Solution to Differential Calculus Test

1. (a) domain $f = \mathbf{R}$.
   (b) $f(-x) = (-x+1)(-x-3)^3$. It's not even nor odd.
   (c) $(0,-27)$, $(-1,0)$, $(3,0)$
   (d) There is no vertical asymptote. Also, since $\lim_{x\to\infty} f(x) = \infty$, there's no horizontal asymptote. Furthermore, $\lim_{x\to\infty} \frac{f(x)}{x} = \infty$, thus, there's no oblique asymptote.
   (e) Using logarithmic differentiation,

$$\begin{aligned}
\ln f(x) &= \ln(x+1) + 3\ln(x-3) \\
\frac{f'(x)}{f(x)} &= \frac{1}{x+1} + \frac{3}{x-3} \\
f'(x) &= \frac{f(x)}{x+1} + \frac{3f(x)}{x-3} \\
&= (x-3)^3 + 3(x+1)(x-3)^2 \\
&= (x-3)^2(x-3+3x+3) \\
&= (x-3)^2 \cdot (4x) \\
\ln f'(x) &= 2\ln(x-3) + \ln 4x \\
\frac{f''(x)}{f'(x)} &= \frac{2}{x-3} + \frac{1}{x} \\
f''(x) &= \frac{2f'(x)}{x-3} + \frac{f'(x)}{x} \\
&= 2(x-3)\cdot 4x + 4(x-3)^2 \\
&= 4(x-3)(2x+x-3) \\
&= 12(x-3)(x-1)
\end{aligned}$$

   (f) Critical points are $x = 3$ and $x = 0$. It is found that $f'(x) > 0$ when $x > 3$ or $0 < x < 3$, and $f'(x) < 0$ when $x < 0$.
   Possible inflexion points are $x = 1$ and $x = 3$. It is found that $f''(x) > 0$ when $x < 1$ or $x > 3$, and $f''(x) < 0$ when $1 < x < 3$.
   (g) $(0,-27)$ is a relative minimum.
   (h) $(1,-16)$ and $(3,0)$ are inflexion points.
   (i) See separate diagram.
2. (a)

$$\begin{aligned}
f'(x) &= -1 + \frac{2x}{2!} - \frac{3x^2}{3!} \\
&= -1 + x - \frac{x^2}{2} \\
&= -\frac{1}{2}(x^2 - 2x + 2) \\
&= -\frac{1}{2}((x-1)^2 + 1) \\
&< 0 \text{ for every } x
\end{aligned}$$

   Hence f is strictly decreasing.
   f is continuous because f is a polynomial. $f(0) = 1$, $f(2) = 1 - 2 + \frac{4}{2} - \frac{8}{6} = -\frac{1}{3}$. By the Intermediate Value Theorem, f has a root between 0 and 2, say α.
   If there's another root, say β, and say $\alpha < \beta$, then $f(\alpha) > f(\beta)$ which is a contradiction (since we assumed $f(\alpha) = f(\beta) = 0$).
   Thus, there is only one root.

(b)

$$
\begin{aligned}
g(x) &= f(x) + \frac{x^4}{4!} \\
g'(x) &= f'(x) + \frac{4x^3}{24} = f'(x) + \frac{x^3}{6} \\
g'(x) &= 0 \\
\text{Thus, } \frac{x^3}{6} &= f'(x) \\
\frac{x^3}{6} &= -1 + x - \frac{x^2}{2} \\
0 &= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} = f(x)
\end{aligned}
$$

Since $f(\alpha) = 0$, thus, $x = \alpha$ is a critical point for g.
When $x < \alpha$, $g' < 0$, when $x > \alpha$, $g' > 0$. Thus $x = \alpha$ is an absolute minimum.
Also, $g(\alpha) = f(\alpha) + \frac{\alpha^4}{4} = \frac{\alpha^4}{4} > 0$. Thus, $g(x) > 0$ for every x. Hence, g does not have a root.

3. (a) This is of the form $\frac{0}{0}$:

$$
\begin{aligned}
\lim_{x\to 0} \frac{a^x - 1}{x} &= \lim_{x\to 0} \frac{\frac{d}{dx}(e^{x \ln a} - 1)}{1} \\
&= \lim_{x\to 0} \ln a \cdot a^x \\
&= \ln a
\end{aligned}
$$

(b) This is of the form $\infty - \infty$. So we need to 'pull' out something to make it into the form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

$$
\begin{aligned}
\lim_{x\to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right) &= \lim_{x\to 1} \frac{1}{\ln x} \left(\frac{x \ln x}{x-1} - 1 \right) \\
&= \lim_{x\to 1} \frac{\frac{x \ln x - x + 1}{x-1}}{\ln x} \\
&= \lim_{x\to 1} \frac{x \ln x - x + 1}{(x-1)\ln x} \text{ this is of the form } \frac{0}{0} \\
&= \lim_{x\to 1} \frac{\ln x + 1 - 1}{\ln x + 1 - \frac{1}{x}} \text{ apply one more time} \\
&= \lim_{x\to 1} \frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}} \\
&= \frac{1}{1+1} \\
&= \frac{1}{2}
\end{aligned}
$$

(c) This is of the form 1^∞.

$$
\begin{aligned}
\lim_{x\to 1} x^{\frac{1}{1-x}} &= \lim_{x\to 1} e^{\ln x^{\frac{1}{1-x}}} \\
&= \lim_{x\to 1} e^{\frac{\ln x}{1-x}} \\
&= e^{\lim_{x\to 1} \frac{\ln x}{1-x}} \\
&= e^{\lim_{x\to 1} \frac{\frac{1}{x}}{-1}} \\
&= e^{-1}
\end{aligned}
$$