# Helen Liang Memorial Secondary School (Shatin)

## Half-yearly Examination 2000/2001

## PURE MATHEMATICS

Date: 11-1-2001

Max. Marks: 100
Time allowed: 2 hours (8:45 to 10:45 a.m.)

Secondary 6S    Name: ____________________    Class No.: ______

1. This paper consists of Section A and Section B.
2. Answer ALL questions in Section A.
3. Answer ALL questions in Section B.

### FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$2\sin A \cos B = \sin(A+B) + \sin(A-B)$$
$$2\cos A \cos B = \cos(A+B) + \cos(A-B)$$
$$2\sin A \sin B = \cos(A-B) - \cos(A+B)$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$
$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

**SECTION A (40 marks)**
**Answer ALL questions in this section.**

1. Solve the inequality
$$\frac{x-1}{(x-2)(2x+1)} \leq 0.$$

2. Let $n$ be a positive integer. Let
$$(1+x)^n = \sum_{r=0}^{n} C_r^n x^r,$$
and
$$P_n = \prod_{r=0}^{n} C_r^n.$$
Show that
$$\frac{P_{n+1}}{P_n} = \frac{(n+1)^n}{n!}.$$

3. By considering the numbers $\frac{1}{1\cdot 2}, \frac{1}{2\cdot 3}, \ldots, \frac{1}{n(n+1)}$, show that
$$(n!)^2 > (n+1)^{n-1}.$$

4. (a) Let $k$ and $n$ be positive integers. If $k > 1$, when $(1+k)^n$ is divided by $k$, show that the remainder is 1.

   (b) If today is Tuesday, what day of the week is $8^{96}$ days after?

5. Given that $f(x)$ is odd and $g(x)$ is even. Determine $f(x)$ and $g(x)$ such that
$$f(x) + g(x) = 2001x\sqrt{5-x^2} + x^{2000}.$$

**SECTION B (60 marks)**
**Answer ALL questions in this section. Each question carries 20 marks.**

6. Let $\alpha, \beta$, and $\gamma$ be the roots of
$$x^3 - 2x^2 + x - 3 = 0.$$
Form a cubic equation whose roots are $\alpha^3, \beta^3, \gamma^3$.

7. (a) Prove by mathematical induction that
$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$
for all positive integers $n$. Hence, deduce the sums
   (i) $\frac{1+2+3+\cdots+n}{2}$;
   (ii) $\frac{2+3+\cdots+(n+1)}{2}$.

   (b) Show that, for any positive integer $n$,
   (i) $\sqrt{n(n+1)} < \frac{n+(n+1)}{2}$;
   (ii) $\sqrt{n(n+1)} > n$.

   (c) Using the results in (a), (b), show that, for any positive integer $n$,
$$\frac{n(n+1)}{2} < \sqrt{1 \cdot 2} + \sqrt{2 \cdot 3} + \cdots + \sqrt{n(n+1)} < \frac{(n+1)^2}{2}.$$

8. (a) Let $\alpha, \beta, \gamma$ be the roots of the equation $x^3 + 3ax^2 + bx + c = 0$.
   (i) If $\alpha + a, \beta + a, \gamma + a$ are roots of the equation $y^3 + py + q = 0$, show that
$$p = b - 3a^2$$
and
$$q = c - ab + 2a^3.$$

   (ii) Let $y = z - \frac{p}{3z}$, show that
$$z^6 + qz^3 - \frac{1}{27}p^3 = 0.$$

   (b) Hence solve the equation
$$x^3 - 6x^2 + 3x - 18 = 0.$$

**END OF PAPER**