• Discrete Time Signals

c. Classification of sequences

c2.Energy and Power Signals

-total energy of a sequence x[n]

$$E_x = \sum_{n=-\infty}^{\infty} \left| x[n] \right|^2$$

-average power of a periodic sequence

$$P_x = \lim_{k \to \infty} \frac{1}{2k+1} \sum_{n=-k}^{k} \left| x[n] \right|^2$$

-average power of an aperiodic sequence

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} \left| \tilde{x}[n] \right|^2$$

• Discrete Time Signals

c. Classification of sequences

c3.Other types

-bounded sequence

$$\left|x[n]\right| \le B_x < \infty$$

-absolutely summable

$$\sum_{n=-\infty}^{\infty} \left|x[n]\right| < \infty$$

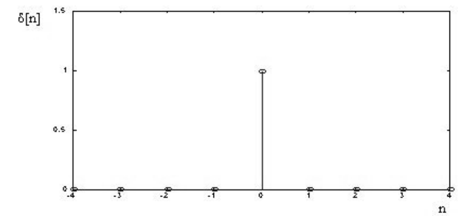
-square summable (finite energy)

$$\sum_{n=-\infty}^{\infty} \left|x[n]\right|^2 < \infty$$

- Sequence รูปแบบเฉพาะ

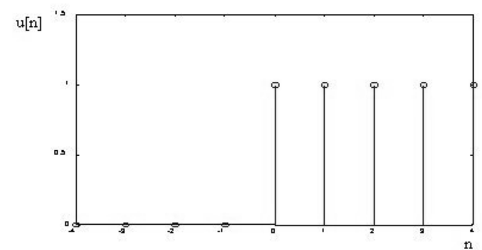
a. unit sample sequence

$$\delta[n]=\begin{cases}1, & n=0\\ 0, & n\neq 0\end{cases} \qquad \delta[n-k]=\begin{cases}1, & n=k\\ 0, & n\neq k\end{cases}$$



b. unit step sequence

$$u[n]=\begin{cases}1, & n\geq 0\\ 0, & n<0\end{cases} \qquad u[n]=\sum_{n=0}^{\infty}\delta[n]$$

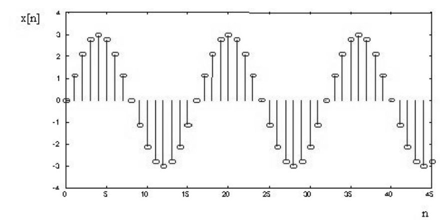


$$\delta[n]=u[n]-u[n-1]$$

- Sequence รูปแบบเฉพาะ

c. Sinusoidal and Exponential sequences

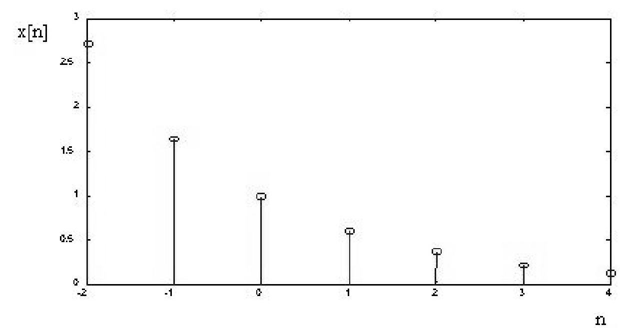
Real sinusoidal sequence



$$x[n] = A\cos(\omega_0 n + \phi), \quad -\infty < n < \infty$$

$$x[n] = x_i[n] + x_q[n] \quad x_i[n] = A\cos(\phi)\cos(\omega_0 n) \quad x_q[n] = -A\sin(\phi)\sin(\omega_0 n)$$

Exponential sequence



$$x[n] = A\alpha^n, \quad -\infty < n < \infty$$

$$\alpha = e^{(\sigma_0 + j\omega_0)}, \quad A = |A|e^{j\phi}$$

$$x[n] = Ae^{(\sigma_0 + j\omega_0)n} = |A|e^{\sigma_0 n}e^{j(\omega_0 n + \phi)}$$

$$x[n] = |A|e^{\sigma_0 n}\cos(\omega_0 n + \phi) + j|A|e^{\sigma_0 n}\sin(\omega_0 n + \phi)$$

- Sequence รูปแบบเฉพาะ

c. Sinusoidal and Expo

Real sinusoidal sequen

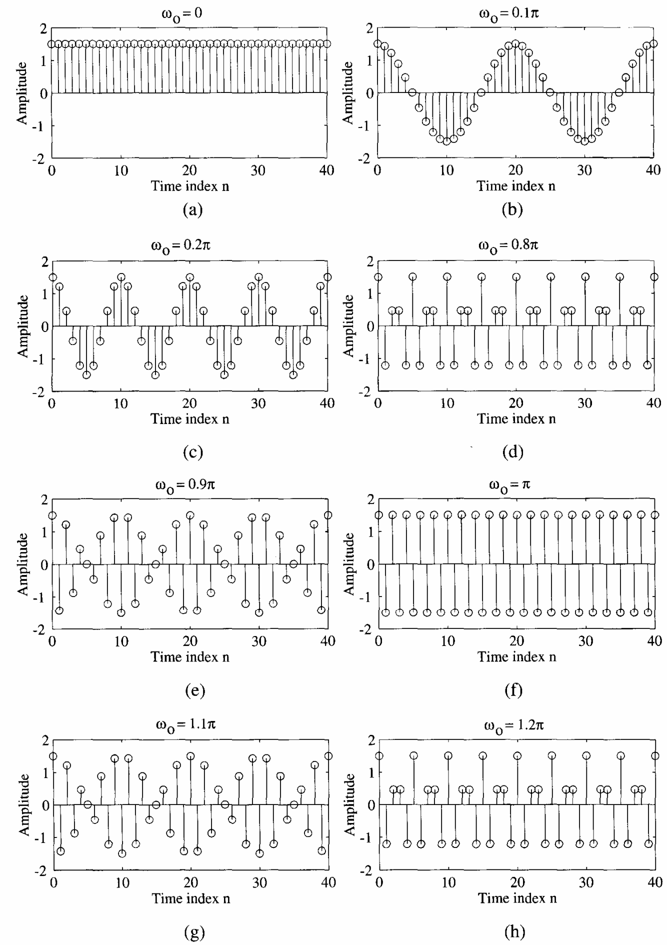


Figure 2.15: A family of sinusoidal sequences given by $x[n] = 1.5\cos\omega_o n$: (a) $\omega_o = 0$, (b) $\omega_o = 0.1\pi$, (c) $\omega_o = 0.2\pi$, (d) $\omega_o = 0.8\pi$, (e) $\omega_o = 0.9\pi$, (f) $\omega_o = \pi$, (g) $\omega_o = 1.1\pi$, and (h) $\omega_o = 1.2\pi$.

- Sequence รูปแ

c. Sinusoidal a

Exponential se

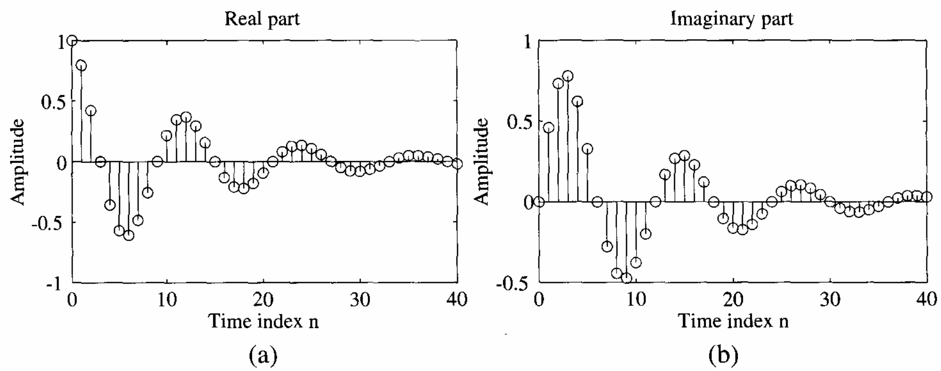


Figure 2.16: A complex exponential sequence $x[n] = e^{(-1/12+j\pi/6)n}$. (a) Real part and (b) imaginary part.

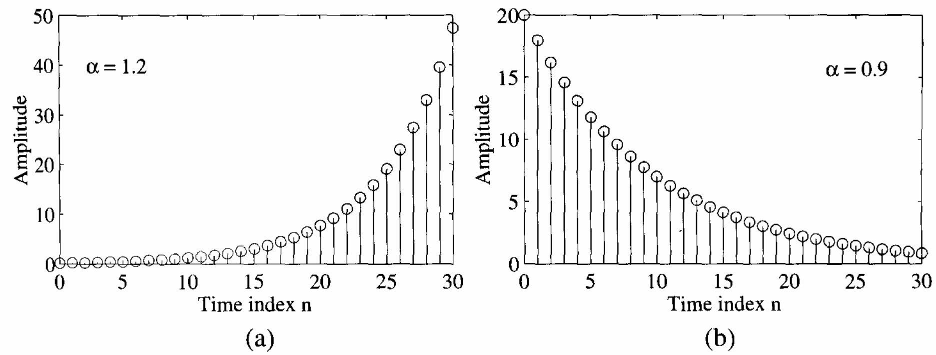


Figure 2.17: Examples of real exponential sequences: (a) $x[n] = 0.2(1.2)^n$, (b) $x[n] = 20(0.9)^n$.

- Sequence รูปแบบเฉพาะ

d. Representation of arbitrary sequence using unit sample sequence

$$x[n] = 0.5\delta[n+2] + 1.5\delta[n-1] - \delta[n-2] + 2\delta[n-4]$$

• Sampling Process

at discrete-time constant t_n

$$t_n = nT = \frac{n}{F_T} = \frac{2\pi n}{\Omega_T}$$

Ex2.12 Determine the discrete-time signal v[n] obtained by uniformly sampling at a sampling rate of 200 Hz a continuous-time signal $v_a(t)$ composed of a weighted Sum of five sinusoidal signals of various frequencies, as given below:

$$v_a(t) = 6\cos(60\pi t) + 3\sin(300\pi t) + 2\cos(340\pi t) + 4\cos(500\pi t) + 10\sin(660\pi t).$$

- Discrete-Time Systems

a. Simple Discrete-Time Systems

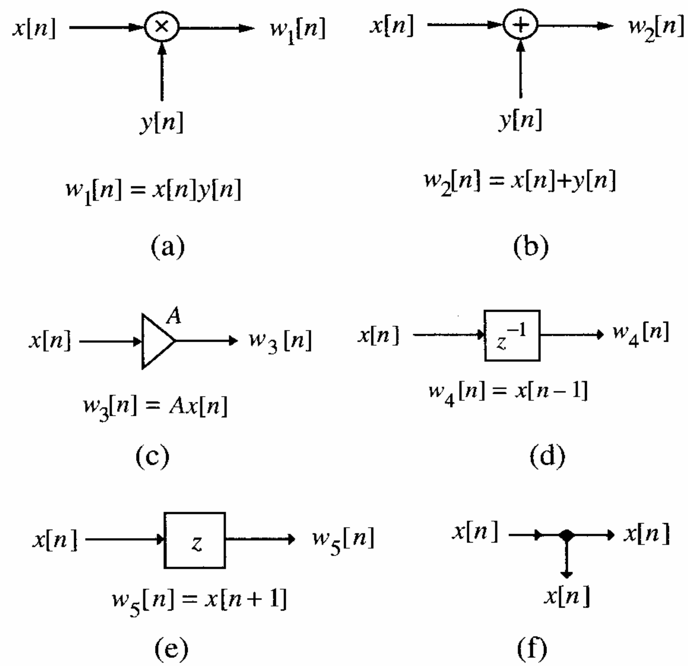
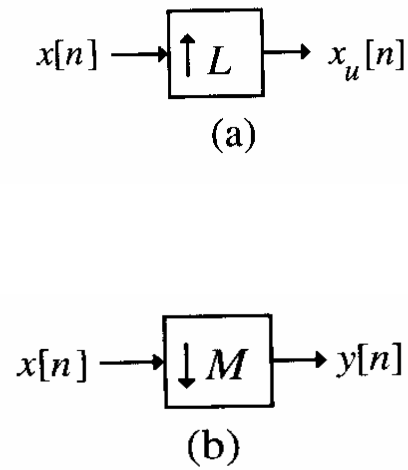


Figure 2.5: Schematic representations of basic operations on sequences: (a) modulator, (2) adder, (c) multiplier, (d) unit delay, (e) unit advance, and (f) pick-off node.



- Discrete-Time Systems

b.Complex Discrete-Time Systems

MA

AR

ARMA

Accumulator

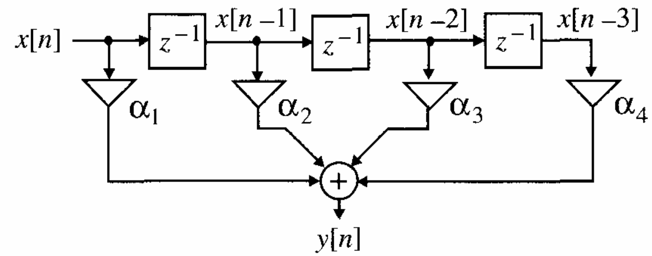


Figure 2.6: Discrete-time system of Example 2.3.

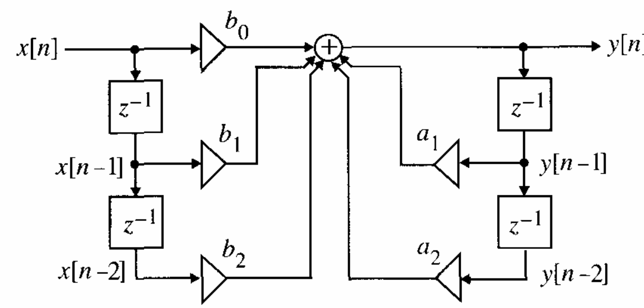


Figure 2.7: Discrete-time system of Example 2.4.

• Discrete-Time Systems

b.Classification of Discrete-Time Systems

b1.Linear System

$x_1[n] \quad produces \quad y_1[n]$

$x_2[n] \quad produces \quad y_2[n]$

$x[n] = \alpha x_1[n] + \beta x_2[n] \quad must\ produce \quad y[n] = \alpha y_1[n] + \beta y_2[n]$

• Discrete-Time Systems

b.Classification of Discrete-Time Systems

b2.Time Invariant System

$$x_1[n] \quad produces \quad y_1[n]$$

$$x[n] = x_1[n-n_0] \quad must\ produce \quad y[n] = y_1[n-n_0]$$

• Discrete-Time Systems

b.Classification of Discrete-Time Systems

b3.Linear Time Invariant (LTI) System

รวมคุณสมบัติของทั้ง linear และ Time invariant

$x_1[n] \quad produces \quad y_1[n]$

$x_2[n] \quad produces \quad y_2[n]$

$x[n] = \alpha x_1[n-n_0] + \beta x_2[n-m_0] \quad must\ produce$

$$y[n] = \alpha y_1[n-n_0] + \beta y_2[n-m_0]$$

• Discrete-Time Systems

b.Classification of Discrete-Time Systems

b4.Causal System

ผลตอบของแต่ละจุดตัวอย่าง จะขึ้นกับขาเข้าก่อนหน้าหรือที่เวลานั้นเท่านั้น

causal $y[n] = \alpha_1 x[n] + \alpha_2 x[n-1] + \alpha_3 \delta[n-2]$

non-causal $y[n] = \alpha_1 x[n] + \alpha_2 x[n+1] + \alpha_3 \delta[n-1]$

• Discrete-Time Systems

b.Classification of Discrete-Time Systems

b5.Stable System

ทุกๆ bounded input จะให้ bounded output

$$\left|x[n]\right| < B_x \quad \Rightarrow \quad \left|y[n]\right| < B_y$$

• Discrete-Time Systems

b.Classification of Discrete-Time Systems

b6.Passive and Lossless Systems

Passive $$\sum_{n=-\infty}^{\infty} |y[n]|^2 \leq \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

Lossless $$\sum_{n=-\infty}^{\infty} |y[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

- Time-Domain Characterization of LTI system

a.Input-Output relationship

$$y[n] = x[n] * h[n]$$

เมื่อ h[n] เป็นผลตอบอิมพัลส์ของ LTI system

- Time-Domain Cha stem

a.Input-Output relat

$$y[n] = x[n] * h[n]$$

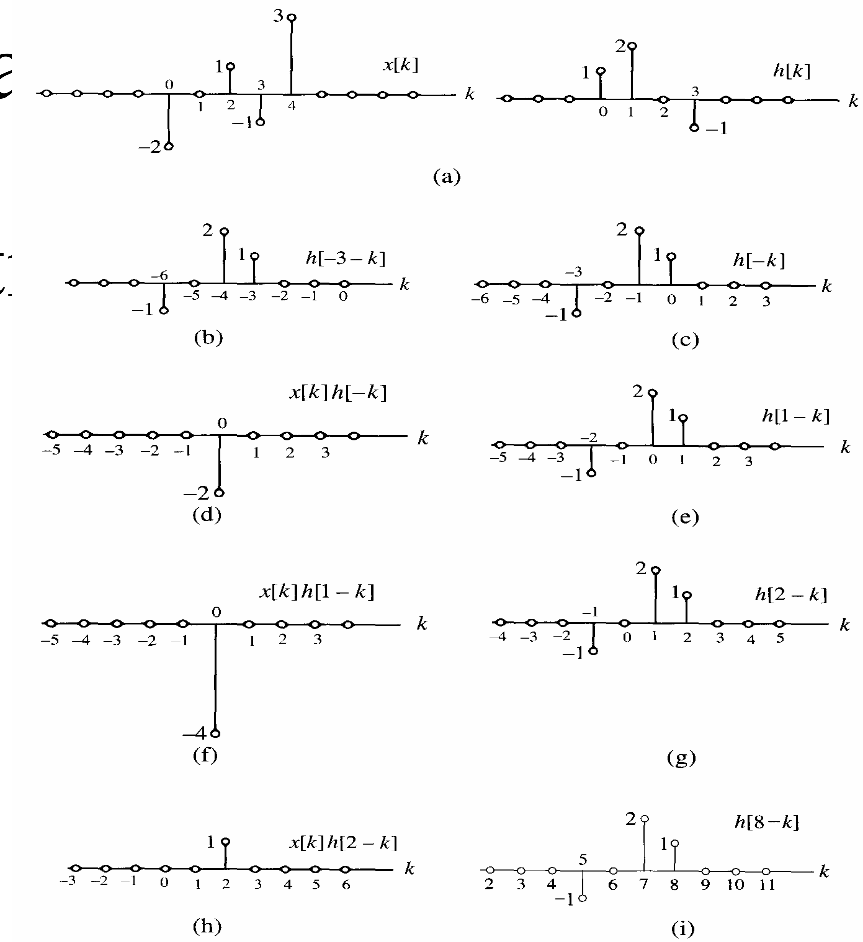


Figure 2.25: Illustration of the convolution process.

• Time-Domain Cha stem

a.Input-Output relat

$$y[n] = x[n] * h[n]$$

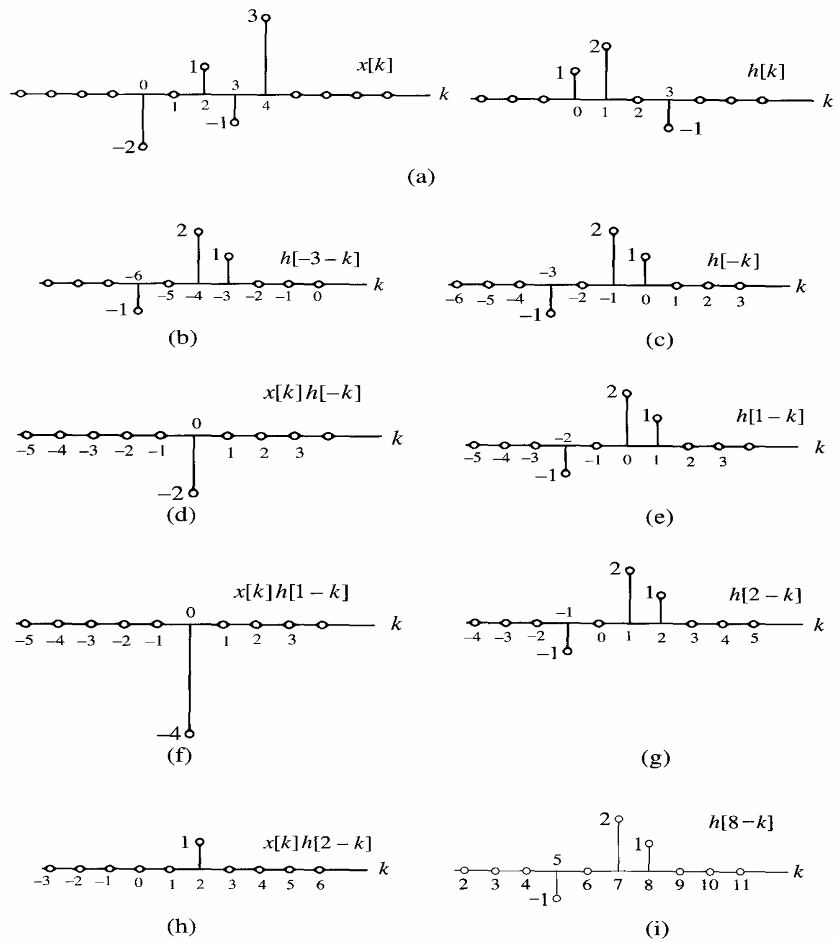


Figure 2.25: Illustration of the convolution process.

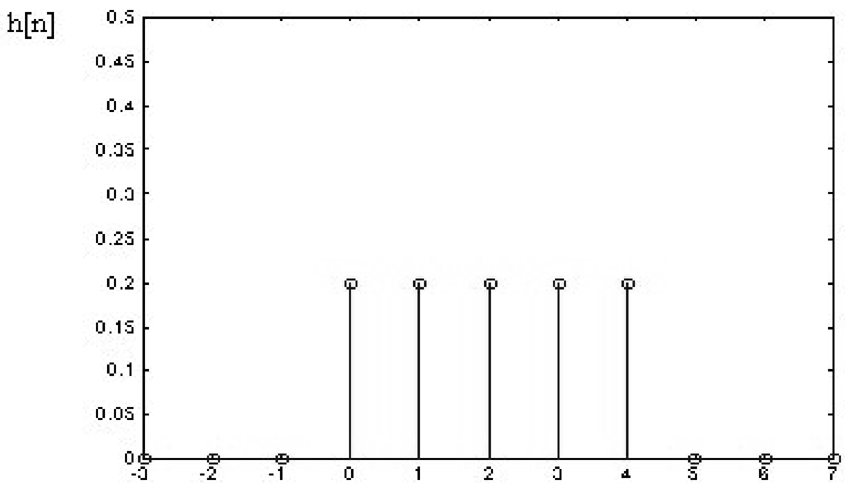
- Time-Domain Characterization of LTI system

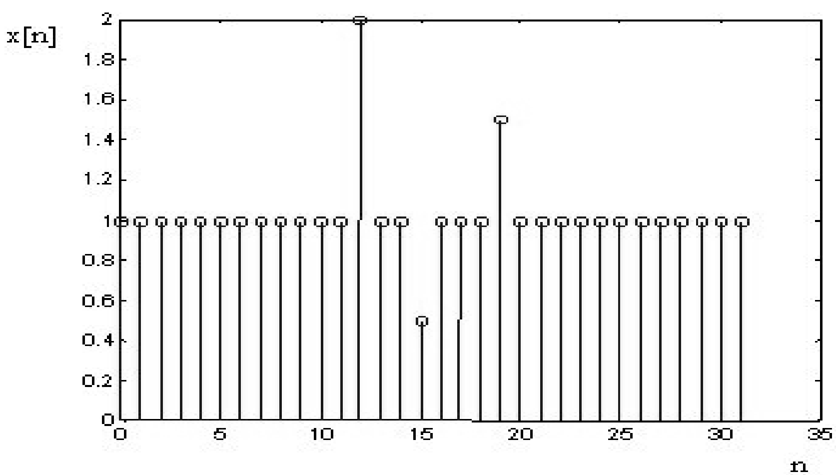
a.Input-Output relationship

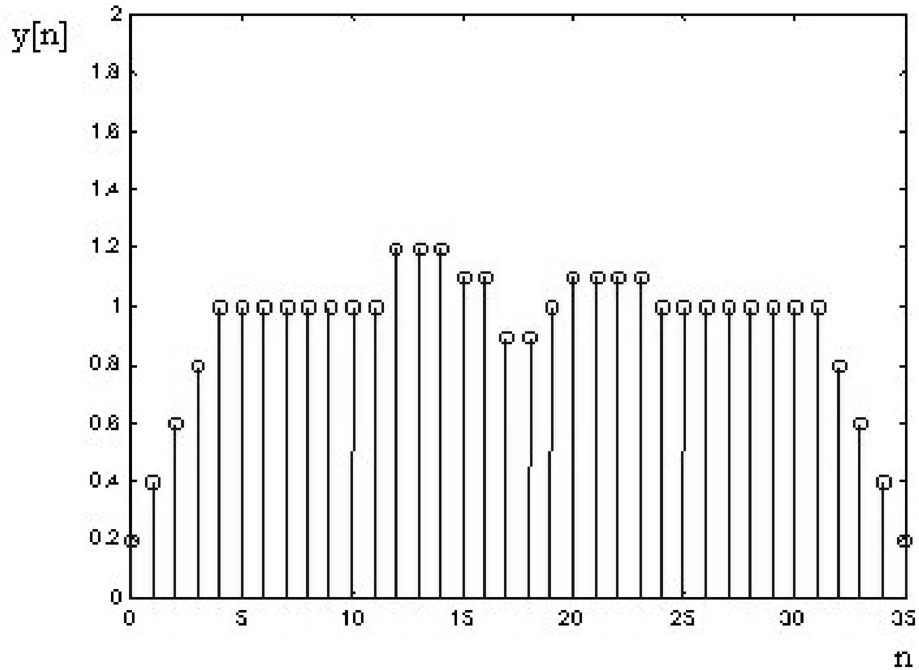
$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]n[n-k]$$

x[n]:			2	1	−2				
h[−n]:	−1	2	1						y[0] = 2
h[1−n]:		−1	2	1					y[1] = 5
h[2−n]:			−1	2	1				y[2] = −2
h[3−n]:				−1	2	1			y[3] = −5
h[4−n]:					−1	2	1		y[4] = 2
h[5−n]:						−1	2	1	y[5] = 0

Discrete Time Signals and Systems in Time-Domain 2







• Time-Domain Characterization of LTI system

a.การเชื่อมต่อระบบ

Cascade

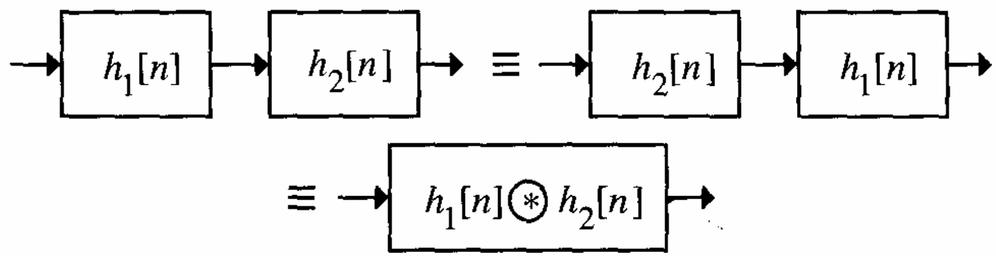


Figure 2.28: The cascade connection.

Parallel

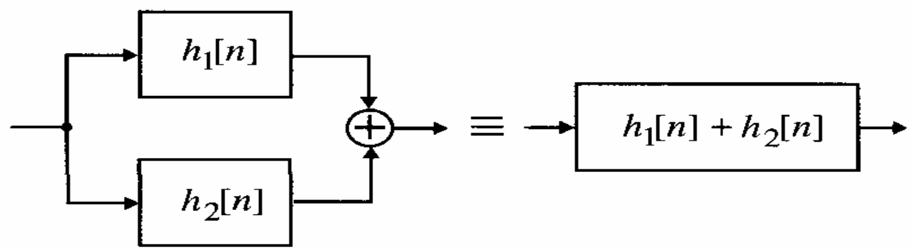


Figure 2.29: The parallel connection.

• Time-Domain Characterization of LTI system

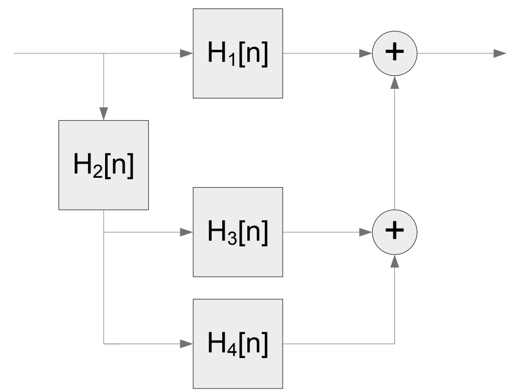
a. การเชื่อมต่อระบบ

Ex 2.27 Consider the discrete-time system composed of 4 discrete-time systems with impulse responses given by

$$h_1[n] = \delta[n] + \frac{1}{2}\delta[n-1], \qquad h_2[n] = \frac{1}{2}\delta[n] - \frac{1}{4}\delta[n-1],$$

$$h_3[n] = 2\delta[n], \qquad h_4[n] = -2\left(\frac{1}{2}\right)^n u[n].$$

Find the overall impulse response h[n].



3.1 Discrete Time Fourier Transform – DTFT

- Fourier Transform ของ Discrete-Time sequence $x[n]$
- เป็นการแสดง sequence ในรูปแบบของ complex exponential sequence $\{e^{-j\omega n}\}$,

  ω เป็นตัวแปรความถี่ค่าจริง

<u>Discrete-time Fourier Transform</u> DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

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Definition

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$= X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega})$$

$$= |X(e^{j\omega})| \; e^{j\theta(\omega)}$$

เมื่อ $\theta(\omega) = \arg\{X(e^{j\omega})\}$

$|\cdot|$ = magnitude function (spectrum)

$e^{j\theta(\omega)}$ = phase function (spectrum)

$-\pi < \theta(\omega) < \pi$ เป็น principal value

Ex 3.1 For $\delta[n]$

$$\Delta\left(e^{j\omega}\right) = \sum_{n=\infty}^{\infty} \delta[n]\, e^{(-j\omega n)} = 1$$

$$\sum_{n=-\infty}^{\infty} \delta[n-k]\, f[n] = f[k]$$

Ex 3.2

$$x[n] = \alpha^{n} u[n] \qquad : |\alpha| < 1$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \alpha^{n} u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \alpha^{n} e^{-j\omega n}$$

$$= \sum_{\alpha=-\infty}^{\infty} \left(\alpha e^{-j\omega}\right)^{n}$$

$$= \frac{1}{1-\alpha e^{-j\omega}}$$

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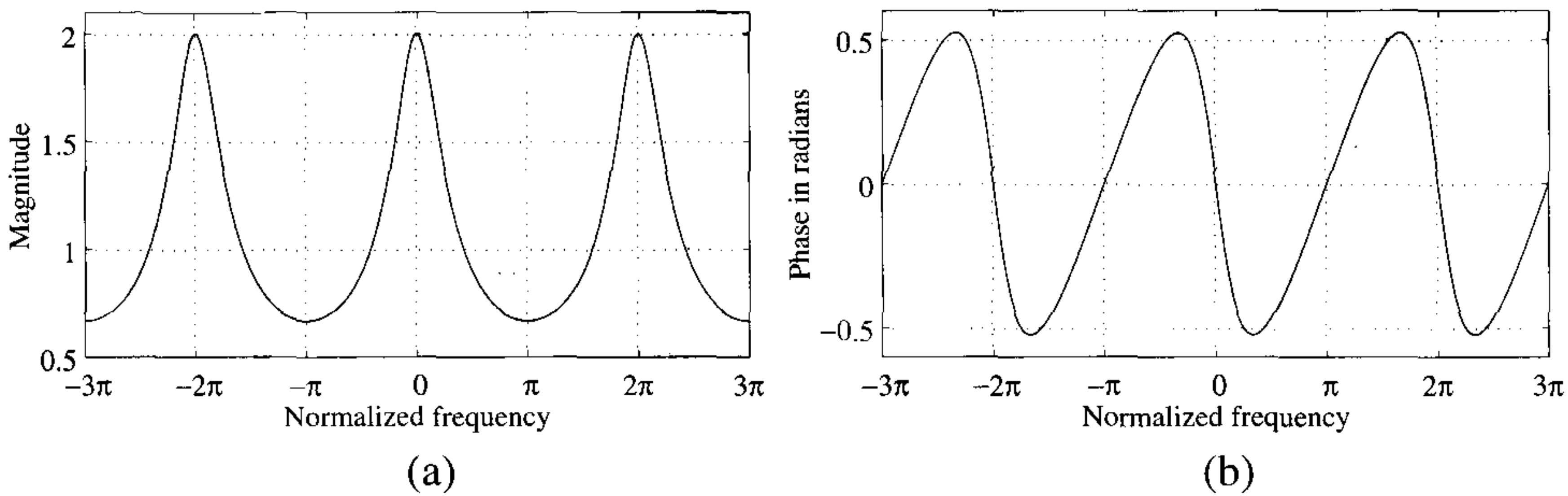


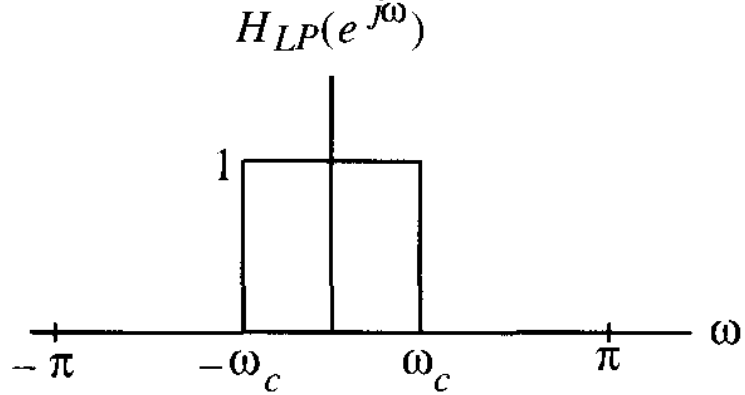
Figure 3.1: Magnitude and phase of $X(e^{j\omega}) = 1/(1 - 0.5e^{-j\omega})$.

<u>Inverse Discrete-time Fourier Transform</u> **IDTFT**

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{-j\omega}\right) e^{j\omega n}\, d\omega$$

Ex 3.3 Consider the DTFT

$$H_{LP}(e^{i\omega}) = \begin{cases} 1, & o \le |\omega| \le \omega_c \\ 0, & \omega_c \le |\omega| \le \pi \end{cases}$$



$$H_M(e^{j\omega}) = \sum_{n=-M}^{M} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

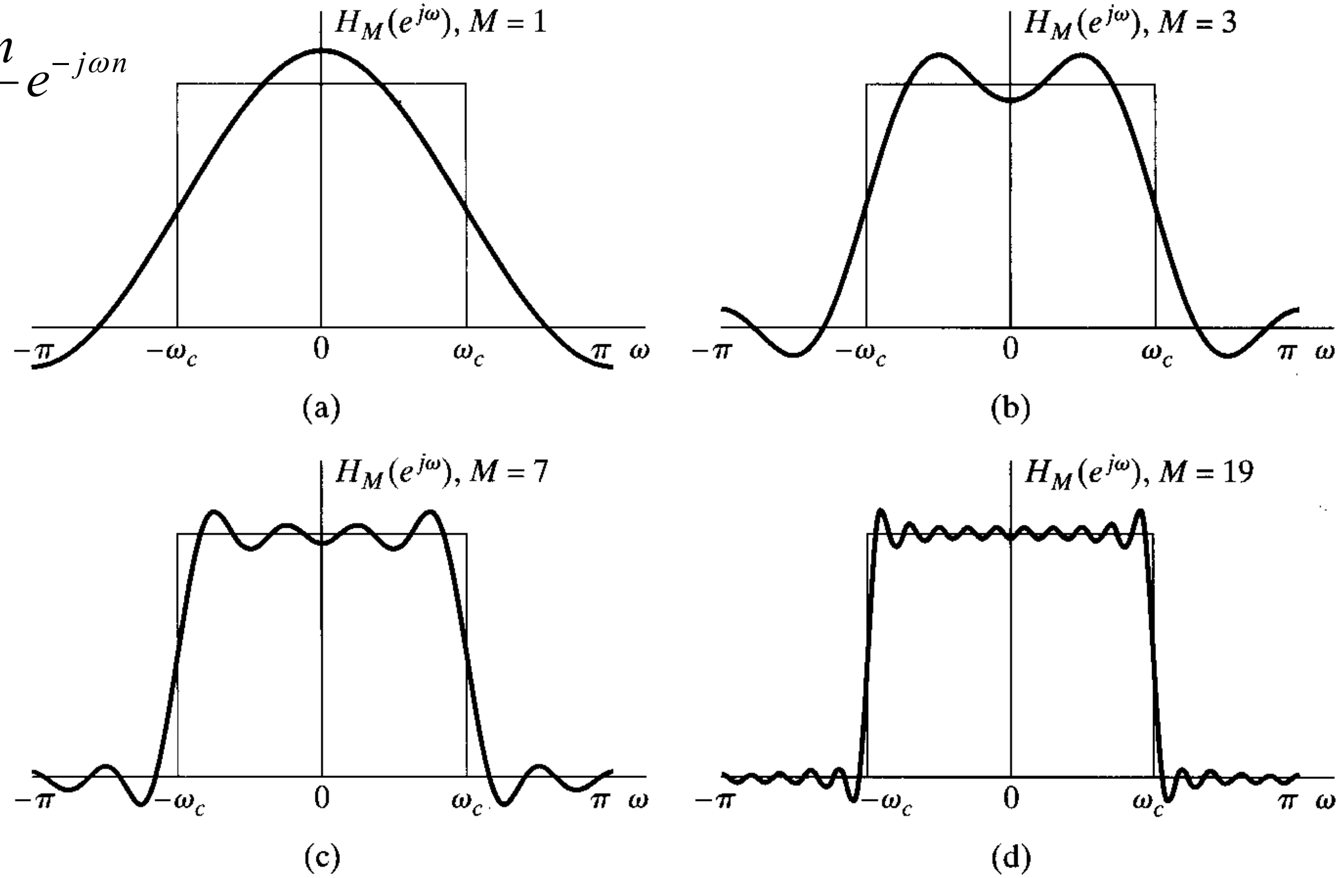


Figure 2.21 Convergence of the Fourier transform. The oscillatory behavior at $\omega = \omega_c$ is often called the Gibbs phenomenon.

Band-limited Signals

- **full-band discrete-time signal** $0 \le |\omega| \le \pi$
- **bandlimited signal**

lowpass discrete-time signal $0 \le |\omega| \le \omega_p \le \pi$

Bandwidth BW=ω_p

bandpass discrete-time signal $0 \le \omega_L \le |\omega| \le \omega_H \le \pi$

Bandwidth BW=$\omega_H - \omega_L$

DTFT Properties

TABLE 2.1 SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\mathcal{R}e\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)
4. $j\mathcal{I}m\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)	$X_R(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)	$jX_I(e^{j\omega}) = j\mathcal{I}m\{X(e^{j\omega})\}$
The following properties apply only when $x[n]$ is real:	
7. Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8. Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9. Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
10. Any real $x[n]$	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ (magnitude is even)
11. Any real $x[n]$	$\sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega})$ (phase is odd)
12. $x_e[n]$ (even part of $x[n]$)	$X_R(e^{j\omega})$
13. $x_o[n]$ (odd part of $x[n]$)	$jX_I(e^{j\omega})$

DTFT Properties

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence $x[n]$ $y[n]$	Fourier Transform $X(e^{j\omega})$ $Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_d]$ (n_d an integer)	$e^{-j\omega n_d} X(e^{j\omega})$
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Parseval's theorem:

8. $$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

9. $$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$$

DTFT Properties

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
4. $a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{1 - ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
6. $(n+1)a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{(1 - ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\|r\| < 1)$	$\frac{1}{1 - 2r\cos\omega_p e^{-j\omega} + r^2 e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \|\omega\| < \omega_c, \\ 0, & \omega_c < \|\omega\| \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\omega} 2\pi\delta(\omega - \omega_0 + 2\pi k)$
11. $\cos(\omega_0 n + \phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega - \omega_0 + 2\pi k) + \pi e^{-j\phi}\delta(\omega + \omega_0 + 2\pi k)]$

Ex 3.5 Determine the DTFT of the sequence $y[n] = (n+1)\alpha^n u[n] \qquad |\alpha| < 1$

Ex 3.6 Determine the DTFT $V(e^{j\omega})$ of the sequence v[n] given by

$$d_0 v[n] + d_1 v[n-1] = p_0 \delta[n] + p_1 \delta[n-1] \qquad |d_1 / d_0| < 1$$

Ex 3.7 Determine the impulse response of stable LTI system which input and output satisfy the constant-coefficient difference equation

$$y[n]-\frac{1}{2}y[n-1]=x[n]-\frac{1}{4}x[n-1]$$

Ex 3.8 Compute the energy of the sequence $h_{LP}[n]$ of Ex 3.3

$$H_{LP}(e^{i\omega}) = \begin{cases} 1, & o \le |\omega| \le \omega_c \\ 0, & \omega_c \le |\omega| \le \pi \end{cases}$$

3.2 Discrete Fourier Transform – DFT (Introduction)

- เฉพาะกรณีของ finite-length sequence x[n], 0 <= n <= N-1
- N ค่าของ $X(e^{j\omega})$ เรียกว่า frequency samples ที่จุดความถี่ต่างๆ กัน
  $\omega = \omega_k, \quad 0 \le k \le N-1$
- N ค่าของ $X(e^{j\omega})$ เพียงพอที่จะหา x[n] ได้

Discrete Fourier Transform **DFT**

$$X[K] = \sum_{n=0}^{N-1} x[n] w_N^{kn} \qquad 0 \le k \le N-1$$