Journal of Banking & Finance 26 (2002) 1253–1272

Journal of BANKING & FINANCE

www.elsevier.com/locate/econbase

Measures of risk

Giorgio Szegö *

Università di Roma "La Sapienza", Roma, Italy

Abstract

The conditions under which the classical measures of risk like the mean, the linear correlation coefficient and VaR can be used are discussed. The definition of risk measure and the main recently proposed risk measures are presented. The problems connected with co-dependence are outlined.

JEL classification: G110; G120

Keywords: Risk measures; Scalar co-dependence measures; Conditional value-at risk; Expected shortfall; Spectral risk measures and acceptable risk weights

1. Origins of risk measures

In the pre-Markowitz era financial risk was considered as a correcting factor of expected return, and risk-adjusted returns were defined on an ad-hoc basis. These primitive measures had the advantage of allowing an immediate preferential order of all investments.

Markowitz [1] proposed to measure the risk associated to the return of each investment by means of the deviation from the mean of the return distribution, the variance, and in the case of a combination (portfolio) of assets, to gauge the risk level via the covariance between all pairs of investments, i.e.:

$$\mathrm{Cov}[X, Y] = E[X, Y] - E[X]E[Y],$$

where X and Y are random returns. The main innovation introduced by Markowitz is to measure the risk of a portfolio via the joint (multivariate) distribution of returns of all assets. Multivariate distributions are characterized by the statistical (marginal)

* Tel.: +06-8632-4395; fax: +06-8621-4805.

E-mail address: e.jbf@tiscalinet.it (G. Szegö).

[1] See Szegö (1980) for a complete mathematical presentation.

0378-4266/02/$ - see front matter © 2002 Published by Elsevier Science B.V.

PII: S0378-4266(02)00262-5

properties of all component random variables and by their dependence structure. Markowitz described the former by the first two moments of the univariate distributions – the asset returns – and the latter via the linear (Pearson) correlation coefficient between each pair of random returns, i.e.:

$$\rho(X,Y) = \mathrm{Cov}[X,Y]/(\sigma_X^2\sigma_Y^2)^{1/2}$$

where σ_X and σ_Y denote the standard deviations of the univariate random variables X and Y, respectively. Note that a measure of dispersion can be adopted as a measure of risk only if the relevant distribution is symmetric.

The correlation coefficient, while allows to fully describe a multivariate distribution by taking into account the dependence structure among all pairs of components only, is strictly related to the slope parameter of a linear regression of the random variable Y on the random variable X, and it measures only the co-dependence between the linear components of X and Y. [2] Indeed,

$$\rho(X,Y)^2 = \sigma_Y^2 - \min E[(Y-(aX+b))^2]/\sigma_Y^2,$$

that is the relative variation of σ_Y^2 by linear regression on X.

It can be proved that for all vectors z and random vectors X, the variance of the linear combination $z^{\mathrm{T}}X$, satisfies the relationship

$$\sigma^2(z^{\mathrm{T}}X) = z^{\mathrm{T}}\mathrm{Cov}(X)z$$

which is essential in Markowitz portfolio theory. Linear correlation co-dependence measure is indeed very intuitive and appealing in its simplicity.

We must recall that Markowitz model goes hand in hand with appropriate utility functions, which allows a subjective preference ordering of assets and their combinations. In the case of non-normal, albeit symmetric, distributions the utility functions must be quadratic. In practice this limitation restricts the use of this model to portfolios characterized by normal joint return distribution, i.e. to the case in which the returns of all assets as well as their dependence structure is normal.

Recently the class of random variables for which linear correlation can be used as an dependence measure has been fully identified. [3] This is the class of *elliptic* [4] distributions characterized by the property that their equi-density surfaces are ellipsoids. Thus Markowitz model is suited only to the case of elliptic distributions, like normal or t-distributions with finite variances. Note that symmetric distributions are not necessarily elliptic.

The linear correlation coefficient, if used in the case of non-elliptic distributions, may lead to incorrect results. [5] The concept of "incorrect" must however be specified, since it requires an agreement on a "correct" dependence measure.

[2] See, for instance, Alexander (2001, pp. 7–19).

[3] See Cambanis et al. (1981).

[4] See Joe (1997).

[5] See, for instance, Embrechts et al. (1999), that compare bivariate normal distributions and Gumbel distributions with the same linear correlation coefficient.

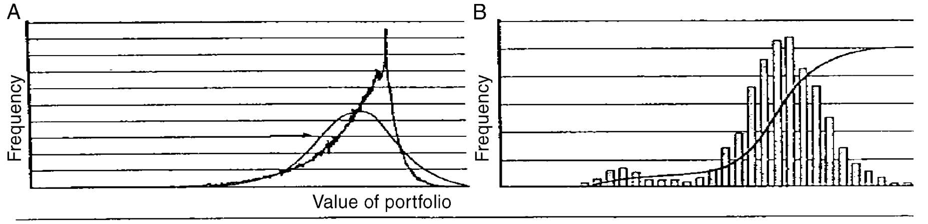

Fig. 1. A. Leptokurtic distribution related to the value of a loan portfolio. B. The distribution of the market value of the credit of an emerging country: Columbia, showing an important occurrence of extreme events.

In absence of such measure (see Section 6) one can compare numerical results achieved via simulations. [6] This comparison is not trivial. Indeed these results have shown that in comparing two different distributions (one normal and one Gumbel) with the same linear correlation coefficient, in the vast majority of cases the two results agree. However most of the points of disagreement [7] lay in the upper right corner of the distribution, corresponding to extreme losses. We can say that if one uses a variance–covariance model for non-elliptic distributions one can severely underestimate extreme events that cause the most severe losses.

In the 1960s the concept of β (volatility) was introduced. This development was motivated by computational reasons. The complexity of the mean–variance approach was considered too high. After almost 40 years, and the gigantic progress in computers, this is no longer the case. The second motivation for the introduction of the β-based portfolio methods was the insufficient data to compute the variance–covariance matrix (the number of data should be at least twice the number of assets). Now bootstrapping techniques allow to circumvent this problem and βs are almost abandoned in portfolio management in favor of complete variance–covariance models.

The measure of the linear dependence between the return of each security and that of the market, β, led to the development of the main pricing models, CAPM, and APT. These models, while extendible to heavy-tailed distributions, [8] have been developed in a "normal world" and lead to misleading results, when applied to every day life situations [9] like the ones shown in Fig. 1.

For instance, the cumulative distribution of not marketed loans is totally asymmetric, and in particular leptokurtic (Fig. 1A), and the distribution of returns of some traded debts of developing countries may contain extreme values (Fig. 1B).

It is unfortunate that the precisely formulated Markowitz model has become a "solution in search of a problem" and incorrectly applied to many cases in which

[6] This is the technique used by Embrechts et al. (1999).

[7] In the simulations performed by Embrechts et al. (1999), 0.8% of the cases.

[8] See, for instance Huschens and Kim (2000), Silvapulle and Granger (2001).

[9] Both Mandelbrot as well as Fama stipulated that the return of most assets can be well approximated by a t-distribution with infinite variance.

risk cannot be described by variance, dependence cannot be measured by linear correlation coefficient, and utility function does not even dream to be quadratic. [10]

2. New research development

Multivariate normal distribution-based models are very appealing, because the association between any two random variables can be fully described by their marginal distributions and the linear correlation coefficient. It is evident that these models are only a very initial step towards more realistic ones, better tuned to grasp real life situations, i.e. the case in which investment return cumulative distributions of individual assets are skewed, leptokurtic and/or heavy tailed. The introduction of these model have been hampered by the lack of a suitable theoretical framework.

Probabilistic models for univariate returns have been investigated and extended to the multivariate case under the assumption that all the combined returns and their dependence structure have the same probabilistic structure. This severe drawback can be overcome with the use of copula functions (see Section 7) suitable to the analysis multivariate distributions with almost arbitrary univariate components and dependence structure. Only recently [11] the problem of the study of extreme events, i.e. of the tails of the distribution has received due attention. Most of the research on new risk measures has been stimulated by "dependent extreme events". [12]

In the last five years, namely from 1997 [13] there has been a great momentum in research on this subject, which has touched five different, but interconnected aspects:

- definition of risk measure;
- construction of (coherent) risk measures;
- rationality of insurance premia;
- "good deals";
- generalized hyperbolic Lévy processes;
- copulas for the study of dependence in multivariate distributions.

The first line of research was started by an international group of scholars: Philippe Artzner, Freddy Delbaen, Jean-Marc Eber, and David Heath. [14] Their results will be presented in the next section of this paper.

[10] I could quote a large number such examples, but... I have already a sufficient number of enemies!

[11] See Embrechts et al. (1997), Reiss and Thomas (2001). See also the following sections.

[12] Essentially catastrophic events unexpectedly connected. The most typical example has been the increase of spread among sovereigns of countries due to join the EURO in January 1999, following the Russian crisis of August 1998.

[13] Year in which the first results by Artzner, Delbaen, Eber, and Heath, on coherent risk measures were published. In the same year Wang, Young, and Panjer published their work on the axiomatic characterization of insurance prices.

[14] The results on coherent risk measures were first published by Philippe Artzner, Freddy Delbaen, Jean-Marc Eber, and David Heath, in 1996 with the title *A Characterization of Measures of Risk,* as Preprint 1996/14 of the Department of Mathematics, ULP Strasbourg, followed by a short paper entitled *Thinking coherently*, published in Risk, in 1997. See also Artzner et al. (2000, 2001). The definitive version was published in Mathematical Finance in 1999. A complete presentation of the state of the art of this theory is contained in the paper by Frittelli and Rosazza Gianin (2002).

Independently, practically at the same time, Shuan Wang, Virginia Young, and Harry Panjer, [15] presented similar conclusions on a closed related subject related to insurance premia.

Again, independently from the other groups, but practically at the same time, Stewart Hodges developed the theory of "good deals". [16]

The hyperbolic distributions were introduced in finance by Eberlein and Keller in 1995. They provide a very precise fit to distributions of daily prices of equities. This research is related to that on generalized Lévy motions. [17]

The last line of research is centered on the application of a concept developed in the middle 1970s in the study of multivariate distributions, [18] the copula, to the investigation of dependent tail events, i.e. of the possibility of simultaneous occurrence of different uncommon events. In this case the level of the possible losses could be very large. Neither linear correlation nor other more recent dependence measures [19] can fully describe these events. For this reason, many researchers [20] have applied the theory of copulas to the analysis of the general dependence structure among random variables. We shall present some results in Section 7 of this paper.

The recent research on new risk measures has been possibly enhanced by the new trends in regulation of financial institutions that require the use of very sophisticated risk control models and by the reaction of the academic community to the attempt of regulators to impose incorrect and misleading risk measures. [21]

In 1994 the concept of Value at Risk, VaR, was introduced with drums and cymbals [22] with the precise task of answering to the following very relevant and precise question: how much one can expect to lose in one day, week, year, … with a given probability? What is the percentage of the value of the investment that is at risk?

For a given time horizon and a probability level k, $0 < k < 1$, VaR_k, is simply the loss that is exceeded over this specified period with probability $1 - k$, i.e. VaR_k is the maximum loss in a specified period with probability level k (Fig. 2).

[15] The results on the *Axiomatic characterization of insurance prices* were published by Wang et al. (1997) in Insurance: Mathematics and Economics. The most recent results in this line of research are due to Landsman and Sherris (2001) in Insurance: Mathematics and Economics.

[16] These are (net) trades X with no cash flows at time 0 and a random flow at time 1, which are acceptable, i.e. such that $\rho(X) \leqslant 0$, where ρ is a weakly coherent risk measure and in addition $E_P(X) \geqslant 0$ hold for all valuations (probability measures) P. This definition is based on the concept of weakly coherent risk measures that will be presented in the next section. The first results on good deals are contained in Hodges (1998). See also Černý and Hodges (2000). A recent contribution on the same wavelength due to Carr et al. (2001) seems to be uninformed about the work of Hodges.

[17] See Geman (2002). See also Eberlein (2001a,b) and Eberlein and Keller (1995).

[18] See Sklar (1973). See also Joe and Hu (1996), Joe (1997) and Lee (1993).

[19] Like, for instance, the Kendall tau and the Spearman rho, see Nelsen (1999).

[20] See, for instance, Frey and McNeil (2001, 2002), where additional references can be found.

[21] This is the case of VaR for which Daníelsson et al. (2001) expressed the view that "VaR can destabilize an economy and induce crashes when they would not otherwise occur".

[22] JP Morgan (1994, 1997), see also Phelan (1997), Crouhy et al. (2000, 2001), Cumperayot et al. (2001), Caouette et al. (1998), Jorion (2000) and Manganelli and Engle (2001).

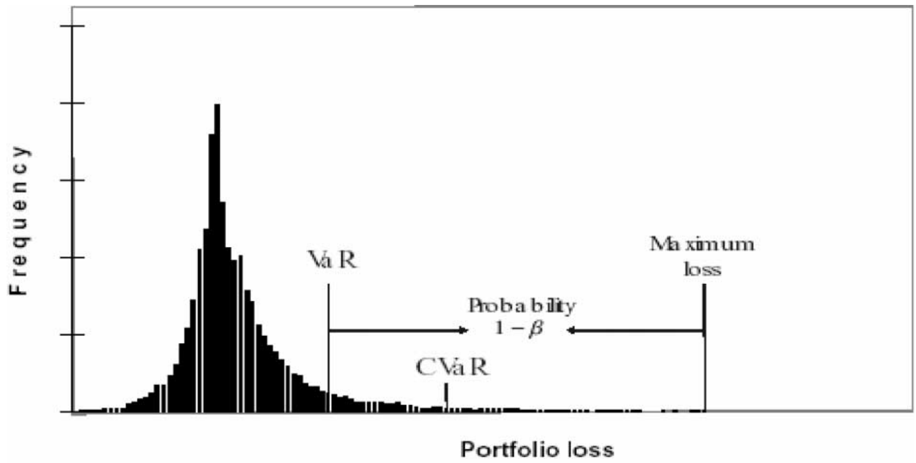

Fig. 2. VaR and CVaR.

The exact definition of the VaR_k of a random variable X is based on the k-quantile, taken with a negative sign of the distribution function F_X, i.e.

$$\mathrm{VaR}_k = -F_X^{-1}(k).$$

F_X^{-1} denotes the inverse of the distribution function F_X. [23]

While VaR sounds like a great idea, when the distribution is multimodal, for some values of k, VaR_k is not even defined. Indeed in this case the inverse of $F_X(k)$ does not exist and the inverse image of $F_X(k)$ is not even connected. [24] In order to overcome this difficulty, VaR is defined [25] as the lowest number belonging to the set $F_X^{-1}(k)$, or as the k-quantile of the generalized inverse of F_X, i.e.

$$\mathrm{VaR}_k = \inf\{-F_X^{-1}(k)\}.$$

Taking this caveat in mind, VaR could be used, albeit only in the case of the original question. Unfortunately, just like Markowitz approach, with the explicit encouragement of regulators, VaR has become another "solution in search of a problem" and was wrongly adopted as a risk measure. In particular the major inspiring principles of the 2001 proposal of the Basel Banking Supervisory Committee are: [26]

- VaR is assumed as risk measure;
- the risk of each loan must be portfolio invariant, i.e. must be measured by its own characteristic only, not taking into account those of portfolio in which the loan is held;

[23] In case in which there does not exist a unique inverse a slightly modified concept can be used. See Embrechts et al. (1997).

[24] In Fig. 1B there exist a range of values for which this inverse image is composed by three points.

[25] See Rockafellar and Uryasev (2002).

[26] See Gordy (2000a,b, 2002). See Basel Committee on Banking Supervision (1999, 2001). See also Daníelsson (2002), Daníelsson et al. (2002), Daníelsson and de Vries (1998), Daníelsson et al. (1998a,b) and Jones (2000).

- the regulatory capital for a loan must be correlated to its marginal contribution to VaR.

3. How to measure risk

To measure risk is equivalent to establishing a correspondence ρ between the space X of random variables (for instance the returns of a given set of investments) and a non-negative real number, i.e. $\rho : X \to R$. Scalar measures of risk allow to order and to compare investments according to their respective risk value. These correspondences cannot be without restrictions (in this case they would not have any property) that can take the form of binding conditions. [27] Any risk measure lacking such properties may lead to inconsistencies.

In order to better understand the role of proper conditions that need to be satisfied by a scalar risk measure, we recall the three conditions that any functional $\rho : X \to R$, defining the distance between two points in the space X must satisfy:

- the distance between a point and itself is zero;
- the distance does not change by inverting the two points;
- given three points, the distance between any pair cannot be larger than the sum of the distances between the other two pairs.

Any functional that satisfy these conditions is a measure of distance. Note these restrictions do not define a precise measure, but only the class of possible measures.

In the case of risk measures, analogous conditions have been proposed by various scholars. [28] In our presentation we shall follow these advanced by Artzner, Delbaen, Ebner, and Heath, [29] in their more precise formalization due to Frittelli and Rosazza Gianin. [30]

Any acceptable risk measure $\rho : X \to R$ must satisfy the following properties:

(a) Positive homogeneity: $\rho(\lambda x) = \lambda \rho(x)$ for all random variables x and all positive real numbers λ.
(b) Subadditivity: $\rho(x + y) \leqslant \rho(x) + \rho(y)$ for all random variables x and y;
It can be proved that any positively homogeneous functional ρ, is convex if and only it is subadditive.

If, in addition, the following two properties are satisfied:

[27] Readers must be reminded that in many circumstances it is necessary to impose restrictions in order to obtain meaningful definitions, like in the case of the definition of a distance between two points or of a dynamical system!

[28] Albanese (1997), Frittelli (2000), Artzner et al. (1997, 1999), Carr et al. (2001). A parallel line of research has been developed in order to solve the problem of risk pricing in insurance. See Landsman and Sherris (2001).

[29] Artzner et al. (1997, 1999).

[30] Frittelli and Rosazza Gianin (2002).

(c) Monotonicity: $x \leqslant y$ implies $\rho(x) \leqslant \rho(y)$ for all random variables x and y;
(d) Transitional invariance: $\rho(x + \alpha r_0) = \rho(x) - \alpha$ for all random variables x and real numbers α, and all riskless rates r_0,

then ρ is a (coherent) risk measure. Indeed the word coherent is redundant: Any measure of risk must satisfy these conditions!

Let us make some comments on the economic significance of these conditions.

Subadditivity, if ρ would not be subadditive, then $\rho(x) + \rho(y) < \rho(x+y)$, this would imply, for instance, that in order to decrease risk, it could be convenient to split up a company into different distinct divisions. From the regulatory point of view this would allow to reduce capital requirements. Note that covariance is subadditive, and this property turned out to be essential in Markowitz portfolio theory: Indeed no new investment increases risk.

Note that the subadditivity requirement implies that $\rho(\lambda y) \leqslant \lambda\rho(x)$, thus positive homogeneity implies that $\rho(\lambda y) \geqslant \lambda\rho(x)$, which combined with the previous inequality leads to the equality sign. The latter inequality can be justified by liquidity considerations [31]: an investment (λx) could be less liquid, and therefore more risky, that the total λx of λ smaller investments x.

Transitional invariance implies that by adding a sure return αr_0 to a random return x the risk $\rho(x)$ decreases by α.

Finally note that monotonicity rules out any semi-variance type of risk measure.

Some authors have replaced the first two conditions of coherence with the condition that ρ be convex, i.e. that

$$\rho(\lambda X + (1-\lambda)Y) \leqslant \lambda\rho(X) + (1-\lambda)\rho(Y), \quad 0 \leqslant \lambda \leqslant 1.$$

Note that since convexity does not necessarily imply positive homogeneity, a risk measure that is only transitionally invariant, monotonous and convex has weaker properties than coherent measures and it can be called *weakly coherent*. [32]

4. VaR does not measure risk

VaR in general turns out to be not even weakly coherent and in particular not subadditive. To try to measure risk without this property is like measuring the distance between two points using a rubber band instead of a ruler! Only in the special case in which the joint distribution of return is elliptic [33] VaR is subadditive, i.e.:

$$\mathrm{VaR}_k(P_1 + P_2) \leqslant \mathrm{VaR}_k(P_1) + \mathrm{VaR}_k(P_2),$$

where P_1 and P_2 denote the returns of two portfolios.

Note, however, that in this case a VaR-minimizing portfolio coincides with the Markowitz variance-minimizing portfolio. [34] Thus VaR, that was introduced in

[31] Artzner et al. (1997, 1999).
[32] This concept has been presented in the paper by Carr et al. (2001), and further developed by Frittelli and Rosazza Gianin (2002).
[33] Embrechts et al. (1999, p. 72).
[34] Embrechts et al. (1999, p. 72).

the attempt of measuring risk for weird distributions, can be used only when the computationally simpler variance can also be used!

Indeed VaR, if applied to most (not elliptical) return distributions is not an acceptable risk measure:

- it does not measure losses exceeding VaR;
- a reduction of VaR may lead to stretch the tail exceeding VaR;
- it may provide conflicting results at different confidence levels;
- non-sub-additivity implies that portfolio diversification may lead to an increase of risk and prevents to add up the VaR of different risk sources;
- non-convexity make it impossible to use VaR in optimization problems;
- VaR has many local extremes leading to unstable VaR ranking. [35]

Thus VaR is an inadequate risk measure, and as shown in the following Fig. 3, the lack of convexity makes it unsuitable to measure risk in a real life portfolio.

If regulatory agencies will insist on its use, some very damaging consequences will follow. Indeed, as pointed out by Daníelsson et al., [36] "VaR can destabilize an economy and induce crashes when they would not otherwise occur".

Various simple numerical proofs of the total inadequacy of VaR as a measure of risk have been proposed in the literature. [37]

The computational difficulties connected with the estimation of VaR_α, that can be essentially performed through three different methods: analytical, historical or Monte Carlo, are also not irrelevant. In spite of these problems, this measure sponsored by a leading bank, [38] has met the favor of regulatory agencies, and has become part of financial regulations for the following reasons:

- it is a compact representation of risk level;
- it measures downside risk.

Thus, the consultative document issued by the Basel Committee in January 2001 [39] falls short of its goals by not encouraging diversification.

The major inspiring principles of the new proposal are: [40]

- VaR is assumed as risk measure;
- the risk of each loan must be portfolio invariant i.e. must be measured by its own characteristic only, not taking into account those of portfolio in which the loan is held;

[35] See Rockafellar and Uryasev (2002).

[36] Daníelsson et al. (2001) as well as Szegö (1993, 1997, 1999, 2000, 2001a,b) and Szegö and Varetto (1999).

[37] Acerbi and Tasche (2002), Frey and McNeil (2001, 2002).

[38] JP Morgan in its Riskmetric approach has been one of the main prophets of VaR.

[39] Basel Committee on Banking Supervision (2001).

[40] See Gordy (2000a,b, 2002).

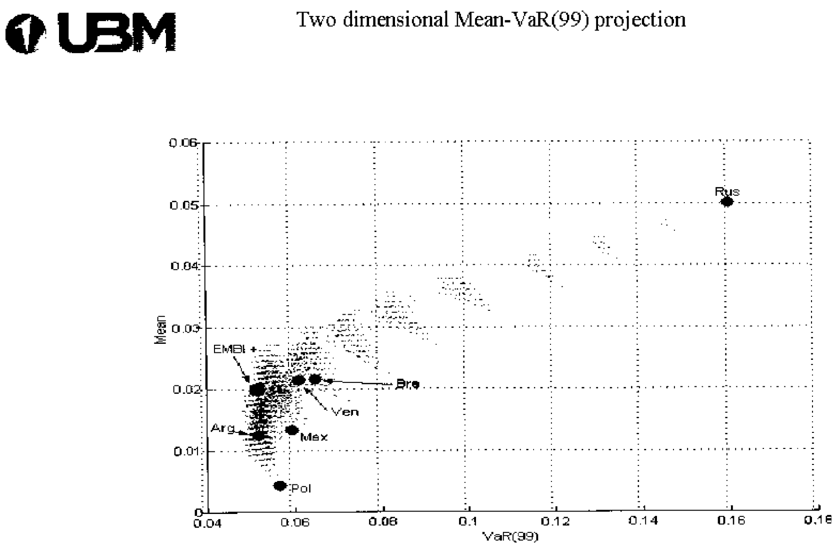

Fig. 3. The mean–VaR_{99} efficient frontier shows the existence of multiple maxima.

- the regulatory capital for a loan must be correlated to its marginal contribution to VaR.

5. Some correct measures of risk

The risk measures that have so far been presented (variance, linear correlation, VaR) in the case of non-elliptic (but possibly symmetric) joint probability distributions

- are not convex and lead to absurd results;
- do not allow to measure the degree of co-dependence (positive or negative) between these random variables.

The first problem has well illustrated in Fig. 1, in particular by the non-symmetric fat-tailed distribution describing the market value of an emerging country debt. The second has become quite relevant with the increasing frequency of occurrence of "dependent defaults", i.e. of unexpectedly connected tail events of different random variables, that happened to be "tail dependent". The present section will be devoted to the first problem. The dependence measures will be discussed in the next one.

In order to investigate tail events in 1997 Embrechts, Küppelberg and Mikosch introduced the concept of k-expected shortfall or k-tail mean. This contribution was followed in 2000 by the presentation of a similar measure, conditional value at risk (CVaR) due to Uryasev. As we shall see, [41] the main contribution of Uryasev, has been to propose a simple linear programming algorithm.

In the sequel, I shall consider the following measures, that satisfy the conditions presented in Section 4, [42] and can therefore be used to measure risk:

- expected regret (ER),
- CVaR,
- expected shortfall (ES),
- tail conditional expectation (TCE) and tail mean (TM),
- worst conditional expectation (WCE),
- spectral risk measures.

Spectral risk measures are a generalization of the previous risk measures, in which the distribution function is pre-multiplied with a admissible risk aversion function which allow to introduce a subjective risk weight.

5.1. Expected regret

ER, that is a risk measure closely related to CVaR, [43] is defined as the expected value of the (loss) distribution beyond a threshold α, i.e.

$$G_\alpha(x) = \int_{y \in R^m} [f(x,y) - \alpha]^+ p(y) \mathrm{d}y$$

with $[u]^+ = \max\{0, u\}$.

ER can be computed via a linear programming model based on a scenario approach.

$$\min_{\mathbf{x}} \mathbf{p}^\mathrm{T}[\mathbf{y} - \alpha]^+$$

under the conditions:

$y^\mathrm{T} = x^\mathrm{T}L > \alpha e^\mathrm{T}$, where e denotes the unit vector,
$x^\mathrm{T}q = \chi$, where $\chi = e^\mathrm{T}q$,
$x^\mathrm{T}(r - \pi)q^\mathrm{T} \geqslant 0^\mathrm{T}$, where 0 denotes the zero column vector,
$x^\mathrm{T}r \geqslant \pi$,
$l \leqslant x \leqslant u$,

where the variables are defined in the following Table 1.

Few comments are due.

We consider $i = 1, \dots, n$ (assets) and $j = 1, \dots, m$ (scenarios).

[41] Rockafellar and Uryasev (2002). See also Andersson et al. (2000).

[42] The most direct proof of this fact is presented in this issue in the paper by Acerbi and Tasche. Spectral risk measures are due to Acerbi (2002).

[43] See Testuri and Uryasev (2000). See also Embrechts (2000), Embrechts et al. (2001) and Franke et al. (2000).

Table 1
Variables definitions

	Definition	Dimension
x	Portfolio weights	$n \times 1$
y	Portfolio losses exceeding α for each scenario j	$m \times 1$
q	Market value of assets	$n \times 1$
b	Future value of each asset with fixed risk level	$n \times 1$
D	Future value of each asset for each scenario and possible changes in risk level	$n \times m$
l	Lower trading limit	$n \times 1$
u	Upper trading limit	$n \times 1$
p	Probability associated at each scenario	$m \times 1$
L	Losses due to increase of risk for each asset and each scenario	$n \times m$
R	Expected assets returns without any change in risk level	$n \times 1$
π	Minimal acceptable expected portfolio return	1×1

$p^{\mathrm{T}}[\mathbf{y} - \alpha]^+$ is the objective function. It represents the mean with respect to all possible scenarios of the portfolio losses higher than the threshold α. It is the weighted average weighted with the probability of each scenario of the portfolio regrets.

L is the $n \times m$ loss matrix, loss due to a variation in the value of any of the n assets for all possible scenario. Thus $l_{ij} = b_i - d_{ij}$. The portfolio loss that must be minimized for all scenarios j $(j = 1, \dots, m)$ is given by $y^{\mathrm{T}} = x^{\mathrm{T}}L > \alpha e^{\mathrm{T}}$.

Equality $x^{\mathrm{T}}q = \chi$ where $\chi = e^{\mathrm{T}}q$, e being the unit vector, provides the budget constraint.

Inequality $x^{\mathrm{T}}(r - \pi)q^{\mathrm{T}} \geqslant 0^{\mathrm{T}}$ imposes a constraint on portfolio return.

Inequalities $l \leqslant x \leqslant u$ and/or $x \geqslant 0$ sets position limits.

5.2. Conditional value at risk

5.2.1. Definition of CVaR

For continuous random variables, CVaR is the expected value of the losses exceeding VaR_k, i.e.

$$\mathrm{CVaR}_k = \phi_k(x) = (1-k)^{-1}\int_{f(x,y) \geqslant \alpha_k(x)} f(x,y)p(y)\mathrm{d}y$$

that can also be defined as

$$\mathrm{CVaR}_k = \mathrm{VaR}_k + E[f(x,y) - \mathrm{VaR}_k | f(x,y) > \mathrm{VaR}_k].$$

This definition does not provide any hint of how to compute this measure, without knowing VaR. Thus we replace CVaR with an auxiliary simpler function F_k.

5.2.2. An equivalent simpler problem

Define F_k as

$$F_k(x,\alpha) = \alpha + (1-k)^{-1}\int_{y \in R^m} [f(x,y) - \alpha]^+ p(y)\mathrm{d}y$$

where $[u]^+ = u$ if $u > 0$; and $[u]^+ = 0$ if $u \leqslant 0$.

It can be proved that [44] $F_k(\mathbf{x}, \alpha)$ is convex and differentiable and that k-CVaR can then be computed as follows:

$$\phi_k(x) = \min_{\alpha \in R} F_k(x, \alpha).$$

In addition

$$\min_{x \in X} \phi_k(x) = \min_{(x,\alpha) \in X \times R} F_k(x, \alpha).$$

5.2.3. Computing CVaR via linear programming

In the case in which the analytical representation of the density function is not available but we can consider m different scenarios, the function $F_k(\mathbf{x}, \alpha)$ can be approximated.

Since $F_k(\mathbf{x}, \alpha)$ and $\phi_k(\mathbf{x})$ (k-CVaR) are both convex, if the admissible set X is convex, then the previous minimization problem can be formulated as the following LP problem:

$$\min_{x,\alpha} \alpha + (1-k)^{-1} p^{\mathrm{T}} [y - \alpha]^+$$

under the conditions:

$y^{\mathrm{T}} = x^{\mathrm{T}} \mathrm{L} > \alpha e^{\mathrm{T}}$, where e denotes the unit vector,
$x^{\mathrm{T}} q = \chi$, where $\chi = \sum_{i=1}^{n} q_i$,
$x^{\mathrm{T}}(r - \pi) q^{\mathrm{T}} \geqslant 0^{\mathrm{T}}$, where 0 denotes the zero column vector,
$x^{\mathrm{T}} r \geqslant \pi$,
$l \leqslant x \leqslant u$.

It has been pointed out [45] that CVaR can be related to ER as follows:

$$F_k(x, \alpha) = \alpha + (1-k)^{-1} G_\alpha(x).$$

5.3. Expected shortfall

In the case of continuous random variables, but only in that case [46] the definition of ES coincides with that of CVaR.

6. Scalar measures of dependence

We are left with the task of defining the concept of dependence among random variables when their joint distribution is not elliptic. As already mentioned, in this

[44] Rockafellar and Uryasev (2002).
[45] Testuri and Uryasev (2000).
[46] See the papers by Acerbi and by Acerbi and Tasche in this issue.

case linear correlation cannot be used. We must therefore define the concept of dependence in a general form. Let us start from the definition of concordance (or positive dependence).

Two distinct observations (x', y') and (x'', y'') of a vector (X, Y) of continuous random variables are said to be concordant (or positively dependent) if

$$(x' - x'')(y' - y'') > 0, \text{ i.e.}$$

$$\text{if } x' > x'', \text{ then } y' > y''$$

and discordant (or negatively dependent) if

$$(x' - x'')(y' - y'') < 0, \text{ i.e.}$$

$$\text{if } x' > x'', \text{ then } y' < y''.$$

Note that dependence is a property of observations of a pair of random variables. This feature is in sharp contrast with the simple linear correlation measure.

We shall next introduce a classic measure of concordance, called Kendall τ.

Consider for that a random sample of n pairs of observations (x^1, y^1) $(x^2, y^2) \ldots (x^n, y^n)$ from a vector (X, Y) of continuous random variables. In this sample there are $\binom{n}{2}$ distinct pairs of observations, each of which can either be concordant or discordant according to the previous definition. If we designate with c the number of concordant and with d the number of discordant pairs, respectively, Kendall's τ is defined as

$$\tau = (c - d)/(c + d) = (c - d) \Big/ \binom{n}{2}.$$

Thus Kendall's τ is equal to the probability of concordance minus that of discordance for pairs of distinct observations randomly chosen from the sample of n observations. Like the linear correlation coefficient ρ, Kendall τ provides a scalar measure of the degree of dependence, a more precisely of the degree of monotonic dependence. Note that this definition is particularly well suited to numerical analysis of dependence properties.

In the same way in which we have defined the desired in risk measures, we must impose suitable requirements on dependence measures $\delta \to R$. Thus:

A functional $\delta(X, Y) \to R$ is a dependence measure of the random variables X and Y if has the following properties:
(a) existence, $\delta(X, Y)$ is defined for every pairs of continuous random variables X, Y;
(b) symmetry, $\delta(X, Y) = \delta(Y, X)$;
(c) normalization I°, $-1 \leqslant \delta(X, Y) \leqslant 1$.

These properties are, for instance satisfied, by linear correlation. It would be very important to include also the property that:
(d) $\delta(X, Y) = 0$ if and only if X and Y are independent.

It can be proved [47] that the condition (c) cannot co-exist with condition (d). In order to adopt the independence condition (d), we must replace (c) with

(e) normalization II°, $0 \leqslant \delta(X, Y) \leqslant 1$. The conditions (a), (b), and (e) must be integrated with the following two monotonicity conditions:

(f) $\delta(X, Y) = 1 \Longleftrightarrow X$ is almost surely a monotone function of Y and Y of X.

(g) if ϕ and θ are almost surely strictly monotone functions on the range of X and Y, respectively, then $\delta(\phi(X), \theta(Y)) = \delta(X, Y)$.

Note that linear correlation, does not satisfy property (d). The same is true for Kendall τ.

The pairwise definition of co-dependence, Kendall τ, can be modified into a global population in the case of a random vector (X, Y) of continuous random variables with joint distribution H by using the following concept of *copula*.

Joint distribution functions are defined by the marginal behavior and the dependence structure of the random variables. The copula is a joint distribution function is obtained by transforming the margins of each variable into standard uniform form. Thus the identification of the copula of a joint random variable is organized along two consequent steps:

- identification of the marginal contribution of each random variable,
- definition of the appropriate transformations.

For that assume a joint distribution function $H(x, y)$ is given and that its marginal distributions $F(x) = \lim_{y \to +\infty} H(x, y)$ and $G(y) = \lim_{y \to +\infty} H(x, y)$ have been computed.

In this case $H(x, y)$ can be represented as a function of the marginal distributions:

$$H(x, y) = C(F(x), G(y))$$

The function $C(.,.)$, which is called copula and provide a complete description of the association properties among the random variables x and y, is monotonically increasing on [0,1) i.e.:

$$C(0, v) = 0 = C(u, 0),$$

$$C(1, v) = v \text{ and } C(u, 1) = u,$$

on the rectangle u_1, u_2, v_1, v_2, with $u_1 < u_2$, $v_1 < v_2$ it is:

$$[C(u_2, v_2) - C(u_2, v_1)] - [C(u_1, v_2) - C(u_1, v_1)] \geqslant 0.$$

It has been proved [48] that if $F(x)$ and $G(y)$ are continuous, then $C(.,.)$ is unique and that

$$C(u, v) = H(F^{-1}(u), G^{-1}(v)).$$

[47] Embrechts et al. (1999, p. 15).

[48] Sklar (1973). See also Nelsen (1999) for a very readable presentation.

The last relationships allows to compute the copula C from the marginals F and G. We recall that this copula is unique if F and G are continuous.

Copulas provide a complete description of the association, the co-dependence properties of random variables at each point of a distribution. This representation allows to take into account the problems connected with dependent extreme events that cannot be characterized via linear correlation or VaR.

The importance of the copula approach is emphasized by the serious doubts [49] expressed about the possibility of developing a scalar measure of dependence.

7. How to construct copulas, analyze dependent extreme events, and live happily thereafter

The construction of copulas is not a trivial problem. The determination of the marginals and of the related copula that allow to identify the dependence properties of random variables can be assimilated to the search for the pair of eyeglasses that provide the best view!

Two basic methodologies have been proposed:

- parametric methods;
- non-parametric methods.

The first approach pre-defines the type of copula (Gaussian, *t*-student, Gumbel, etc.) and try to fit the parameters to the data, the second, based on some proposals put forward by Deheuvels (1981) [50] allows to deduce from the data an empirical copula.

8. Conclusions

In spite of the stubbornness of regulators, and the VaR-machine, [51] if VaR is used to measure risk in most of financial situations, leads to disastrous results. The problem of risk measure can be easily solved by using CVaR or ER and portfolio problems in the mean–CVaR plane can be solved by linear programming methods. This can be effectively implemented in standard commercial software such as Microsoft Office: we have run portfolios of up to 40 assets via Excel! Spectral risk measures that are as easy to compute as CVaR, allow the construction of tailor-made risk measure that fit the need of each investor. Unfortunately this is only one side of the story: dependent extreme events have been a major killer of banks. Co-dependence mea-

[49] Ané and Kharoubi (2001).

[50] See also Nelsen (1999), Durrleman et al. (2000) and Frees and Valdez (1998), Genest and Rivest (1993), Bouyé et al. (2000) and Sancetta and Satchell (2001).

[51] See Jorion (2000) to have some fun.

sures, possibly based on copulas, still not very user friendly, are the next frontier of research.

Acknowledgements

This research has been supported under the CNR research grant no. 01.00370. PF25 on "Metodi Previsivi del Rischio di Insolvenza e Analisi della Rischiosità con Metodologie Quantitative. The author wishes to thank Carlo Acerbi and Giorgio Consigli for the constructive comments.

References

Acerbi, C., 2002. Spectral measures of risk: A coherent representation of subjective risk aversion. In: Szegö, G. (Ed.), "Beyond VaR" (special issue). Journal of Banking & Finance 26 (July). Available from <www.gloriamundi.org/var/wps.html>.

Acerbi, C., Tasche, D., 2002. On the coherence of expected shortfall. In: Szegö, G. (Ed.), "Beyond VaR" (special issue). Journal of Banking & Finance 26 (July). Available from <www.gloriamundi.org/var/wps.html>.

Albanese, C., 1997. Credit exposures, diversification risk, and coherent VaR, WP, Department of Mathematics, University of Toronto, September.

Alexander, C. (Ed.), 2001. Mastering Risk. Financial Times Prentice Hall, London.

Andersson, F., Mausser, H., Rosen, D., Uryasev, S., 2000. Credit risk optimization with conditional value-at-risk criterion. Mathematical Programming.

Ané, T., Kharoubi, C., 2001. Dependence structure and risk measure, Presented at the Conference on Quantitative Methods in Finance, Sydney, 12–14 December. Available from <thierry.ane@hec.unil.ch>, forthcoming in the Journal of Business.

Artzner, P., Delbaen, F., Eber, J.-M., Heath, D., 1997. Thinking coherently. Risk 10, 33–49.

Artzner, P., Delbaen, F., Eber, J.-M., Heath, D., 1999. Coherent measures of risk. Mathematical Finance 9, 203–228.

Artzner, P., Delbaen, F., Eber, J.-M., Heath, D., 2000. Risk management and capital allocation with coherent measures of risk, October. Available from <www.math.ethz.ch/finance>.

Artzner, P., Delbaen, F., Eber, J.-M., Heath, D., Ku, H., 2001. Coherent multiperiod risk adjusted values, October. Available from <www.math.ethz.ch/finance>.

Basel Committee on Banking Supervision, 1999. Capital Requirements and Bank Behavior: The Impact of the Basel Accord, Working Paper no.1, BIS, Basel, April.

Basel Committee on Banking Supervision, 2001. The New Basel Capital Accord, Consultative Document, BIS, Basel, January.

Bouyé, E., Durrleman, V., Nikeghbali, A., Riboulet, G., Roncalli, T., 2000, Copulas for finance: A reading guide and some applications. WP, Group de Recherche Opérationnelle, Credit Lyonnais, Paris. Available from <http://gro.creditlyonnais.fr/content/rd/home_copulas.htm>.

Cambanis, S., Huang, S., Simons, G., 1981. On the theory of elliptically contoured distributions. Journal of Multivariate Analysis 11, 368–385.

Caouette, J.B., Altman, E.I., Narayanan, P., 1998. Managing Credit Risk. Wiley, New York.

Carr, P., Geman, H., Madan, D., 2001. Pricing and hedging in incomplete markets. Journal of Financial Economics 62, 131–168.

Černý, A., Hodges, S., 2000. The theory of good-deal pricing in financial markets. Imperial College Management School, London.

Crouhy, M., Galai, D., Mark, R., 2000. A comparative analysis of current credit risk models. In: Jackson, P., Perraudin, W. (Eds.), "Credit Modelling and Regulatory Issues" (special issue). Journal of Banking & Finance 24 (1), 59–117.

Crouhy, M., Galai, D., Mark, R., 2001. Risk Management. McGraw-Hill, New York.

Cumperayot, P.J., Daníelsson, J., Jorgensen, B.N., de Vries, C.G., 2001. On the (ir)relevance of value at risk regulation. In: Franke, F., Haerde, W., Stahl, G. (Eds.), Measuring Risk in Complex Stochastic Systems. Springer, Berlin, pp. 99–117.

Daníelsson, J., 2002. The emperor has no clothes: Limits to risk modelling. In: Szegö, G. (Ed.), "Beyond VaR" (special issue). Journal of Banking & Finance 26 (July).

Daníelsson, J., de Vries, C.G., 1998. Value at risk and extreme returns. London School of Economics, Financial Markets Group, Discussion Paper No. 273.

Daníelsson, J., Hartmann, P., de Vries, C.G., 1998a. The cost of conservatorism: Extreme value returns, value at risk, and the Basel multiplication factor. Risk, 11.

Daníelsson, J, Jorgensen, B.N., de Vries, C.G., 1998b. The value at risk: Statistical, financial, and regulatory considerations. In: Proceedings of the Conference on "Financial Services at the Crossroad: Capital Regulation in the Twenty-First Century", Economic Policy Review, Federal Reserve Bank of New York, vol. 4, no. 3.

Daníelsson, J., Embrechts, P., Goodhart, C., Keating, C., Muennich, F., Renault, O., Shin, H.-S., 2001. An academic response to Basel II, Special Paper No. 130, FMG and ESRC, London, May. Available from <www.riskresearch.org>.

Daníelsson, J., Jorgensen, B.N., de Vries, C.G., 2002. Incentives for effective risk management, Presented at the Conference on "Statistical and Computational Problems in Risk Management", University of Rome, "La Sapienza", June 14–16. In: Szegö, G.. (Ed.), "Beyond VaR" (special issue). Journal of Banking & Finance 26 (July).

Deheuvels, P., 1981. A non-parametric test for independence. Publications de l'Institut de Statistique de l'Universitéde Paris 26, 29–50.

Durrleman, V., Nikeghbali, A., Roncalli, T., 2000. Which copula is the right one? Operations Research Group Credit Lyonnais, Paris, August 25. Available from <gro.creditlyonnais.fr/content/rd/rd_math/copulas.htm>.

Eberlein, E., 2001a. Recent advances in more realistic market risk management: The hyperbolic model. In: Alexander, C. (Ed.), Mastering Risk, vol. 2. Financial Times Prentice Hall, London, pp. 59–72. Available from <www.fdm.uni-freiburg.de>.

Eberlein, E., 2001b. Application of generalized Lévy motions to finance. In: Barndorff-Nielsen, O.E., Mikosch, T., Resnick, S. (Eds.), Lévy Processes: Theory and Applications. Birkhauser-Verlag, Basel.

Eberlein, E., Keller, U., 1995. Hyperbolic distributions in finance. Bernoulli 1, 261–299. Available from <www.neyman.mathematik.uni-freiburg.de/homepages/eberlein/papers/eberlein.keller>.

Embrechts, P. (Ed.), 2000. Extremes and Integrated Risk Management. Risk Books, London.

Embrechts, P., Küppelberg, C., Mikosch, T., 1997. Extremal Events in Finance and Insurance. Springer, Berlin.

Embrechts, P., McNeil, A., Straumann, D., 1999. Correlation and dependence in risk management: Properties and pitfalls, August. Available from <www.math.ethz.ch/finance>.

Embrechts, P., Lindskog, F., McNeil, A., 2001. Modelling dependence with copulas and applications to risk management, September. Available from <www.math.ethz.ch/finance>.

Franke, F., Härde, W., Stahl, G., 2000. Measuring Risk in Complex Stochastic Systems. Springer, Berlin.

Frees, E.W., Valdez, E., 1998. Understanding relationships using copulas. North American Actuarial Journal 2, 1–25. Available from <http://www.soa.org/library/naaj/1997-09/naaj9801_1.pdf>.

Frey, R., McNeil, A.J., 2001. Modelling dependent defaults, Presented at the Conference on "Statistical and Computational Problems in Risk Management", University of Rome "La Sapienza", June 14–16. Available from <http://www.math.ethz.ch/~frey/credit-paper.pdf>.

Frey, R., McNeil, A., 2002. VaR and expected shortfall in credit portfolios: Conceptual and practical insights, Presented at the Conference on "Statistical and Computational Problems in Risk Management", University of Rome, "La Sapienza", June 14–16. In: Szegö, G.. (Ed.), "Beyond VaR" (special issue). Journal of Banking & Finance 26 (July).

Frittelli, M., 2000. Representing sublinear risk measures and pricing rules, Mimeo, July. Available from <http://www.dimequant.unimib.it/main.htm>.

Frittelli, M., Rosazza Gianin, E., 2002. Putting order in risk measures. In: Szegö, G. (Ed.), "Beyond VaR" (special issue). Journal of Banking & Finance 26 (July).

Geman, H., 2002. Pure jump processes for asset price modelling, Presented at the Conference on "Statistical and Computational Problems in Risk Management", University of Rome, "La Sapienza", June 14–16. In: Szegö, G.. (Ed.), "Beyond VaR" (special issue). Journal of Banking & Finance 26 (July).

Genest, C., Rivest, L.P., 1993. Statistical inference procedures for bivariate Archimedean copulas. Journal of the American Statistical Association 88, 1034–1043.

Gordy, M.B., 2000a. A comparative anatomy of credit risk models. Journal of Banking & Finance 24, 119–149. Available from <http://mgordy.tripod.com/index.html>.

Gordy, M.B., 2000b. A risk-factor model foundation for ratings-based bank capital rules. Federal Reserve Board.

Gordy, M.B., 2002. Calculation of higher moments in CreditRisk$^+$ with applications, Presented at the Conference on "Statistical and Computational Problems in Risk Management", University of Rome, "La Sapienza", June 14–16. In: Szegö, G. (Ed.), "Beyond VaR" (special issue). Journal of Banking & Finance 26 (July).

Hodges, S., 1998. A generalization of the Sharpe ratio and its application to valuation bounds and risk measures. FORC Preprint 98/88, University of Warwick.

Huschens, S., Kim, J.-R., 2000. A stable CAPM in the presence of heavy-tailed distributions. In: Franke, J., Härdle, W., Stahl, G. (Eds.), Measuring Risk in Complex Stochastic Systems. Lecture Notes on Statistics, vol. 147. Springer, Berlin.

Joe, H., 1997. Multivariate Models and Dependence Concepts. Chapman and Hall, London.

Joe, H., Hu, T., 1996. Multivariate distributions from mixtures of max-infinetely divisible distributions. Journal of Multivariate Analysis 57, 240–265.

Jones, D., 2000. Emerging problems with the Basel capital accord: Regulatory capital arbitrage and related issues. In: Jackson, P., Perraudin, W. (Eds.), "Credit Risk Modelling and Regulatory Issues" (special issue). Journal of Banking & Finance 24, 35–58.

Jorion, P., 2000. VaR: The New Benchmark for Managing Financial Risk. McGraw Hill, New York.

JP Morgan, 1994. Riskmetrics. New York, October and November 1995.

JP Morgan, 1997. Credit Metrics, New York.

Landsman, Z., Sherris, M., 2001. Risk measures and insurance premium principles. Insurance: Mathematics and Economics 29, 103–115.

Lee, J., 1993. Generating random binary deviates having fixed marginal distributions and specified degrees of association. Statistical Computing 47, 209–215.

Manganelli, S., Engle, R.F., 2001. Value at risk models in finance, WP N° 75, ECB, Frankfurt, August. Available from <http://www.gloriamundi.org/var/pub/maneng.pdf>.

Nelsen, R.B., 1999. An Introduction to Copulas. Springer, Berlin.

Phelan, M.J., 1997. Probability and statistics applied to the practice of financial risk management: The case of JP Morgan's RiskMetrics. Journal of Financial Services Research 12, 175–200. Available from <http://fic.wharton.upenn.edu/fic/papers/95/9519.pdf>.

Reiss, R.D., Thomas, M., 2001. Statistical Analysis of Extreme Values. Birkhauser, Basel.

Rockafellar, R.T., Uryasev, S., 2002. Optimization of conditional value-at-risk, Presented at the Conference on "Statistical and Computational Problems in Risk Management", University of Rome, "La Sapienza", June 14–16. In: Giorgio, Szegö. (Ed.), "Beyond VaR" (special issue). Journal of Banking & Finance 26 (July).

Silvapulle, P., Granger, C.W., 2001. Large returns, conditional portfolio diversification: Value-at-risk approach. Quantitative Finance 1, 542–551.

Sancetta, A., Satchell, S., 2001. Bernstein approximations to the copula function and portfolio optimization, Mimeo, Trinity College, Cambridge, June. Available from <http://logec.hhs.se/scripts/paper-stat.pl?h = repec:cam:camdae:0105>.

Sklar, A., 1973. Random variables, joint distributions, and copulas. Kybernetica 9, 449–460.

Szegö, G., 1980. Portfolio Theory. Academic Press, New York.

Szegö, G., 1993. Pseudo risk-adjusted capital requirements: many thorns and little flowers, Seminar at Harvard University, May 21.

Szegö, G., 1997. A critique of the Basel regulation, or how to enhance (im)moral hazards. In: Proceedings of the International Conference on Risk Management and Regulation in Banking, Bank of Israel, Jerusalem, May 17–19. Kluwer, Dordrecht, 1999.

Szegö, G., 1999. Corporate governance and the role of capital in the banking industry. WP 1/99 Department of Accounting and Finance, Monash University, Melbourne, January.

Szegö, G., 2000. News and gossips on bank regulation. Lecture given at City University, May 17.

Szegö, G., 2001a. (Bad) News on bank regulation. Lecture presented at the International Conference on "Financial and Real Markets, Risk Management and Corporate Governance: Theory and International Evidence", Sousse, Tunisia, March 15–17, 2001. Available from <w3.uniroma1/bancaefinanza>.

Szegö, G., 2001b. Bubble control, and the twelve wise guys on the Rhine, Presented at the XII Annual Meeting of EFMA, Lugano, June 17–30. Available from <w3.uniroma1/bancaefinanza>.

Szegö, G., Varetto, F., 1999. Il Rischio Creditizio: Misura e Controllo. UTET Libreria, Torino.

Testuri, C.E., Uryasev, S., 2000, On relation between expected regret and conditional value-at-risk, Mimeo, ISE University of Florida, August. Available from <www.ise.ufl.edu/uryasev/testuri.pdf>.

Wang, S.S., Young, V.R., Panjer, H.H., 1997. Axiomatic characterization of insurance prices. Insurance: Mathematics and Economics 21, 173–183.