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Fuzzy capital budgeting

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Abstract

Fuzzy equivalents of all the classical capital budgeting (investment choice) methods are generalised or proposed. These equivalents can be used to evaluate and compare projects in which the cash flows, duration time and required rate of return (cost of capital) are given imprecisely, in the form of a fuzzy number. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

The aim of this paper is to propose a practical tool of incorporating uncertainty into capital budgeting (investment decisions) in its simplest form – when we want to be able to decide for a single investment decision whether to reject it or to accept it (or, when there are alternative projects, which one to choose).

There are a variety of methods used in capital budgeting (see e.g. [2, 6, 18]) a survey of which we give in Section 2. However, the classical forms of these methods do not take into account the uncertainty which may be inherent in the information used in them. This information (see e.g. [18, 19]) includes: future cash in- and outflows, the required rate of return of the investment or cost of capital, the duration of the project.

There exist several ways of incorporating uncertainty in the capital budgeting decision making (see e.g. the items referred to above and [7, 15, 16, 20, 21]). Some of them are based on intuitive modification of the data used in the classical methods (like the "risk adjustment", which consists in increasing the present cost of capital by a constant in later stages of projects being analysed), some replace the deterministic data with their random equivalents. The intuitive methods are widely used in practice, but they have the disadvantage of depending too much on the intuition of the decision maker. The probabilistic approach does propose formalised approaches, but, as it is pointed out in [7, 15, 16, 22], there are other problems connected with it. The probability theory is much less flexible than the fuzzy sets theory: it requires the fulfilment of several assumptions about probabilistic distributions and operations on them, and these assumptions are

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seldom fulfilled for the distributions used in capital budgeting. Moreover, in order to be applied it requires much more information than is usually available about the investments an enterprise considers to undertake – and fuzzy sets can be applied even with rather scanty information.

That is why several authors (see e.g. [4, 19, 7, 15, 16]) have used another approach, which we can call the possibilistic or fuzzy one. In this approach instead of probability distributions possibilistic distributions or fuzzy numbers are used.

Although Buckley in [4] proposed a fuzzy capital budgeting theory already more than 10 years ago, and some other authors contributed to it too [7, 8, 17], recent papers discussing how firms deal with risk/uncertainty in practice (e.g. [20, 21]) show no trace at all of the possibilistic/fuzzy approach (an exception is [19]). The question arises why it is so, given the fact that literature contains no explicit criticism of this method either.

In our opinion the reasons might include the following:

- The authors of [4, 17] indicate some formal problems (e.g. in the internal rate of return method), claiming that certain capital budgeting decision methods cannot be used in the fuzzy approach.
- Buckley in [4] does not take into account some methods which are widely used in practice (e.g. the payback period method, very popular in practice (see e.g. [7, 13])).
- In the same paper some assumptions are made which do not have to be fulfilled in practice (e.g. that the cash flows are either positive or negative fuzzy numbers).

The aim of this paper is to invalidate the reasons listed above or at least to weaken their force. In other words, we will try to complete Buckley's approach, hoping to make it in this way more apt to be used in practice.

We start by defining the problem and listing the capital investment methods used in practice in their classical form. Further in the paper all of them will be given a "fuzzified" form.

2. A survey of capital budgeting methods in the classical, crisp case

We consider the following problem:

There is a project or another type of investment which might be interesting for the company to undertake. We want to get a numerical estimation of this investment which will allow us to say whether to accept it or not and to compare this project with alternative ones.

The following data for the project must be given (in this section we assume them to be real numbers, further in the paper they will be fuzzified).

(a) n – the duration of the project in assumed time units (here years), in this section n is integer;

(b) COF_i (a non-negative number) – cash outflow at the end of the ith year ($i = 0, 1, \ldots, n$) and CIF_i (a non-negative number) – cash inflow at the end of the ith year ($i = 1, \ldots, n$). Both yearly cash flows will be often summarised as a single cash flow CF_i, occurring at the end of the ith year ($i = 0, 1, \ldots, n$), where $CF_0 = -COF_0$, $CF_i = CIF_i - COF_i$ ($i = 1, \ldots, n$).

(c) r (a positive real number) – the required rate of return of the investment or cost of capital (for the way the r is determined and various assumptions that are made see e.g. [2, 3]).

Although generally we assume that the duration of the project is n years, where n is an integer, additionally we will consider a situation when the project finishes before the end of the nth year, after the α-part of the nth year has elapsed. In this case we take into account only a fraction of the final cash flows which would take place at the end of the year. These fractions would be equal to $\alpha COF_n, \alpha CIF_n, \alpha CF_n$. We also assume that the rate of return corresponding to the α-part of an arbitrary year is αr, where r is the rate of return for the whole year.

This supplement to the normal case of an integer duration may be interesting, per se, but here its main role is to illustrate the reasoning which we will use in the case of fuzzy duration time and in the definitions of sums in which the upper value of the summation index will be an interval or a fuzzy number.

Here are several methods used in practice and permitting to evaluate a project given by the data described above. E will denote the output of each method, i.e. the numerical estimation of the given project according to this method. Several estimations are possible: E may be the difference between all the revenues and all the expenditures connected with the project (i.e. a kind of profit delivered by the project) or the time it is going to take for the project to pay off the money which were initially invested in it. E will serve as a basis for the rejection or acceptance decision.

(1) *Revenues per one dollar method*:

$$E = \sum_{i=1}^{n} CIF_i \Big/ \sum_{i=0}^{n} COF_i.$$

Obviously, the bigger E, the better the project.

Alternatively, it is possible to calculate other ratios, like

$$E = \left(\sum_{i=1}^{n} CIF_i - \sum_{i=0}^{n} COF_i\right) \Big/ \sum_{i=0}^{n} COF_i.$$

When the project finishes after the α-part of the nth year has elapsed, the ratios become

$$E|_\alpha = \left(\sum_{i=1}^{n-1} CIF_i + \alpha CIF_n\right) \Big/ \left(\sum_{i=0}^{n-1} COF_i + \alpha COF_n\right),$$

$$E|_\alpha = \left[\left(\sum_{i=1}^{n-1} CIF_i + \alpha CIF_n\right) - \left(\sum_{i=0}^{n-1} COF_i + \alpha COF_n\right)\right] \Big/ \left(\sum_{i=0}^{n-1} COF_i + \alpha COF_n\right).$$

(2) *Payback period method*:

$$E = \min_{k=1,\dots,n} \left\{ k : \sum_{i=0}^{k} CF_i \geqslant 0, \infty \right\} \quad \text{or} \quad E = \min_{k=1,\dots,n} \left\{ k : \sum_{i=0}^{k} \frac{CF_i}{(1+r)^i} \geqslant 0, \infty \right\},$$

(the division of the cash flow in the ith year by $(1+r)^i$ expresses the value of this cash flow in the monetary units of the present moment).

In this case, the smaller E, the better the project.

When the project finishes after the α-part of the nth year has elapsed, the respective evaluations become

$$E|_\alpha = \min_{k=1,\dots,n} \left\{ k\colon \sum_{i=0}^{k} CF'_i \geqslant 0 \right\}, \infty \quad \text{where } CF'_i = CF_i \ (i = 0, 1, \dots, n-1), \quad CF'_n = \alpha CF_n,$$

$$E|_\alpha = \min_{k=1,\dots,n} \{k\colon S_k \geqslant 0, \infty\}, \quad \text{where } S_k = \sum_{i=0}^{k} \frac{CF_i}{(1+r)^i} \text{ for } k = 1, \dots, n-1$$

and

$$S_n = \sum_{i=0}^{n-1} \frac{CF_i}{(1+r)^i} + \frac{\alpha CF_n}{(1+r)^{n-1}(1+\alpha r)}.$$

(3) *Net present value*:

$$E = CF_0 + \sum_{i=1}^{n} \frac{CF_i}{(1+r)^i} \quad \text{(usually denoted as } NPV\text{)}.$$

E represents the value of the project at the beginning of the year it is started. A bigger E means here that the project is better. Usually, there is some limit, minimal value of E required for the acceptance of the project (e.g. 0).

If the project finishes when the α-part of the nth year has elapsed, the respective evaluation becomes

$$E|_\alpha = NPV|_\alpha = \sum_{i=0}^{n-1} \frac{CF_i}{(1+r)^i} + \frac{\alpha \cdot CF_n}{(1+r)^{n-1}(1+\alpha \cdot r)}.$$

(4) *Net future value*: Here we calculate the value of the project at the end of the mth year. The mth year can be the last year of the project or any following one (i.e. $m \geqslant n$). Although usually projects are evaluated on the basis of their present value, also the future value (in a future year common for the projects being compared) can be used for this purpose (see e.g. [10]).

$$E(m) = \sum_{i=0}^{n} CF_i (1+r)^{m-i} \quad \text{for any integer } m \geqslant n.$$

In case of a project finishing before the end of the nth year, after the α-part of the nth year, we get

$$E(m)|_\alpha = \sum_{i=0}^{n-1} CF_i (1+r)^{m-i} + \alpha \cdot CF_n (1 + (1-\alpha) \cdot r)(1+r)^{m-n} \quad \text{if } m \geqslant n.$$

(5) *Method of utility of the net present value*:

$$E = 1 - \frac{1}{e^{b_U \cdot NPV}} \quad \text{or} \quad E|_\alpha = 1 - \frac{1}{e^{b_U \cdot NPV|_\alpha}},$$

where b_U (a positive real number) is the risk aversion constant of the decision maker. The constant b_U is usually equal to a fraction of the equity of the enterprise and expresses the degree to which the decision maker is ready to take a risk (the bigger the constant, the less cautious the decision maker is).

Like in the previous method, the bigger the C, the better.

(6) *The internal rate of return method*: E is a solution of the following equation:

$$CF_0 + \sum_{i=1}^{n} \frac{CF_i}{(1+E)^i} = 0.$$

In order for this solution to be unique and for the interpretation of this solution to be reasonable (see e.g. [3]), we assume that $CF_i \geqslant 0$ for $i = 1, \ldots, n$.

When the project finishes after the α-part of the nth year has elapsed, $E|_\alpha$ is a solution of the following equation:

$$CF_0 + \sum_{i=1}^{n-1} \frac{CF_i}{(1+E|_\alpha)^i} + \frac{\alpha \cdot CF_n}{(1+E|_\alpha)^{n-1}(1+\alpha \cdot E|_\alpha)} = 0.$$

(7) *The modified internal rate of return method* [3]: In this case no additional assumptions about cash flows CF_i are necessary. E is a solution of the following equation:

$$\sum_{i=0}^{n} \frac{COF_i}{(1+r)^i} = \frac{\sum_{i=0}^{n} CIF_i (1+r)^{n-i}}{(1+E)^n}$$

and $E|_\alpha$ of the following one:

$$\sum_{i=0}^{n-1} \frac{COF_i}{(1+r)^i} + \frac{\alpha \cdot COF_n}{(1+\alpha \cdot r)(1+r)^{n-1}} = \frac{\sum_{i=0}^{n-1} CIF_i(1+r)^{n-i} + \alpha \cdot CIF_n(1+(1-\alpha)r)}{(1+E|_\alpha)^{n-1}(1+\alpha E|_\alpha)}.$$

On the left-hand side of the above equations we have the present value of all the outflows, and on the right-hand side the future value of all the inflows reported afterwards to the present moment through the unknown modified internal rate of return.

The evaluations E for all the capital budgeting methods presented here can be easily found using, e.g. spreadsheets.

3. Notation, interval and fuzzy numbers and operations performed on them

We start our presentation with interval numbers (using [12] as a basis), which will be necessary in defining fuzzy numbers and operations on the latter. The operations on interval numbers can be interesting also per se, as the whole content of the paper may be easily referred just to the interval numbers as another way of representing uncertainty in capital budgeting decisions (see e.g. [9]).

3.1. Interval numbers and operations performed on them

Let $\overline{A} = [a_1, a_2]$, $\overline{B} = [b_1, b_2]$, etc., denote real, closed intervals, which we will call interval numbers.

Let us start with the following definition:

Definition 3.1. Let s be an arbitrary real number. Then

(a) $[a_1, a_2] > s \Leftrightarrow a_1 > s$, $[a_1, a_2] < s \Leftrightarrow a_2 < s$.

(b)

$$s\,[a_1, a_2] = \begin{cases} [sa_1, sa_2] & \text{if } s > 0, \\ [0, 0] & \text{if } s = 0, \\ [sa_2, sa_1] & \text{if } s < 0. \end{cases}$$

Moreover, the interval exponential function will be defined in the following way:

(c) $e^{[a_1, a_2]} = [e^{a_1}, e^{a_2}]$.

Let us now go over to operations on two interval numbers. The basic idea of these operations is contained in the following formula, where * denotes an arbitrary operation defined for real numbers:

$$\overline{A} * \overline{B} = \{a * b \quad \text{s.t. } a \in \overline{A},\ b \in \overline{B}\}.$$

By the way, it is worth noticing in this place that in some cases, if $\overline{A} = \overline{B}$, the above concept may lead to unnecessarily wide result intervals (whether this problem occurs, depends on the given situation and on the interpretation of $\overline{A}$ and $\overline{B}$). For detailed information, see e.g. [1]. However, we assume that in our decision situations it is really so that the result interval should always contain all the possible results of the corresponding operation carried out on all the elements of the argument intervals.

The following definition is a consequence of the above formula.

Definition 3.2. (a) $[a_1, a_2] + [b_2, b_2] = [a_1 + b_1, a_2 + b_2]$, $[a_1, a_2] - [b_1, b_2] = [a_1 - b_2, a_2 - b_1]$.

(b) $[a_1, a_2] \cdot [b_2, b_2] = [\min\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}, \max\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}]$.

(c) Let $0 \notin [b_1, b_2]$. Then $[a_1, a_2]/[b_1, b_2] = [a_1, a_2] \cdot [1/b_2, 1/b_1]$.

The following lemma is proved in [12].

Lemma 3.1. *Let* $\overline{A} = [a_1, a_2]$, $\overline{B} = [b_1, b_2]$, $\overline{C} = [c_1, c_2]$ *be arbitrary intervals such that* $\overline{B} \cdot \overline{C} > 0$. *Then*

$$\overline{A} \cdot (\overline{B} + \overline{C}) = \overline{A} \cdot \overline{B} + \overline{A} \cdot \overline{C}$$

holds.

Let us now define the sum of a series of intervals, where the upper limit of the sum is an interval. The symbol $[x]$ stands for the integer part of a given real number x.

We will need two kinds of sums. The idea behind them is the same as that behind the evaluations $E|_\alpha$ in Section 2. Each time we will obtain an interval containing all possible values of the respective real-valued sums. Both methods of summation will be used in Definition 3.10, which in its turn is necessary for Section 6.

Definition 3.3. We assume that the following interval numbers are given:

$$[a_1, a_2] > 0, \; [b_{1,i}, b_{2,i}] \quad (i = 0, 1, \ldots, [a_2] + 1)$$

(if a_2 is integer, we assume $[b_{1,a_2+1}, b_{2,a_2+1}] = [0, 0]$).

Then we define

$$\sum_{i=0(1)}^{[a_1,a_2]} [b_{1,i}, b_{2,i}] = [C_1, C_2],$$

where

$$C_1 = \min \left\{ \begin{array}{l} \sum_{i=0(1)}^{[a_1]} b_{1,i} + (a_1 - [a_1]) b_{1,[a_1]+1}, \; \sum_{i=0(1)}^{k} b_{1,i} \; (k = [a_1]+1, \ldots, [a_2]), \\ \sum_{i=0(1)}^{[a_2]} b_{1,i} + (a_2 - [a_2]) b_{1,[a_2]+1} \end{array} \right\},$$

$$C_2 = \max \left\{ \begin{array}{l} \sum_{i=0(1)}^{[a_1]} b_{2,i} + (a_1 - [a_1]) b_{2,[a_1]+1}, \; \sum_{i=0(1)}^{k} b_{2,i} \; (k = [a_1]+1, \ldots, [a_2]), \\ \sum_{i=0(1)}^{[a_2]} b_{2,i} + (a_2 - [a_2]) b_{2,[a_2]+1} \end{array} \right\}.$$

Definition 3.4. We assume that the following interval numbers are given:

$$[a_1, a_2] > 0, \; [b_{1,i}, b_{2,i}] \quad (i = 0, 1, \ldots, [a_2] + 1), \; [r_1, r_2] > 0$$

(if a_2 is integer, we assume $[b_{1,a_2+1}, b_{2,a_2+1}] = [0, 0]$).

Apart from that we fix an integer number $m \geqslant [a_2]$ and denote

$$\alpha_1 = a_1 - [a_1], \qquad \alpha_2 = a_2 - [a_2].$$

Then we define (the union symbol will be used for this summation, in order to distinguish it from the one defined in the previous definition):

$$\bigcup_{i=0(1)}^{[a_1,a_2]} \frac{[b_{1,i}, b_{2,i}]}{(1 + [r_1, r_2])^i} = \left[D_1^a, D_2^a \right],$$

where

$$D_1^a = \min\left\{\begin{array}{l} \sum_{i=0(1)}^{[a_1]} \dfrac{b_{1,i}}{(1+r_x)^{m-i}} + \dfrac{\alpha_1 b_{1,[a_1]+1}}{(1+\alpha_1 r_x)(1+r_x)^{[a_1]}}, \\ \sum_{i=0(1)}^{k} \dfrac{b_{1,i}}{(1+r_x)^{i}} \ (k=[a_1]+1,\ldots,[a_2]), \\ \sum_{i=0(1)}^{[a_2]} \dfrac{b_{1,i}}{(1+r_x)^{i}} + \dfrac{\alpha_2 b_{1,[a_2]+1}}{(1+\alpha_2 r_x)(1+r_x)^{[a_2]}} \end{array}\right\},$$

with

$$r_x = \begin{cases} r_2 & \text{if the numerator of the fraction is non-negative,} \\ r_1 & \text{if the numerator of the fraction is negative;} \end{cases}$$

$$D_2^a = \max\left\{\begin{array}{l} \sum_{i=0(1)}^{[a_1]} \dfrac{b_{2,i}}{(1+r_x)^{m-i}} + \dfrac{\alpha_1 b_{2,[a_1]+1}}{(1+\alpha_1 r_x)(1+r_x)^{[a_1]}}, \\ \sum_{i=0(1)}^{k} \dfrac{b_{2,i}}{(1+r_x)^{i}} \ (k=[a_1]+1,\ldots,[a_2]), \\ \sum_{i=0(1)}^{[a_2]} \dfrac{b_{2,i}}{(1+r_x)^{i}} + \dfrac{\alpha_2 b_{2,[a_2]+1}}{(1+\alpha_2 r_x)(1+r_x)^{[a_2]}} \end{array}\right\},$$

where

$$r_x = \begin{cases} r_1 & \text{if the numerator of the fraction is non-negative,} \\ r_2 & \text{if the numerator of the fraction is negative} \end{cases}$$

and

$$\bigcup_{i=0(1)}^{[a_1,a_2]} [b_{1,i}, b_{2,i}](1+[r_1,r_2])^{m-i} = \left[D_1^b, D_2^b\right],$$

where

$$D_1^b = \min\left\{\begin{array}{l} \sum_{i=0(1)}^{[a_1]} b_{1,i}(1+r_x)^{m-i} + \alpha_1 b_{1,[a_1]+1}(1+(1-\alpha_1)r_x)(1+r_x)^{m-[a_1]-1}, \\ \sum_{i=0(1)}^{k} b_{1,i}(1+r_x)^{m-i} \ (k=[a_1]+1,\ldots,[a_2]), \\ \sum_{i=0(1)}^{[a_2]} b_{1,i}(1+r_x)^{m-i} + \alpha_2 b_{1,[a_2]+1}(1+(1-\alpha_2)r_x)(1+r_x)^{m-[a_2]-1} \end{array}\right\},$$

with

$$r_x = \begin{cases} r_1 & \text{if the whole component is non-negative,} \\ r_2 & \text{if the whole component is negative;} \end{cases}$$

$$D_2^b = \max\left\{\begin{array}{l} \sum_{i=0}^{[a_1]} b_{2,i}(1+r_x)^{m-i} + \alpha_1 b_{2,[a_1]+1}(1+(1-\alpha_1)r_x)(1+r_x)^{m-[a_1]-1}, \\ \sum_{i=0}^{k} b_{2,i}(1+r_x)^{m-i} \ (k=[a_1]+1,\ldots,[a_2]), \\ \sum_{i=0}^{[a_2]} b_{2,i}(1+r_x)^{m-i} + \alpha_2 b_{2,[a_2]+1}(1+(1-\alpha_2)r_x)(1+r_x)^{m-[a_2]-1} \end{array}\right\},$$

where

$$r_x = \begin{cases} r_2 & \text{if the whole component is non-negative,} \\ r_1 & \text{if the whole component is negative.} \end{cases}$$

3.2. Fuzzy numbers and operations performed on them

In this paper we will use a notation similar to that used in [4] and will adopt the following definition of fuzzy number:

Definition 3.5. A fuzzy number is a sextuplet of the following form:

$$\widetilde{r} = (r_1, r_2, r_3, r_4, f_1^r(\lambda), f_2^r(\lambda))$$

or shorter,

$$\widetilde{r} = (r_1, r_2, r_3, r_4, f_1^r, f_2^r)$$

where

- r_1, r_2, r_3, r_4 are real numbers s.t. $r_1 \leqslant r_2 \leqslant r_3 \leqslant r_4$.
- f_1^r is a continuous non-decreasing real function defined on the interval $[0,1]$, such that $f_1^r(0) = r_1$, $f_1^r(1) = r_2$.
  f_2^r is a continuous non-increasing real function defined on the interval $[0,1]$, such that $f_2^r(1) = r_3$, $f_2^r(0) = r_4$.

Let us recall now the following definition:

Definition 3.6. Let $\widetilde{r} = \left(r_1, r_2, r_3, r_4, f_1^r, f_2^r\right)$ be an arbitrary fuzzy number and λ an arbitrary number such that $0 \leqslant \lambda \leqslant 1$. The symbol r^λ stands for the λ-level of the fuzzy number $\widetilde{r}$, defined as

$$r^\lambda = \left[f_1^r(\lambda), f_2^r(\lambda)\right].$$

The symbols $\left(r^\lambda\right)_1, \left(r^\lambda\right)_2$ or r_1^λ, r_2^λ stand for, respectively, the lower and upper end of the interval r^λ.

The following well-known lemma is obvious.

Lemma 3.2. *A fuzzy number is unequivocally determined by the set of its λ-levels* $(0 \leqslant \lambda \leqslant 1)$. *Thus, we can write* $\widetilde{r} = \left\{r^\lambda, 0 \leqslant \lambda \leqslant 1\right\}$.

Let us now introduce (after [11]) definitions analogous to those from Section 3.1. The basic idea of these definitions is that any relation with a fuzzy number or operation on fuzzy numbers is verified or performed for the corresponding λ-levels $(0 \leqslant \lambda \leqslant 1)$, according to the definitions from the previous section. The resulting λ-levels $(0 \leqslant \lambda \leqslant 1)$ determine unequivocally the resulting fuzzy number. So, all the following definitions are in fact a consequence of the following one:

Definition 3.7. (a) An arbitrary fuzzy number $\widetilde{r}$ fulfils a relation $\Re$ iff for each $0 \leqslant \lambda \leqslant 1$ λ-level r^λ fulfils this relation.

(b) If * is an arbitrary operation and $\widetilde{r}$ and $\widetilde{p}$ any two fuzzy numbers, then

$$\widetilde{r} * \widetilde{p} = \left\{r^\lambda * s^\lambda, \text{ for such } \lambda \text{ that } 0 \leqslant \lambda \leqslant 1 \text{ and operation } * \text{ is defined.}\right.$$

Definition 3.8. Let s be an arbitrary real number. Then

(a) $(r_1,r_2,r_3,r_4,f_1^r,f_2^r) > s \Leftrightarrow r_1 > s$, $(r_1,r_2,r_3,r_4,f_1^r,f_2^r) < s \Leftrightarrow r_4 > s$.

(b)

$$s\,(r_1,r_2,r_3,r_4,f_1^r,f_2^r) = \begin{cases} (sr_1,sr_2,sr_3,sr_4,sf_1^r,sf_2^r) & \text{if } s > 0, \\ (0,0,0,0,0,0) & \text{if } s = 0, \\ (sr_4,sr_3,sr_2,sr_1,sf_2^r,sf_1^r) & \text{if } s < 0. \end{cases}$$

Moreover, the fuzzy exponential function will be defined in the following way:

(d) $e^{(r_1,r_2,r_3,r_4,f_1^r,f_2^r)} = (e^{r_1},e^{r_2},e^{r_3},e^{r_4},e^{f_1^r},e^{f_2^r})$

Definition 3.9. (a)

$$(r_1,r_2,r_3,r_4,f_1^r,f_2^r) + (p_1,p_2,p_3,p_4,f_1^p,f_2^p)$$
$$= (r_1+p_1,r_2+p_2,r_3+p_3,r_4+p_4,f_1^r+f_1^p+f_2^r+f_2^p).$$
$$(r_1,r_2,r_3,r_4,f_1^r,f_2^r) - (p_1,p_2,p_3,p_4,f_1^p,f_2^p)$$
$$= (r_1-p_4,r_2-p_3,r_3-p_2,r_4-p_1,f_1^r-f_2^p,f_2^p-f_1^p).$$

(b)

$$(r_1,r_2,r_3,r_4,f_1^r,f_2^r)\cdot(p_1,p_2,p_3,p_4,f_1^p,f_2^p)$$
$$= \begin{pmatrix} \min\{r_1p_1,r_1p_4,r_4p_1,r_4p_4\},\min\{r_2p_2,r_2p_3,r_3p_2,r_3p_3\}, \\ \max\{r_2p_2,r_2p_3,r_3p_2,r_3p_3\}, \\ \max\{r_1p_1,r_1p_4,r_4p_1,r_4p_4\}, \\ \min\{f_1^rf_1^p,f_1^rf_2^p,f_2^rf_1^p,f_2^rf_2^p\},\max\{f_1^rf_1^p,f_1^rf_2^p,f_2^rf_1^p,f_1^rf_2^p\} \end{pmatrix}.$$

(c) If $0 \notin [p_1,p_4]$ then

$$(r_1,r_2,r_3,r_4,f_1^r,f_2^r)/(p_1,p_2,p_3,p_4,f_1^p,f_2^p)$$
$$= (r_1,r_2,r_3,r_4,f_1^r,f_2^r)\cdot(1/p_4,1/p_3,1/p_2,1/p_1,1/f_2^p,1/f_1^p).$$

It is easy to show that the results of the operations from Definitions 3.8 and 3.9 are fuzzy numbers.

The following lemma is a direct consequence of Lemma 3.1 and Definition 3.7.

Lemma 3.3. *Let $\widetilde{r}$, $\widetilde{p}$, $\widetilde{t}$ be arbitrary fuzzy numbers such that $\widetilde{p}\cdot\widetilde{t} > 0$. Then*

$$\widetilde{r}\cdot(\widetilde{p}+\widetilde{t}) = \widetilde{r}\cdot\widetilde{p}+\widetilde{r}\cdot\widetilde{t}$$

holds.

Let us now define the sum of a series of fuzzy numbers, where the upper limit of the sum is a fuzzy number.

Definition 3.10. Let us assume that the following fuzzy numbers are given:

$$\widetilde{s} > 0, \widetilde{p}_i \quad (i = 1,2,\ldots[s_4]+1)$$

(if s_4 is integer, we assume $\widetilde{p}_{s_4+1} = (0,0,0,0,0,0)$), $\widetilde{r} > 0$, as well as an integer $m \geqslant [s_4]+1$ if s_4 is not integer and $m \geqslant s_4$ if s_4 is integer.

Then we define

(a) $\sum_{i=0(1)}^{\tilde{s}} \widetilde{p}_i = \{T^\lambda\colon \lambda \in [0,1]\}$, where $T^\lambda = \sum_{i=0}^{s^\lambda} p_i^\lambda$ is defined by Definition 3.3.

(b)

$$\bigcup_{i=0(1)}^{\tilde{s}} \frac{\widetilde{p}_i}{(1+\widetilde{r})^i} = \{U^\lambda\colon\ \lambda \in [0,1]\},$$

where $U^\lambda = \bigcup_{i=0(1)}^{s^\lambda} p_i^\lambda/(1+r^\lambda)^i$ is defined by Definition 3.4.

(c) $\bigcup_{i=0(1)}^{\tilde{s}} \widetilde{p}_i(1+\widetilde{r})^{m-i} = \{W^\lambda\colon\ \lambda \in [0,1]\}$, where $W^\lambda = \bigcup_{i=0(1)}^{s^\lambda} p_i^\lambda(1+r^\lambda)^{m-i}$ is defined by Definition 3.4.

4. General concept of fuzzy capital budgeting

We assume that for a given project the following information is available:

(a) The duration of the project:
- in Section 5: an integer number n,
- in Section 6: a fuzzy number $\widetilde{n}$.

The number $\widetilde{n}$ is a fuzzy number according to Definition 3.5. This is a difference with respect to the approach presented in [4], where it is assumed that the project must have an integer duration. We assume that it is possible for the project to break off at any moment of time within a certain interval. We still assume that the cash flows occur at the end of each year, except possibly the last cash flow: if the project is broken when the α-part of the ith year has elapsed ($0 < \alpha < 1$), we assume that the terminal cash flow, occurring at the end of the project, is αCF_i.

(b) Fuzzy cash flows:
- in case of crisp duration (Section 5):

$$\widetilde{COF}_i\ (i = 0,\dots,n) \quad \text{and} \quad \widetilde{CIF}_i\ (i = 1,\dots,n) \quad \text{or} \quad \widetilde{CF}_i\ (i = 0,\dots,n),$$

where

$$\widetilde{COF}_i \geqslant 0\ (i = 0,\dots,n) \quad \text{and} \quad \widetilde{CIF}_i \geqslant 0\ (i = 1,\dots,n),$$

$$\widetilde{CF}_0 = -\widetilde{COF}_0,\ \widetilde{CF}_i = \widetilde{CIF}_i - \widetilde{COF}_i\ (i = 1,\dots,n).$$

- in case of fuzzy duration (Section 6):

$$\widetilde{COF}_i\ (i = 0,\dots,[n_4]+1) \quad \text{and} \quad \widetilde{CIF}_i\ (i = 1,\dots,[n_4]+1)$$

or

$$\widetilde{CF}_i\ (i = 0,\dots,[n_4]+1),$$

where

$$\widetilde{COF}_i \geqslant 0\ (i = 0,\dots,[n_4]+1) \quad \text{and} \quad \widetilde{CIF}_i \geqslant 0\ (i = 1,\dots,[n_4]+1),$$

$$\widetilde{CF}_0 = -\widetilde{COF}_0, \quad \widetilde{CF}_i = \widetilde{CIF}_i - \widetilde{COF}_i\ (i = 1,\dots,[n_4]+1),$$

where $[n_4]$ denotes the integer part of the biggest real characteristics of the fuzzy number $\widetilde{n}$.

Generally in the paper we assume that if n_4 is integer, then

$$\widetilde{COF}_{n_4+1} = \widetilde{CIF}_{n_4+1} = \widetilde{CF}_{n_4+1} = 0.$$

Let us remark that, contrary to Buckley [4], we do not require the fuzzy cash flows $\widetilde{CF}_i$ $(i = 1,\dots,n)$ to be positive or negative. The 0-levels of these fuzzy numbers may contain zero. It seems that frequently we can encounter a situation when future cash flows will be estimated as being "around zero", so the limitation of Buckley's approach may be significant.

(c) $\widetilde{r}$ – a fuzzy required rate of return of the investment or cost of capital, $\widetilde{r} > 0$.

We assume that the fuzzy rate of return corresponding to the α-part of a year is $\alpha\widetilde{r}$ $(0 < \alpha < 1)$.

One possible interpretation of the fuzzy data may be that the λ-levels of these data are estimates given by different experts with the degree λ $(\lambda \in [0,1])$ of presumption (see [14]). For example, the expert corresponding to the 0-level may be the one who is the least certain about the characteristics of the problem and his estimations are rather wide intervals and that corresponding to the 1-level may be the one who is the most certain (has most information). Of course, the estimations of various experts do not always form a ready fuzzy number, so that they may have to be rediscussed or processed in a certain way. We do not address this problem here.

In Section 2 we presented several capital investment methods, each of them leading to a numerical evaluation of a given crisp project. For a given method symbolised by the letter M and for a given crisp project, defined as presented in Section 2, let us denote this evaluation by $E_M(n, CF_i, r)$.

In the next two sections we will present fuzzy equivalents of the methods from Section 2. For a given crisp method M, let $\widetilde{M}$ denote its fuzzy equivalent.

The result of applying an arbitrary, "fuzzified" method to a given fuzzy project will be a formula or a procedure permitting to determine, for each given $\lambda \in [0,1]$, a possibly small closed interval, denoted by $E^{*\lambda}_{\widetilde{M}}(n, \widetilde{C}F_i, \widetilde{r})$ for crisp duration of the project and by $E^{*\lambda}_{\widetilde{M}}(\widetilde{n}, \widetilde{C}F_i, \widetilde{r})$ for fuzzy duration, such that

$$E^{*\lambda}_{\widetilde{M}}\left(n, \widetilde{C}F_i, \widetilde{r}\right) \supset \{E_M(n, y_i, z) \text{ s.t. } y_i \in (CF_i)^{\lambda} \quad (i = 0,\dots,n),\ z \in r^{\lambda}\}. \tag{1}$$

and

$$E^{*\lambda}_{\widetilde{M}}\left(\widetilde{n}, \widetilde{C}F_i, \widetilde{r}\right) \supset \{E_M(x, y_i, z) \text{ s.t. } x \in n^{\lambda},\ y_i \in (CF_i)^{\lambda} \quad \left(i = 0,\dots,\left[\left(n^{\lambda}\right)_2\right] + 1\right),\ z \in r^{\lambda}\}, \tag{2}$$

where $\left[\left(n^{\lambda}\right)_2\right]$ is the integer part of the upper end of the interval n^{λ}.

In many cases the interval found in the above way will be the smallest one with the above property; in some cases the inclusion will be an obvious equality. Even in those cases, where for the interval found, we will not be able to prove that it is the smallest one with the above property, it will still provide a useful estimation of the set of all possible estimations of the project at a given level λ.

The intervals $E^{*\lambda}_{\widetilde{M}}$ $(\lambda \in [0,1])$ or their set $\widetilde{E}^{*}_{M}$ will form a fuzzy evaluation of the project being considered, although we will not always be able to show that this set of intervals is a fuzzy number according to Definition 3.5 (hence we use a star in the symbol), to give an explicit form of the functions $f^r_1(\lambda), f^r_2(\lambda)$ or even to determine those intervals for all the parameter values at one time. But it will always be possible to determine the intervals $E^{*\lambda}_{\widetilde{M}}$ for selected values of λ. These intervals or their set will constitute a basis for investment decisions (for comparing, rejecting or accepting investment projects). Of course, it would then be necessary to have a method of evaluating an investment project (i.e. of finding its absolute worthiness) and of comparing several investment projects on the basis of their fuzzy evaluation. Numerous methods of ranking fuzzy and interval numbers can be helpful here. We will not discuss this problem here in detail and recommend e.g. [5] or [9] to those who are interested.

In the approach proposed in this paper, for no capital budgeting method it is possible to say that it does not extend to the fuzzy case (what the authors of [4, 17] affirmed about the internal rate of return method). The fuzzy equivalent of any capital budgeting method is simply the possibility for the decision maker to determine a family (indexed by the parameter $\lambda \in [0,1]$) of closed intervals, as small as possible, each one of them containing all possible crisp evaluations (according to the method chosen) of the project when the parameters of the project take on all the values admitted by the estimations at a given level λ. We will not worry about whether this family of intervals is a fuzzy number according to any formal definition, nor about whether the intervals are "too numerous", i.e. contain some values which do not belong to the set of all possible crisp evaluations of the project in question. Sometimes we will not be able yet to present in this paper a method of determining the whole family at one time (because of the lack of corresponding parametric methods) and we will have to be content with the possibility of determining the intervals for any individual chosen values of λ, but it seems this may be already sufficient for practical purposes.

Formulae for the fuzzy evaluations $\widetilde{E}^*_M$, proposed in the following two sections, are obtained simply by determining, for a given λ, a lower and an upper bound for the values of the corresponding crisp evaluations.

We will start by "fuzzifying" the methods from Section 2 for integer n, i.e. we assume at the beginning that the duration of the project is fixed and integer. In the following we will drop the lower index M from the symbols $E^{*\lambda}_{\widetilde{M}}$ and, $\widetilde{E}^*_M$, as it will not cause any misunderstanding.

5. Capital budgeting in the case of fuzzy cash flows and fuzzy rate of return and a crisp, fixed project duration

For each of the methods listed in Section 2, we will give a formula or a procedure permitting to calculate the fuzzy evaluation $\widetilde{E}^*$ or $E^{*\lambda}$ ($\lambda \in [0,1]$) of the given project, according to the concept presented in Section 4. We assume that the project to be evaluated is described by the data specified also in Section 4.

(1) *Revenues per one dollar method*:

$$\widetilde{E}^* = \sum_{i=1}^{n} C\widetilde{IF}_i \Big/ \sum_{i=0}^{n} C\widetilde{OF}_i \quad \text{or} \quad \widetilde{E}^* = \left(\sum_{i=1}^{n} C\widetilde{IF}_i - \sum_{i=0}^{n} C\widetilde{OF}_i \right) \Big/ \sum_{i=0}^{n} C\widetilde{OF}_i,$$

where $\widetilde{E}^*$ is a fuzzy number and an exact formula for calculating it is given by Definition 3.9.

In both cases we obtain the smallest interval fulfilling relation (1)

Let us illustrate the method with the following example:

Example 1.

$$C\widetilde{OF}_0 = (900,\ 1000,\ 1000,\ 1100,\ 100\lambda + 900,\ -100\lambda + 1100),$$

$$C\widetilde{IF}_1 = (90,\ 100,\ 100,\ 110,\ 10\lambda + 90,\ -10\lambda + 110),$$

$$C\widetilde{IF}_2 = (180,\ 200,\ 200,\ 220,\ 20\lambda + 180,\ -20\lambda + 220),$$

$$C\widetilde{IF}_3 = (1800,\ 2000,\ 2000,\ 2200,\ 200\lambda + 1800,\ -200\lambda + 2200),$$

$$n = 3, \quad \widetilde{r} = (0.09, 0.1, 0.1, 0.11, 0.01\lambda + 0.09, -0.01\lambda + 0.11).$$

According to the first formula we obtain (in this method the information about the fuzzy rate of return is not needed):

$$\widetilde{E}^* = \left(1.88,\ 2.3,\ 2.3,\ 2.81,\ \frac{230\lambda + 2070}{-100\lambda + 1100},\ \frac{-230\lambda + 2530}{100\lambda + 900} \right).$$

The calculations according to the second formula are analogous.

(2) *Payback period method*: For each $\lambda \in [0,1]$, $E^{*\lambda}$ will be calculated according to the following formula:

$$E^{*\lambda} = \left[\min_{k=1,\dots,n} \left\{ k\colon \sum_{i=0}^{k} \left(CF_i^{\lambda}\right)_2 \geqslant 0, \infty \right\}, \min_{k=1,\dots,n} \left\{ k\colon \sum_{i=0}^{k} \left(CF_i^{\lambda}\right)_1 \geqslant 0, \infty \right\} \right]$$

or

$$E^{*\lambda} = \left[\min_{k=1,\dots,n} \left\{ k\colon \sum_{i=0}^{k} \frac{\left(CF_i^{\lambda}\right)_2}{\left(1+r_x^{\lambda}\right)^i} \geqslant 0, \infty \right\}, \min_{k=1,\dots,n} \left\{ k\colon \sum_{i=0}^{k} \frac{\left(CF_i^{\lambda}\right)_1}{\left(1+r_y^{\lambda}\right)^i} \geqslant 0, \infty \right\} \right]$$

where

$$r_x = \begin{cases} r_1 & \text{if the numerator of the fraction is non-negative,} \\ r_2 & \text{if the numerator of the fraction is negative,} \end{cases}$$

$$r_y = \begin{cases} r_2 & \text{if the numerator of the fraction is non-negative,} \\ r_1 & \text{if the numerator of the fraction is negative.} \end{cases}$$

We will use the following example to illustrate this method.

Example 2.

$$\widetilde{COF}_0 = (1500,\ 1500,\ 1500,\ 1500,\ 1500,\ 1500),$$

$$\widetilde{CIF}_1 = \widetilde{CIF}_2 = \widetilde{CIF}_3 = (500,\ 1000,\ 1000,\ 1500,\ 500\lambda + 500, -500\lambda + 1500),$$

$$n = 3, \quad \tilde{r} = (0.1,\ 0.1,\ 0.1,\ 0.1,\ 0.1,\ 0.1).$$

Given below are the results for some values of λ according to the first formula:

$$E^{*0} = [1,3]; \quad E^{*0.25} = [2,3]; \quad E^{*0.5} = [2,2]; \quad E^{*1} = [2,2].$$

In order to obtain the respective values for the second formula we used, like in further examples, the spreadsheet EXCEL:

$$E^{*0} = [2,4]; \quad E^{*0.25} = [2,3]; \quad E^{*0.75} = [2,3]; \quad E^{*1} = [2,2].$$

Before we present a fuzzy version of the following methods from Section 2, let us make the following remark: In the fuzzy version of most of these methods it is necessary to ensure that the future value of a fuzzy amount $\tilde{A}$ in n periods, which is – according to the normal understanding of how this future value is created in the case when the interests are calculated at the end of each year and added to the capital – equal to $\tilde{A} + \tilde{A}\cdot\tilde{r} + \left(\tilde{A} + \tilde{A}\cdot\tilde{r}\right)\cdot\tilde{r} + \cdots + \left(\sum_{i=1}^{n} \tilde{A}\cdot\tilde{r}^{i-1}\right)\cdot\tilde{r}$, can also be expressed by the more convenient formula $\tilde{A}\cdot(1+\tilde{r})^n$. This equality, obvious in the crisp case, has been justified in the fuzzy case by Buckley [4], but only for $\tilde{A} \geqslant 0$. Let us just remark that by repeating the same reasoning and using Lemma 3.3, the equality can be justified for any $\tilde{A}$. That is why we can simply extend the corresponding formulae for the evaluations E from the crisp case to the fuzzy one:

(3) *Net present value*:

$$\tilde{E}^* = N\tilde{P}V = \sum_{i=0}^{n} \frac{\tilde{C}F_i}{(1+\tilde{r})^i},$$

where $\tilde{E}^*$ is a fuzzy number determined by Definition 3.9 (a natural power of a fuzzy number is obviously defined as a repeated multiplication).

Example 3. Here we use the project from Example 1. The fuzzy net present value of the project is as follows:

$$\widetilde{NPV} = \begin{pmatrix} 443,\ 759,\ 759,\ 1085, \\ 100\lambda - 1100 + \dfrac{10\lambda + 90}{-0.01\lambda + 1.11} + \dfrac{20\lambda + 180}{(-0.01\lambda + 1.11)^2} + \dfrac{200\lambda + 1800}{(-0.01\lambda + 1.11)^3}, \\ -100\lambda - 900 + \dfrac{-10\lambda + 110}{0.01\lambda + 1.09} + \dfrac{-20\lambda + 220}{(0.01\lambda + 1.09)^2} + \dfrac{-200\lambda + 2200}{(0.01\lambda + 1.09)^3} \end{pmatrix}.$$

(4) *Net future value*:

$$\widetilde{E}^*(m) = \sum_{i=0}^{n} \widetilde{CF}_i (1 + \widetilde{r})^{m-i} \quad \text{for any integer } m \geqslant n,$$

where $\widetilde{E}^*(m)$ is a fuzzy number determined by Definition 3.9.

Example 4. Here again we use the project from Example 1 and assume $m = n = 3$:

$$\widetilde{E}^*(3) = \begin{pmatrix} 599,\ 1010,\ 1010,\ 1414 \\ (100\lambda - 1100)(-0.01\lambda + 1.11)^3 + (10\lambda + 90)(0.01\lambda + 1.09)^2 \\ \quad +(20\lambda + 180)(0.01\lambda + 1.09) + 200\lambda + 1800, \\ (-100\lambda - 900)(-0.01\lambda + 1.09)^3 + (-10\lambda + 110)(0.01\lambda + 1.11)^2 \\ \quad +(-20\lambda + 220)(0.01\lambda + 1.11) - 200\lambda + 2200 \end{pmatrix}$$

(5) *Method of utility of the net present value*:

$$\widetilde{E}^* = 1 - \frac{1}{\mathrm{e}^{\tilde{b}_U \cdot N\tilde{P}V}},$$

where $\widetilde{b}_U$ $\left(\widetilde{b}_U > 0\right)$ is the risk aversion constant of the decision maker and $\widetilde{NPV}$ is calculated as indicated in point 3 of this section. $\widetilde{E}^*$ is a fuzzy number defined by Definitions 3.8(d) and 3.9.

(6) *The internal rate of return method*: Analogously to the crisp case, we assume that $\widetilde{CF}_i \geqslant 0$ for $i = 1, \ldots, n$. For each $\lambda \in [0, 1]$, $E^{*\lambda}$ will be calculated according to the following formula:

$$E^{*\lambda} = \left[\left(E^{*\lambda}\right)_1, \left(E^{*\lambda}\right)_2\right],$$

where $\left(E^{*\lambda}\right)_1, \left(E^{*\lambda}\right)_2$ are, respectively, solutions of the following equations $\left(\left(CF_i^{\lambda}\right)_1\right.$ and $\left(CF_i^{\lambda}\right)_1$ denote the lower and the upper end of the interval $\left.CF_i^{\lambda}\right)$:

$$\left(CF_0^{\lambda}\right)_1 + \sum_{i=1}^{n} \frac{\left(CF_i^{\lambda}\right)_1}{\left(1 + \left(E^{*\lambda}\right)_1\right)^i} = 0, \qquad \left(CF_0^{\lambda}\right)_2 + \sum_{i=1}^{n} \frac{\left(CF_i^{\lambda}\right)_2}{\left(1 + \left(E^{*\lambda}\right)_2\right)^i} = 0.$$

We obtain here the smallest interval fulfilling relation (1).

Example 5. For the project from Example 1 we obtain the following intervals for three selected values of λ:

$$E^{*0} = [25\%, 45\%], \quad E^{*0.5} = [30\%, 40\%], \quad E^{*1} = [35\%, 35\%].$$

(7) *The modified internal rate of return method*: Analogous to the previous case (but with no additional assumptions about the flows $\widetilde{CF}_i$), for each $\lambda \in [0,1]$ we define

$$E^{*\lambda} = \left[\left(E^{*\lambda}\right)_1, \left(E^{*\lambda}\right)_2\right],$$

where $\left(E^{*\lambda}\right)_1, \left(E^{*\lambda}\right)_2$ are, respectively, solutions of the following equations:

$$\sum_{i=0}^{n} \frac{\left(COF_i^{\lambda}\right)_2}{\left(1+r_1^{\lambda}\right)^i} = \frac{\sum_{i=0}^{n}\left(CIF_i^{\lambda}\right)_1\left(1+r_1^{\lambda}\right)^{n-i}}{\left(1+\left(E^{*\lambda}\right)_1\right)^n}, \qquad \sum_{i=0}^{n} \frac{\left(COF_i^{\lambda}\right)_1}{\left(1+r_2^{\lambda}\right)^i} = \frac{\sum_{i=0}^{n}\left(CIF_i^{\lambda}\right)_2\left(1+r_2^{\lambda}\right)^{n-i}}{\left(1+\left(E^{*\lambda}\right)_2\right)^n}.$$

We obtain here the smallest interval fulfilling relation (1).

Example 6. In this example we use again the project from Example 1:

$$E^{*0} = [24\%, 42\%], \quad E^{*0.5} = [28\%, 37\%], \quad E^{*1} = [33\%, 33\%].$$

6. Capital budgeting in the case of fuzzy cash flows, fuzzy rate of return and fuzzy project duration times

In this section we additionally assume that the duration of the project is a fuzzy number $\tilde{n}$. Like in the previous section, for each method from Section 2 we will present its fuzzy equivalent.

(1) *Revenues per one dollar method*:

$$\widetilde{E}^* = \sum_{i=1}^{\tilde{n}} \widetilde{CIF}_i \Bigg/ \sum_{i=0}^{\tilde{n}} \widetilde{COF}_i \quad \text{or} \quad \widetilde{E}^* = \left(\sum_{i=1}^{\tilde{n}} \widetilde{CIF}_i - \sum_{i=0}^{\tilde{n}} \widetilde{COF}_i\right) \Bigg/ \sum_{i=0}^{\tilde{n}} \widetilde{COF}_i,$$

where the sums in the above formulae are calculated according to Definition 3.10(a), and the division according to Definition 3.7(b). In both cases we obtain the smallest interval fulfilling relation (2).

Let us illustrate the method with the following example.

Example 7. Let us consider the same project as in Example 1 with a fuzzy duration time

$$\tilde{n} = (2,\ 2.5,\ 2.5,\ 3,\ 0.5\lambda + 2, -0.5\lambda + 3).$$

According to the first formula, we obtain for three selected values of λ:

$$E^{*0} = [0.245,\ 2.8], \quad E^{*0.5} = [0.317,\ 1.99], \quad E^{*1} = [1.3,\ 1.3].$$

The calculations according to the second formula are analogous.

(2) *Payback period method*: In the case of a fuzzy duration, the calculations will be carried out in the following way:

$$E^{*\lambda} = [C_1(\lambda), C_2(\lambda)],$$

where

$$C_1(\lambda) = \min_{k=1,2,\ldots,[(n^{\lambda})_2]+1} \left\{k\colon \sum_{i=0}^{k}\left(CF_i^{\lambda}\right)_2' \geqslant 0, \infty\right\},$$

where

$$\left(CF_i^{\lambda}\right)_2' = \left(CF_i^{\lambda}\right)_2 \qquad \text{for } k = 0,1,\ldots,\left[\left(n^{\lambda}\right)_2\right],$$

and

$$\left(CF^{\lambda}_{[(n^{\lambda})_2]+1}\right)'_2 = \left((n^{\lambda})_2 - \left[(n^{\lambda})_2\right]\right)\left(CF^{\lambda}_{[(n^{\lambda})_2]+1}\right)_2,$$

$$C_2(\lambda) = \begin{cases} \infty \text{ if } \sum_{i=0}^{[(n^{\lambda})_1]}\left(CF^{\lambda}_i\right)_1 + \left((n^{\lambda})_1 - \left[(n^{\lambda})_1\right]\right)\left(CF^{\lambda}_{[(n^{\lambda})_1]+1}\right)_1 < 0, \\ \min_{(k=1,\ldots,[(n^{\lambda})_1]+1)}\left\{k\colon \sum_{1=0}^{k}\left(CF^{\lambda}_i\right)_1 \geqslant 0, \infty\right\} \text{ elsewhere} \end{cases}$$

or

$$C_1(\lambda) = \min_{k=1,2,\ldots,[(n^{\lambda})_2]+1}\{k\colon S_k \geqslant 0, \infty\},$$

where

$$S_k = \sum_{i=0}^{k}\frac{\left(CF^{\lambda}_i\right)_2}{\left(1+r^x_{\lambda}\right)^i} \qquad \text{for } k = 1,\ldots,\left[(n^{\lambda})_2\right],$$

$$S_{[(n^{\lambda})_2]+1} = \sum_{i=0}^{[(n^{\lambda})_2]}\frac{\left(CF^{\lambda}_i\right)_2}{\left(1+r^{\lambda}_x\right)^i} + \frac{\left((n^{\lambda})_2 - \left[(n^{\lambda})_2\right]\right)\left(CF^{\lambda}_{[(n^{\lambda})_2]+1}\right)_2}{\left(1+r^{\lambda}_x\right)^{[(n^{\lambda})_2]}\left(1+\left((n^{\lambda})_2 - \left[(n^{\lambda})_2\right]\right)r^{\lambda}_x\right)}$$

and

$$r^{\lambda}_x = \begin{cases} r^{\lambda}_1 & \text{if the numerator of the fraction is non-negative,} \\ r^{\lambda}_2 & \text{elsewhere,} \end{cases}$$

$$C_2(\lambda) = \begin{cases} \infty \text{ if } \sum_{i=0}^{[(n^{\lambda})_1]}\frac{\left(CF^{\lambda}_i\right)_1}{\left(1+r^x_{\lambda}\right)^i} + \frac{\left((n^{\lambda})_1 - \left[(n^{\lambda})_1\right]\right)\left(CF^{\lambda}_{[(n^{\lambda})_1]+1}\right)_1}{\left(1+r^x_{\lambda}\right)^{[(n^{\lambda})_1]}\left(1+\left((n^{\lambda})_1 - \left[(n^{\lambda})_1\right]\right)r^x_{\lambda}\right)^i} < 0 \\ \min_{(k=1,\ldots,[(n^{\lambda})_1]+1)}\left\{k\colon \sum_{i=0}^{k}\frac{\left(CF^{\lambda}_i\right)_1}{\left(1+r^x_{\lambda}\right)^i} \geqslant 0, \infty\right\} \text{ elsewhere} \end{cases}$$

where

$$r^{\lambda}_x = \begin{cases} r^{\lambda}_1 & \text{if the numerator of the fraction is non-negative,} \\ r^{\lambda}_2 & \text{elsewhere.} \end{cases}$$

We will illustrate the method with the following example:

Example 8.

$$C\widetilde{OF}_0 = (1000,\ 1000,\ 1000,\ 1000,\ 1000,\ 1000),$$

$$C\widetilde{IF}_1 = C\widetilde{IF}_2 = C\widetilde{IF}_3 = (500,\ 1000,\ 1000,\ 1500,\ 500\lambda+500, -500\lambda+1500),$$

$$n = (1,2,2,3,\lambda+1,-\lambda+3), \quad \widetilde{r} = (0.1,\ 0.1,\ 0.1,\ 0.1,\ 0.1,\ 0.1).$$

Here are the results for some values of λ according to the first fuzzy payback period method:

$$E^{*0} = [1,\ \infty]; \quad E^{*0.5} = [1,\ 2]; \quad E^{*1} = [1,\ 1].$$

For the second fuzzy payback period method and the same example the results are as follows:

$$E^{*0} = [1, \infty]; \quad E^{*0.5} = [1, 2]; \quad E^{*1} = [2, 2].$$

Let us pass on to the next method:

(3) *Net present value*:

$$\widetilde{E}^* = N\widetilde{P}V = \bigcup_{i=0}^{\tilde{n}} \frac{\widetilde{C}F_i}{(1+\widetilde{r})^i},$$

where $\widetilde{E}^*$ is determined by Definition 3.10(b).

Example 9. Here we use the project from Example 7. The λ-levels of the fuzzy present value for some values of λ are as follows:

$$E^{*0} = [-834.44, 1084.89], \quad E^{*0.5} = [-429.35, 547.23], \quad E^{*1} = [43.29, 43.29].$$

(4) *Net future value*:

$$\widetilde{E}^*(m) = \bigcup_{i=0}^{\tilde{n}} \widetilde{C}F_i(1+\widetilde{r})^{m-i}$$

for any integer m such that, where $m \geqslant n_4$ if n_4 is integer and $m \geqslant [n_4]+1$ otherwise, where $\widetilde{E}^*(m)$ is determined by Definition 3.10(c).

Example 10. Here again we use the project from Example 7 and assume $m = 3$. The λ-levels of the fuzzy future value for some values of λ are as follows:

$$E^{*0} = [-1160, 1414.2], \quad E^{*0.5} = [-581.9, 729.31], \quad E^{*1} = [60, 60].$$

(5) *Method of utility of the net present value*:

$$E^{*\lambda} = 1 - \frac{1}{e^{b_U^\lambda \cdot NPV^{*\lambda}}} \quad (\lambda \in [0,1]),$$

where b_U^λ $\left(b_U^\lambda > 0\right)$ are the λ-levels of the risk aversion constant of the decision maker and $NPV^{*\lambda}$ is calculated as indicated in point 3. $E^{*\lambda}$ is calculated according to Definition 3.1(c).

(6) *The internal rate of return method*: We assume that $\widetilde{C}F_i \geqslant 0$ for $i = 1, \ldots, n$. For each $\lambda \in [0,1]$ we define the internal rate of return as the following interval:

$$E^{*\lambda} = \left[\left(E^{*\lambda}\right)_1, \left(E^{*\lambda}\right)_2\right],$$

where $\left(E^{*\lambda}\right)_1, \left(E^{*\lambda}\right)_2$ are, respectively, the solutions of the following equations:

$$\left(CF_0^\lambda\right)_1 + \sum_{i=1}^{[(n^\lambda)_1]} \frac{\left(CF_i^\lambda\right)_1}{\left(1+\left(E^{*\lambda}\right)_1\right)^i} + \frac{\left(\left(n^\lambda\right)_1 - \left[\left(n^\lambda\right)_1\right]\right)\left(CF^\lambda_{[(n^\lambda)_1]}\right)_1}{\left(1+\left(E^{*\lambda}\right)_1\right)^{[(n^\lambda)_1]}\left(1+\left(\left(n^\lambda\right)_1 - \left[\left(n^\lambda\right)_1\right]\right)\left(E^{*\lambda}\right)_1\right)} = 0,$$

$$\left(CF_0^\lambda\right)_2 + \sum_{i=1}^{[(n^\lambda)_2]} \frac{\left(CF_i^\lambda\right)_2}{\left(1+\left(E^{*\lambda}\right)_2\right)^i} + \frac{\left(\left(n^\lambda\right)_2 - \left[\left(n^\lambda\right)_2\right]\right)\left(CF^\lambda_{[(n^\lambda)_2]}\right)_2}{\left(1+\left(E^{*\lambda}\right)_2\right)^{[(n^\lambda)_2]}\left(1+\left(\left(n^\lambda\right)_2 - \left[\left(n^\lambda\right)_2\right]\right)\left(E^{*\lambda}\right)_2\right)} = 0.$$

We obtain in this way the smallest interval fulfilling relation (2).

Example 11. For the project from Example 7 we obtain the following intervals for three selected values of λ:

$$E^{*0} = [-55\%, 45\%], \quad E^{*0.5} = [-15\%, 30\%], \quad E^{*1} = [12\%, 12\%].$$

A negative internal rate of return means that it is possible in the given project to lose money.

(7) *The modified internal rate of return method*: Here we do not need the assumption $\widetilde{C}F_i \geqslant 0$ for $i = 1, \dots, n$: the cash flows can also be negative. For each $\lambda \in [0,1]$, we define the modified internal rate of return as the following interval:

$$E^{*\lambda} = \left[\left(E^{*\lambda}\right)_1, \left(E^{*\lambda}\right)_2\right],$$

where $\left(E^{*\lambda}\right)_1, \left(E^{*\lambda}\right)_2$ are, respectively, solutions of the following equations:

$$\sum_{i=0}^{[(n^\lambda)_2]} \frac{\left(COF_i^\lambda\right)_2}{\left(1+r_1^\lambda\right)^i} + \frac{\left(\left(n^\lambda\right)_2 - \left[\left(n^\lambda\right)_2\right]\right)\left(COF^\lambda_{[(n^\lambda)_1]+1}\right)_2}{\left(1+r_1^\lambda\right)^{[(n^\lambda)_2]}\left(1+\left(\left(n^\lambda\right)_2 - \left[\left(n^\lambda\right)_2\right]\right) r_1^\lambda\right)}$$

$$= \frac{\left\{\sum_{i=0}^{[(n^\lambda)_1]} \left(CIF_i^\lambda\right)_1 \left(1+r_1^\lambda\right)^{[(n^\lambda)_1]-i}\right\} \cdot \left(1+\left(\left(n^\lambda\right)_1 - \left[\left(n^\lambda\right)_1\right]\right) r_1^\lambda\right) + \left(\left(n^\lambda\right)_1 - \left[\left(n^\lambda\right)_1\right]\right)\left(CIF^\lambda_{[(n^\lambda)_1]+1}\right)_1}{\left(1+\left(E^{*\lambda}\right)_1\right)^{[(n^\lambda)_2]}\left(1+\left(\left(n^\lambda\right)_2 - \left[\left(n^\lambda\right)_2\right]\right)\left(E^{*\lambda}\right)_1\right)},$$

$$\sum_{i=0}^{[(n^\lambda)_1]} \frac{\left(COF_i^\lambda\right)_1}{\left(1+r_2^\lambda\right)^i} + \frac{\left(\left(n^\lambda\right)_1 - \left[\left(n^\lambda\right)_1\right]\right)\left(COF^\lambda_{[(n^\lambda)_1]+1}\right)_1}{\left(1+r_2^\lambda\right)^{[(n^\lambda)_1]}\left(1+\left(\left(n^\lambda\right)_1 - \left[\left(n^\lambda\right)_1\right]\right) r_2^\lambda\right)}$$

$$= \frac{\left\{\sum_{i=0}^{[(n^\lambda)_2]} \left(CIF_i^\lambda\right)_2 \left(1+r_2^\lambda\right)^{[(n^\lambda)_2]-i}\right\} \cdot \left(1+\left(\left(n^\lambda\right)_2 - \left[\left(n^\lambda\right)_2\right]\right) r_2^\lambda\right) + \left(\left(n^\lambda\right)_2 - \left[\left(n^\lambda\right)_2\right]\right)\left(CIF^\lambda_{[(n^\lambda)_2]+1}\right)_2}{\left(1+\left(E^{*\lambda}\right)_2\right)^{[(n^\lambda)_1]}\left(1+\left(\left(n^\lambda\right)_1 - \left[\left(n^\lambda\right)_1\right]\right)\left(E^{*\lambda}\right)_2\right)}.$$

Example 12. In this example we use again the project from Example 7. We proceed in the same way as in Example 11 and obtain the following values:

$$E^{*0} = [-32\%, 69\%], \quad E^{*0.5} = [-11\%, 36\%], \quad E^{*1} = [11\%, 11\%].$$

7. Conclusions

In this paper we have proposed or generalised fuzzy equivalents for all the methods of evaluating and comparing investment projects which are usually presented in the literature and used in practice (in case where there is no capital rationing). These fuzzy equivalents allow to evaluate projects whose cash flows and/or duration are not known precisely, but given in the form of fuzzy numbers. All the computations can be carried out by means of a financial calculator, spreadsheet or any other software with standard financial/mathematical functions. For each project we obtain then a set of closed intervals, constituting its fuzzy evaluation. Each of these intervals may correspond for example to the estimation of the parameters of the project given by one specific expert.

In an actual comparison between various projects, methods of comparing fuzzy numbers or intervals would have to be used. These methods are widely discussed in the literature and are not treated in this paper.

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