-1]$

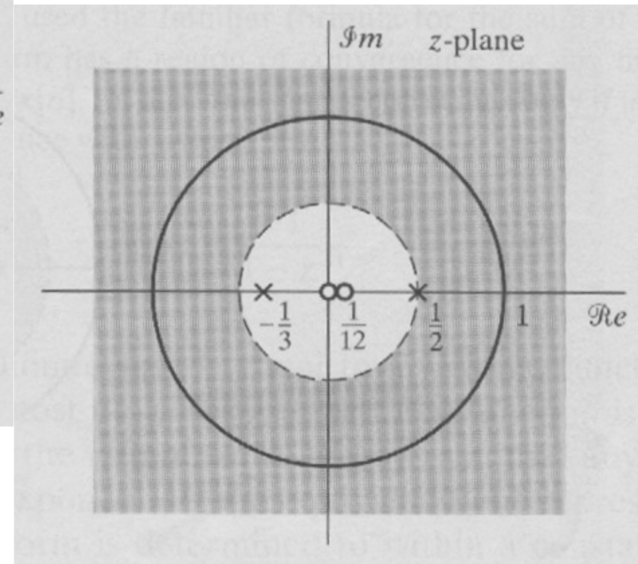


Ex 3.14 Sum of Two Exponential Sequences $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$

Ex 3.14 Sum of Two Exponential Sequences $x[n]=\left(\frac{1}{2}\right)^n u[n]+\left(-\frac{1}{3}\right)^n u[n]$

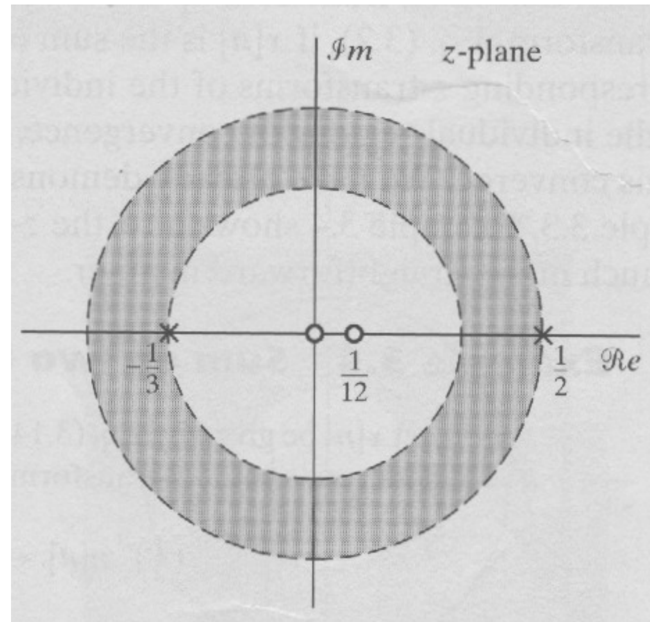


(a) (b)



(c)

Ex 3.15 Two-sided Exponential Sequence $x[n]=\left(-\frac{1}{3}\right)^{n}u[n]-\left(\frac{1}{2}\right)^{n}u[-n-1]$



Ex 3.16 Finite-Length Sequence $x[n] = \begin{cases} a^n & 0 \le n \le N-1 \\ 0 & otherwise \end{cases}$

Z-transform Properties

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m>0$) or ∞ (if $m<0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $[\cos\omega_0 n]u[n]$	$\frac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\frac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0\le n\le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^Nz^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

PROPERTIES OF THE REGION OF CONVERGENCE FOR THE z-TRANSFORM

The examples of the previous section suggest that the properties of the region of convergence depend on the nature of the signal. These properties are summarized next, followed by some discussion and intuitive justification. We assume specifically that the algebraic expression for the z-transform is a rational function and that $x[n]$ has finite amplitude, except possibly at $n = \infty$ or $n = -\infty$.

PROPERTY 1: The ROC is a ring or disk in the z-plane centered at the origin; i.e., $0 \leq r_R < |z| < r_L \leq \infty$.

PROPERTY 2: The Fourier transform of $x[n]$ converges absolutely if and only if the ROC of the z-transform of $x[n]$ includes the unit circle.

PROPERTY 3: The ROC cannot contain any poles.

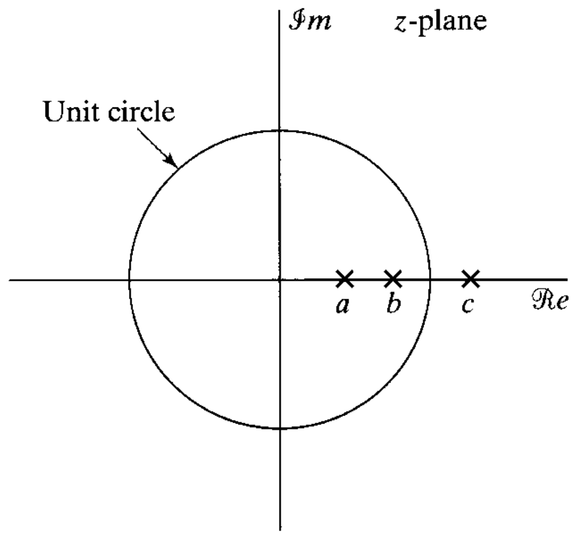
PROPERTY 4: If $x[n]$ is a *finite-duration sequence,* i.e., a sequence that is zero except in a finite interval $-\infty < N_1 \leq n \leq N_2 < \infty$, then the ROC is the entire z-plane, except possibly $z = 0$ or $z = \infty$.

PROPERTY 5: If $x[n]$ is a *right-sided sequence,* i.e., a sequence that is zero for $n < N_1 < \infty$, the ROC extends outward from the *outermost* (i.e., largest magnitude) finite pole in $X(z)$ to (and possibly including) $z = \infty$.

PROPERTY 6: If $x[n]$ is a *left-sided sequence,* i.e., a sequence that is zero for $n > N_2 > -\infty$, the ROC extends inward from the *innermost* (smallest magnitude) nonzero pole in $X(z)$ to (and possibly including) $z = 0$.

PROPERTY 7: A *two-sided sequence* is an infinite-duration sequence that is neither right sided nor left sided. If $x[n]$ is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and, consistent with property 3, not containing any poles.

PROPERTY 8: The ROC must be a connected region.



(a)

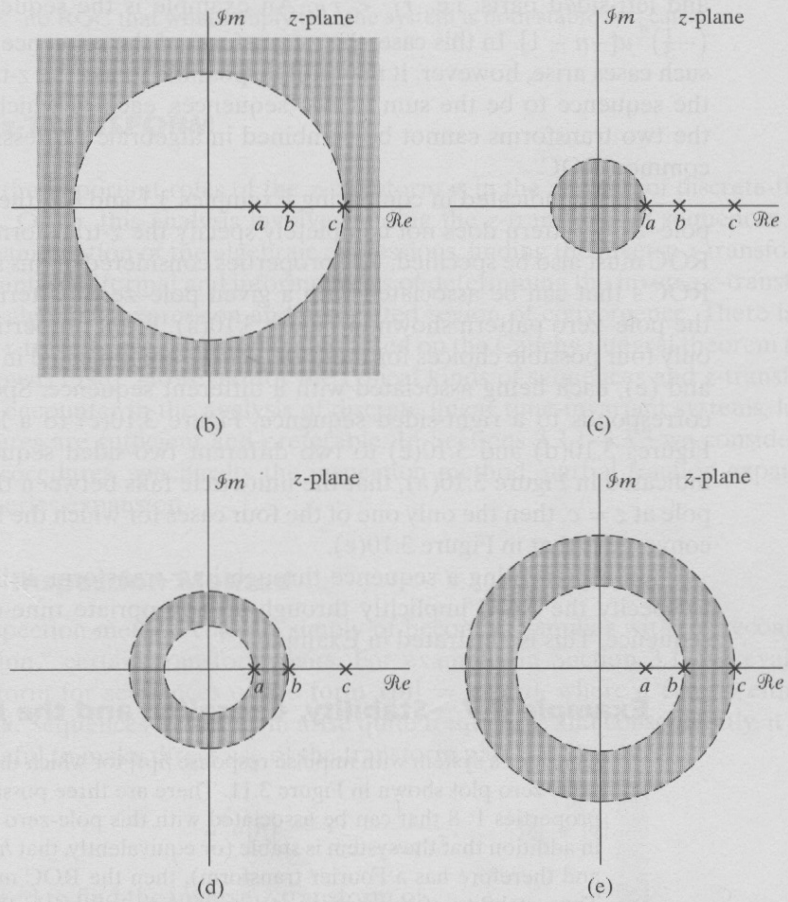
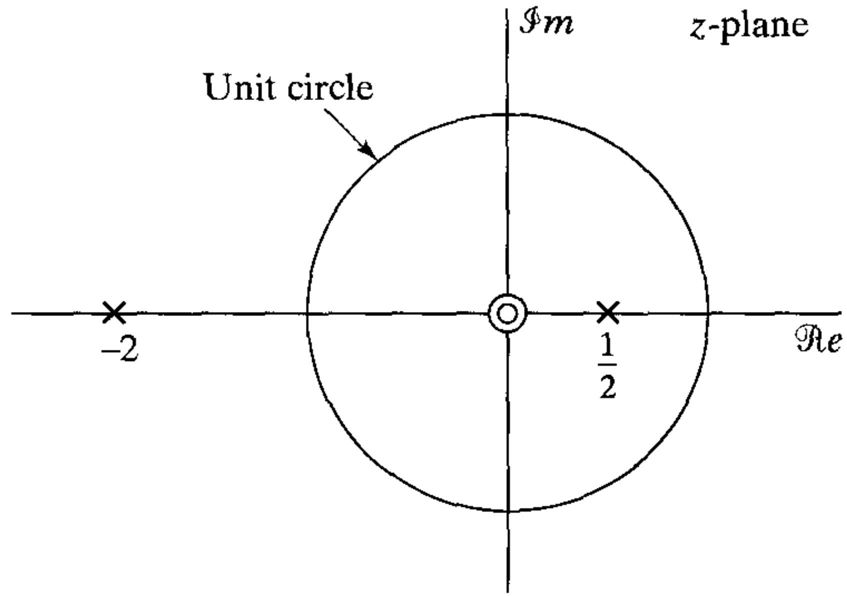


Figure 3.10 Examples of four z- transforms with the same pole–zero locations, illustrating the different possibilities for the region of convergence. Each ROC corresponds to a different sequence: (b) to a right-sided sequence, (c) to a left-sided sequence, (d) to a two-sided sequence, and (e) to a two-sided sequence.

Stability and Causality and The ROC



1. Stability (ROC includes Unit Circle)
2. Causal (h[n]เป็น Right-handed Side)
   (นอกวงกลม property5)

Inverse Z-Transform

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n - n_0]$	$z^{-n_0} X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\| R_x$
4	3.4.4	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x, except for the possible addition or deletion of the origin or ∞
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
7		$\mathcal{J}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
10	3.4.8	Initial-value theorem: $x[n] = 0, \quad n < 0 \qquad \lim_{z\to\infty} X(z) = x[0]$		

Inverse Z-Transform

Inspection Method

Recognizing the certain transform pairs เช่น $a^n u[n] \overset{z}{\longleftrightarrow} \dfrac{1}{1-az^{-1}}$

Partial fraction Expansion

$$X(Z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$X(Z) = \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}}$$

Power Series Expansion

- $$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \ldots + x[2]z^{2} + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \ldots$$

Ex 3.17 Inverse by Partial Fraction $X(z) = \dfrac{1}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})}, \qquad |z| > \frac{1}{2}$

Ex 3.18 Inverse by Partial Fraction $X(z) = \dfrac{(1+z^{-1})^2}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}, \qquad |z| > 1$

Ex 3.19 Finite-Length Sequence $X(z) = z^2(1-\frac{1}{2}z^{-1})(1+z^{-1})(1-z^{-1})$

<u>Note</u> Convolution by Z-Transform $y[n] = x[n]*h[n] = Z^{-1}\{X(z)H(z)\}$

Ex 3.20 Power Series Expansion by Long Division $X(z)=\dfrac{1}{1-az^{-1}}, \qquad |z|>|a|$

INTRODUCTION

For LTI system

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Generalization of Fourier Transform (Z-transform)

$$Y(z) = H(z)X(z)$$

THE FRQUENCY RESPONSE OF LTI SYSTEM

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$$|Y(e^{j\omega})| = |H(e^{j\omega})| \cdot |X(e^{j\omega})|$$

$$\angle Y(e^{j\omega}) = \angle H(e^{j\omega}) + \angle X(e^{j\omega})$$

Ideal Frequency-Selective Filters

Ideal Lowpass

$$H_{lp}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

$$h_{lp}[n] = \frac{\sin \omega_c n}{\pi n}, \qquad -\infty < n < \infty$$

Ideal Highpass

$$H_{hp}(e^{j\omega}) = \begin{cases} 0, & |\omega| < \ \leq \pi \\ 1, & \omega_c < |\omega| \leq \pi \end{cases}$$

$$h_{hp}[n] = \delta[n] - h_{lp}[n] = \delta[n] - \frac{\sin \omega_c n}{\pi n}$$

Phase Distortion and Delay

Ideal Delay

$$\begin{cases} h_{id}[n] = \delta[n-n_d] \\ H_{id}(e^{j\omega}) = e^{-j\omega n_d} \\ \left|H_{id}(e^{j\omega})\right| = 1 \\ \angle H_{id}(e^{j\omega}) = -\omega n_d . \qquad |\omega| < \pi \end{cases}$$

Ideal Lowpass with Linear Phase

$$\begin{cases} H_{lp}(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & |\omega| < \leq\pi \\ 0, & \omega_c < |\omega| \leq \pi \end{cases} \\ h_{lp}[n] = \dfrac{\sin\omega_c(n-n_d)}{\pi(n-n_d)}, \qquad -\infty < n < \infty \\ \angle H(e^{j\omega}) \cong -\phi_0 - \omega n_d \end{cases}$$

Ideal Delay

$$\left\{ \tau(\omega) = grd\left[h(e^{j\omega})\right] = -\frac{d}{d\omega}\left\{\arg\left[H(e^{j\omega})\right]\right\} \right.$$

*

Example 5.1 Effects of Attenuation and Group Delay

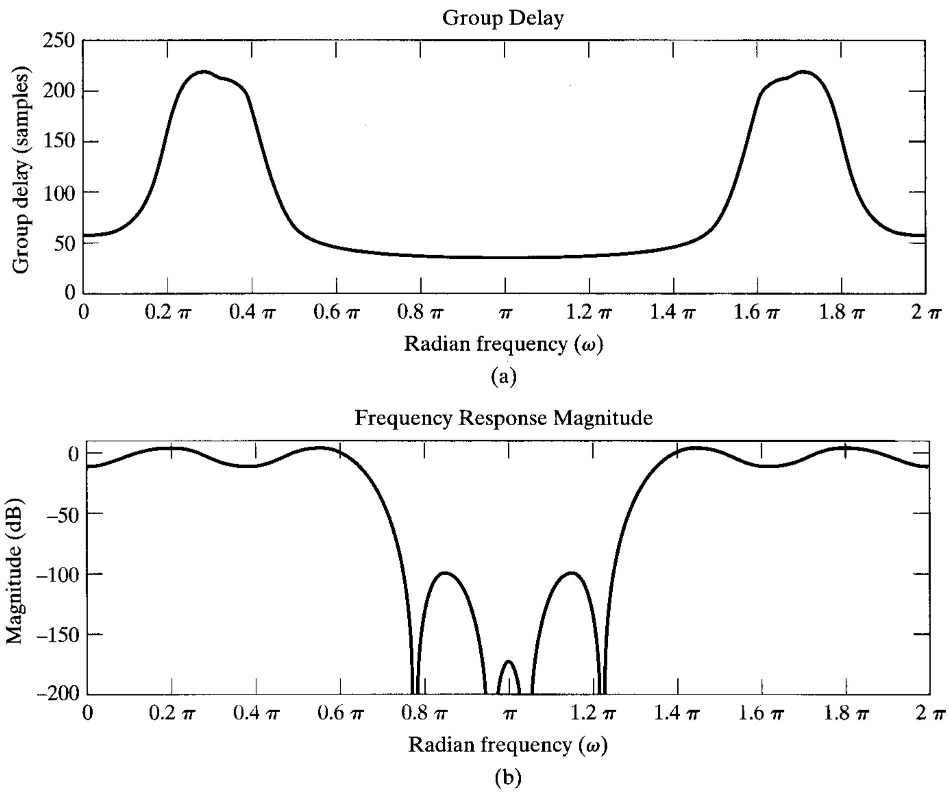


Figure 5.1 Frequency response magnitude and group delay for the filter in Example 5.1.

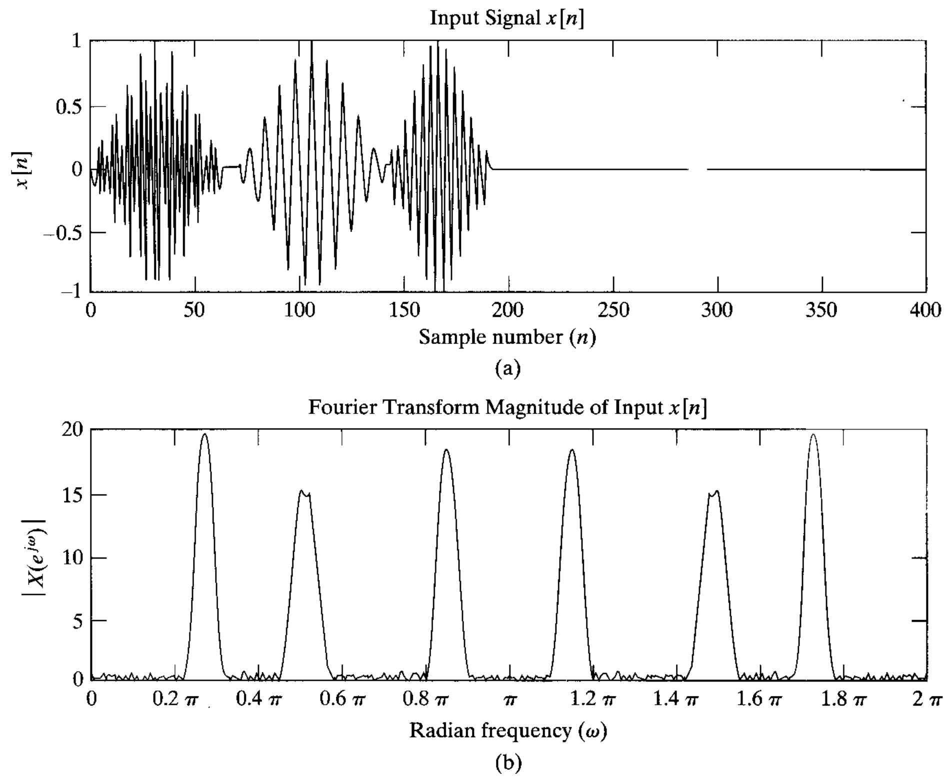
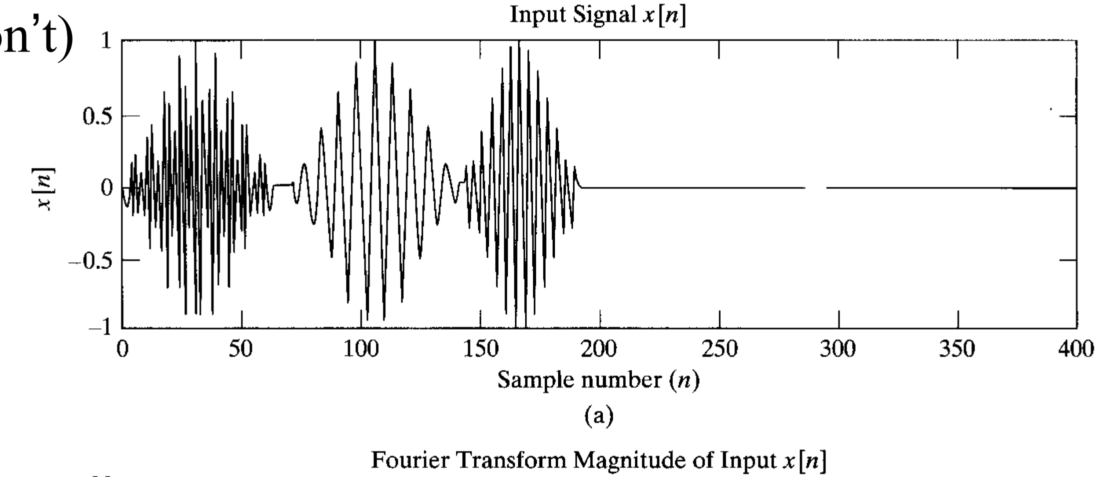


Figure 5.2 Input signal and associated Fourier transform magnitude for Example 5.1.

Example 5.1 (Con't)

Input



Output

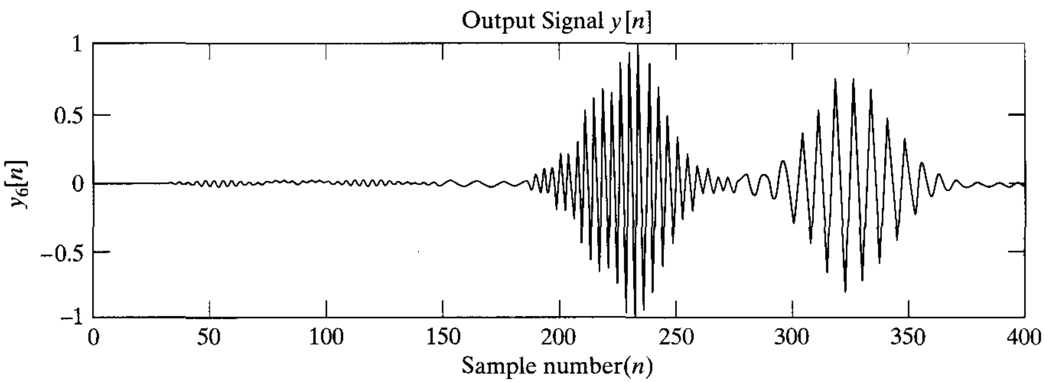


Figure 5.3 Output signal for Example 5.1.

SYSTEM FUNCTIONS FOR SYSTEMS CHARACTERIZED BY LINEAR CONSTANT-COEFFICIENT DIFFERENCE EQUATIONS

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\left(\sum_{k=0}^{N} a_k z^{-k}\right) Y(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right) X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

General Form

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}\left(1 - c_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1 - d_k z^{-1}\right)}$$

Example 5.2 Second-Order System **, find the Difference Equation**

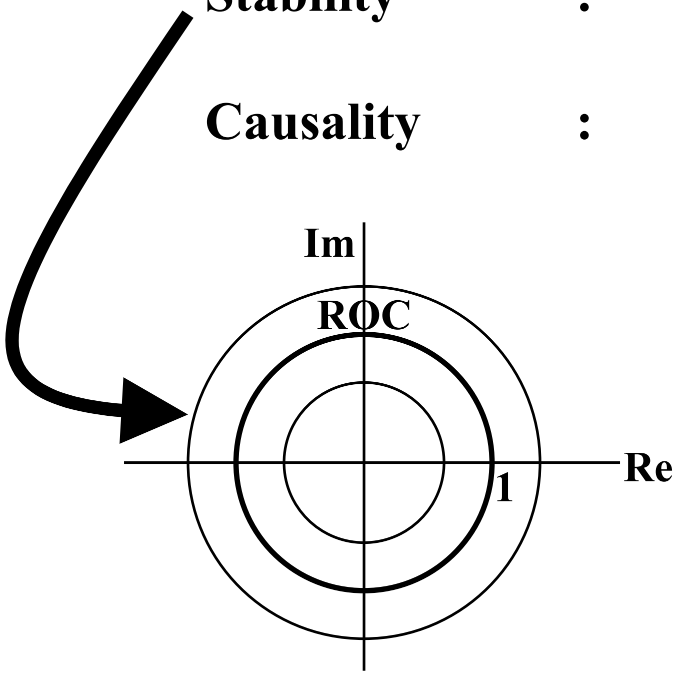
$$H(z)=\frac{\left(1+z^{-1}\right)^2}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{3}{4}z^{-1}\right)}$$

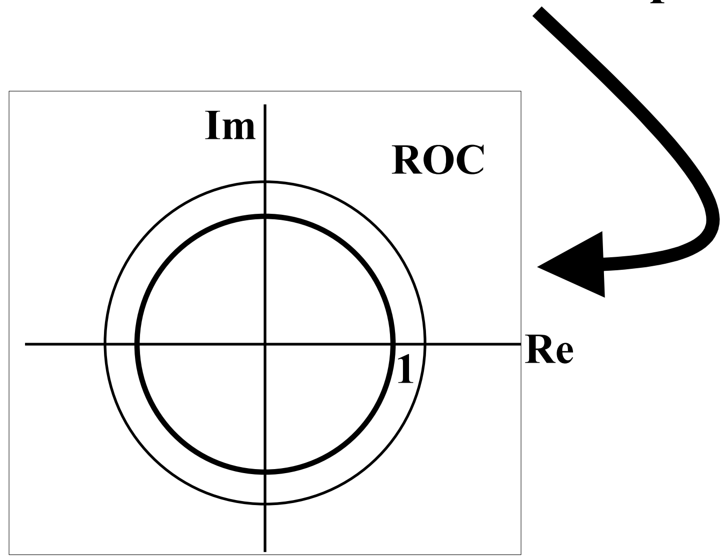
Stability and Causality

Restatement

Stability : ROC includes Unit Circuit

Causality : ROC must be outside the outermost pole





Example 5.3 Determining the ROC **(from transfer function)**

$$y[n]-\frac{5}{2}y[n-1]+y[n-2]=x[n]$$

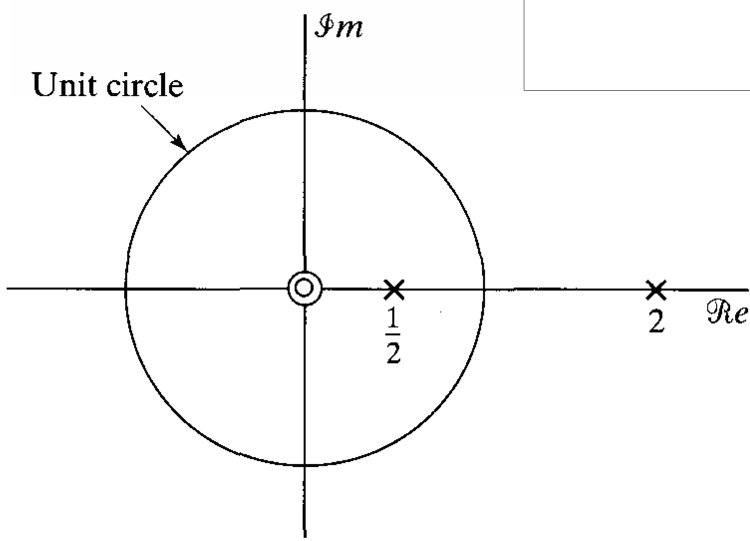


Figure 5.4 Pole–zero plot for Example 5.3.

Inverse Systems

Definition

$$G(z) = H(z)H_i(z) = 1$$

$$H_i(z) = \frac{1}{H(z)}$$

$$g[n] = h[n] * h_i[n] = \delta[n]$$

$$H_i(e^{j\omega}) = \frac{1}{H(e^{j\omega})};$$

System function

$$H(z) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

Inverse System function

$$H_i(z) = \left(\frac{a_0}{b_0}\right)\frac{\prod_{k=1}^{N}(1 - d_k z^{-1})}{\prod_{k=1}^{M}(1 - c_k z^{-1})};$$

$$|z| > \max_k |d_k|$$

Basic condition

Example 5.4 Inverse System for First-Order System

Find $H_i(z)$ and its impulse response

$$H(z) = \frac{1 - 0.5z^{-1}}{1 - 0.9z^{-1}} \quad \textbf{ROC } |z| > 0.9$$

Impulse Response for Rational System Functions

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}}$$

$$h[n] = \sum_{r=0}^{M-N} b_r \delta[n-r] + \sum_{k=1}^{N} A_k d_k^n u[n]$$

Example 5.6 A First-Order IIR System

$$y[n] - ay[n-1] = x[n]$$

Find transfer function and its impulse response

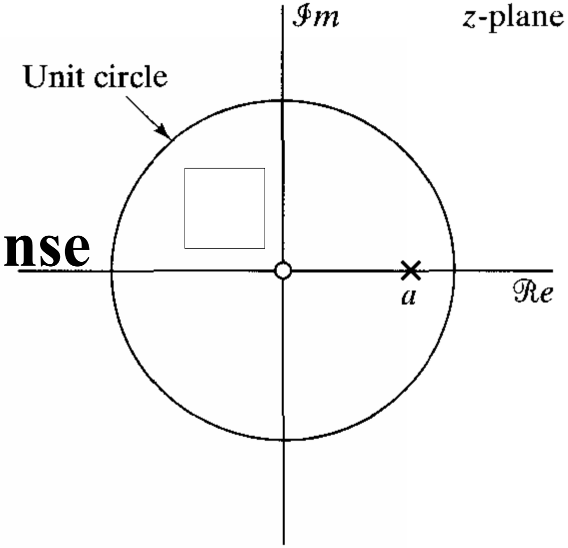
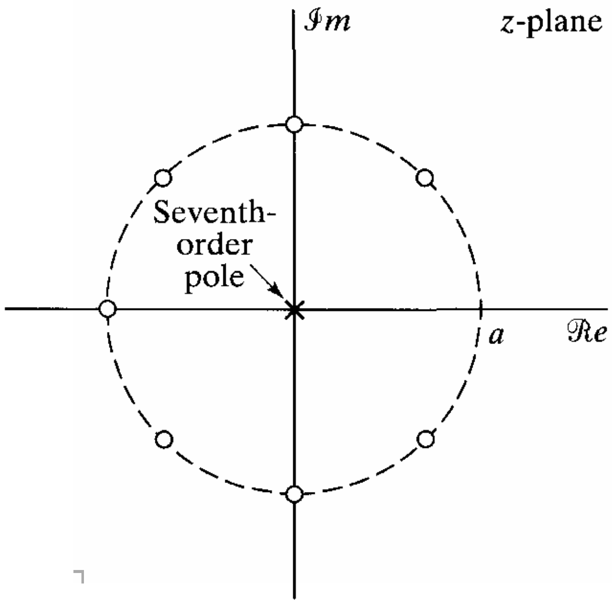


Figure 5.5 Pole–zero plot for Example 5.6.

Example 5.7 A Simple FIR System



$$h[n] = \begin{cases} a^n, & 0 \le n \le M, \\ 0, & \text{otherwise}. \end{cases}$$

$$H(z) = \sum_{n=0}^{M} a^n z^{-n} = \frac{1 - a^{M+1} z^{-M-1}}{1 - az^{-1}}$$

$$z_k = ae^{j2\pi k/(M+1)}, \qquad k = 0,1,\ldots,M,$$

$$y[n] = \sum_{k=0}^{M} a^k x[n-k]$$

$$y[n] - ay[n-1] = x[n] - a^{M+1} x[n-M-1]$$

Figure 5.6 Pole–zero plot for Example 5.7.

FREQUENCY RESPONSE FOR RATIONAL SYSTEM FUNCTIONS

Transfer function

$$H(e^{j\omega}) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}\left(1-c_k e^{-j\omega}\right)}{\prod_{k=1}^{N}\left(1-d_k e^{-j\omega}\right)}$$

Magnitude

$$\left|H(e^{j\omega})\right| = \left|\frac{b_0}{a_0}\right|\frac{\prod_{k=1}^{M}\left|1-c_k e^{-j\omega}\right|}{\prod_{k=1}^{N}\left|1-d_k e^{-j\omega}\right|}$$

Log Magnitude

$$Gain\ in\ dB = 20\log_{10}\left|H(e^{j\omega})\right|$$

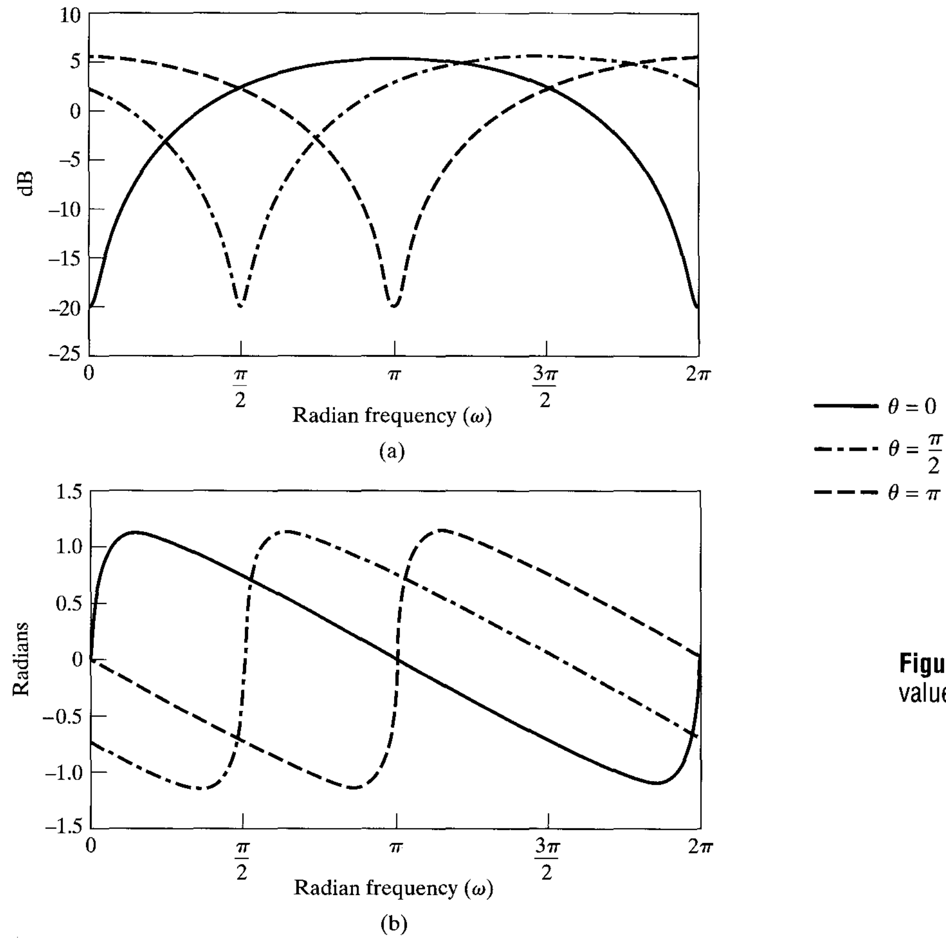
$$Attenuation\ in\ dB = -20\log_{10}\left|H(e^{j\omega})\right|$$

$$= -Gain\ in\ dB$$

Group Delay

$$grd[H(e^{j\omega})] = -\frac{d}{d\omega}[\angle H(e^{j\omega})]$$

Frequency Response For a Single Zero or Pole $(1 - re^{j\theta} z^{-1})$



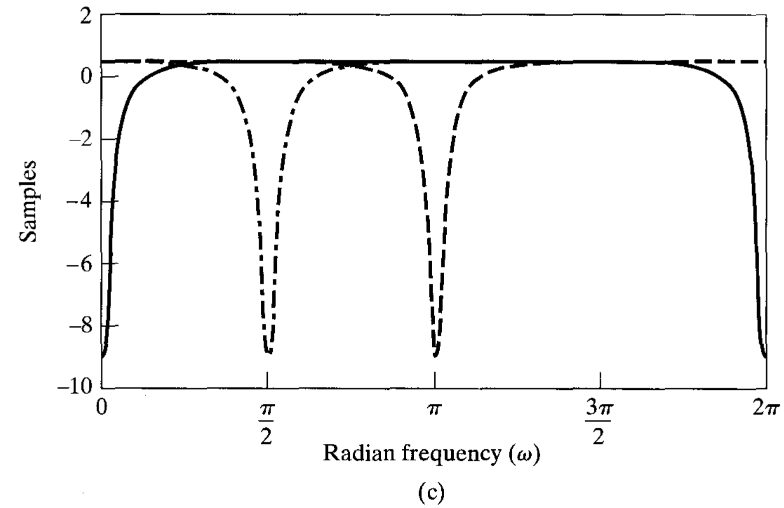


Figure 5.8 Frequency response for a single zero, with $r = 0.9$ and the three values of θ shown. (a) Log magnitude. (b) Phase. (c) Group delay.

ALL-PASS SYSTEMS

$$H_{ap}(z) = \frac{z^{-1} - a^{*}}{1 - az^{-1}}$$

$$H_{ap}(z) = A\prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{\left(z^{-1} - e_k^{*}\right)\left(z^{-1} - e_k\right)}{\left(1 - e_k z^{-1}\right)\left(1 - e_k^{*} z^{-1}\right)}$$