Helen Liang Memorial Secondary School (Shatin)

Annual Examination 2000/2001

PURE MATHEMATICS

Date: 21-6-2001

Max. Marks: 100

Time allowed: 3 hours (8:45 to 11:45 a.m.)

Secondary 6S Name: ______________________ Class No.: ____

1. This paper consists of Section A and Section B.
2. Answer ALL questions in Section A.
3. Answer any FOUR questions in Section B.

FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$
$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$
$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$
$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$
$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$
$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$
$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$
$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

SECTION A (40 marks)
Answer ALL questions in this section.

1. Evaluate
$$\lim_{x\to\infty}\left(\frac{\ln x - 1}{e^x}\right).$$

   (4 marks)

2. Evaluate
$$\lim_{x\to 0}\frac{\int_{2x}^{x^2} x\sqrt{1+\sin t}\,dt}{\sin x}.$$

   (6 marks)

3. Let
$$f(x)=\begin{cases} x\cos(\frac{1}{x}) & \text{for } x\neq 0 \\ 0 & \text{for } x=0 \end{cases}.$$
   Show that $f(x)$ is continuous at $x=0$ but not differentiable at $x=0$.

   (7 marks)

4. Evaluate
$$\lim_{n\to\infty}\frac{\sin e^n}{n^2}.$$

   (4 marks)

5. Evaluate
$$\int \frac{dx}{3+\sin^2 x}.$$

   (7 marks)

6. Evaluate
$$\int_0^{\frac{\pi}{4}} \sec^{10} x dx.$$

   (6 marks)

7. Evaluate
$$\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\cdots+\frac{n-1}{n^2}\right).$$

   (6 marks)

SECTION B (60 marks)

Answer any FOUR questions in this section. Each question carries 15 marks.

8. Let

$$f(x) = e^{2x-x^2}.$$

(a) Find $f'(x)$ and $f''(x)$. (2 marks)

(b) Determine the values of x for each of the following cases:

(i) $f'(x) > 0$,

(ii) $f'(x) < 0$,

(iii) $f''(x) > 0$,

(iv) $f''(x) < 0$.

(4 marks)

(c) Find all relative extrema and inflexion points of $f(x)$. (3 marks)

(d) Find all asymptotes of the graph of $f(x)$. (3 marks)

(e) Sketch the graph of $f(x)$. (3 marks)

9. Let

$$I_n = \int_0^{2a} x^n \sqrt{2ax - x^2} dx,$$

where n is any positive integer and a is any positive real number.

(a) By differentiating $x^{n+\frac{1}{2}}(2a - x)^{\frac{3}{2}}$, or otherwise, show that

$$I_n = \frac{2n+1}{n+2} a I_{n-1}.$$

(4 marks)

(b) Hence, express I_n in terms of I_0. (2 marks)

(c) (i) Show that $I_0 = \frac{a^2\pi}{2}$.

(ii) Hence, prove that

$$I_n = \frac{(2n+1)!\pi a^{n+2}}{n!(n+2)!2^n}.$$

(6 marks)

(d) Evaluate

$$\int_0^1 x^{\frac{11}{2}} \sqrt{1-x} dx.$$

(3 marks)

10. Let $a_1 = 1$, $a_2 = 1$, $a_{n+2} = a_{n+1} + a_n, n = 1, 2, 3, \ldots$.

(a) Two sequences of real numbers $\{x_n\}$ and $\{y_n\}$ are defined as follows:

$$x_n = \frac{a_{2n+1}}{a_{2n} + a_{2n+1}}, y_n = \frac{a_{2n}}{a_{2n+1}}, n = 1, 2, 3, \ldots$$

For $n = 1, 2, 3, \ldots$, prove that

(i) $a_{n+1}^2 - a_{n+1}a_n - a_n^2 = (-1)^n$.

(ii) $x_n > x_{n+1} > y_{n+1} > y_n$

(7 marks)

(b) Prove that both sequences $\{x_n\}$ and $\{y_n\}$ are convergent and have the same limit. (4 marks)

(c) Find the common limit of $\{x_n\}$ and $\{y_n\}$. (4 marks)

11. (a) Let $a > 0$ and k be a positive integer. Show that

(i) $\int_0^a x^{2k+1}e^{-x^2}dx = -\frac{a^{2k}}{2e^{a^2}} + k\int_0^a x^{2k-1}e^{-x^2}dx$.

(ii) $\lim_{x\to+\infty}\frac{x^{2k}}{e^{x^2}} = 0$.

(4 marks)

(b) Prove that

$$\int_0^\infty xe^{-x^2}dx = \frac{1}{2}.$$

(3 marks)

(c) By means of mathematical induction, or otherwise, prove that for any non-negative integer n,

$$\int_0^\infty x^{2n+1}e^{-x^2}dx$$

is convergent. (4 marks)

(d) Evaluate

$$\int_0^\infty x^{2001}e^{-x^2}dx$$

(4 marks)

12. Let $y = f(x)$ such that

$$(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2y = 0,$$

where m is a positive integer.

(a) Show that for any positive integer n,

$$(x^2+1)y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2-m^2)y^{(n)} = 0.$$

(6 marks)

(b) Let $g(x) = f^{(m+1)}(x)$. Show that

$$\frac{g'(x)}{g(x)} = -\frac{(2m+1)x}{x^2+1}.$$

(5 marks)

(c) Show that

$$g(x) = \frac{C}{(x^2+1)^{\frac{(2m+1)}{2}}}$$

where C is a constant. (4 marks)

END OF PAPER