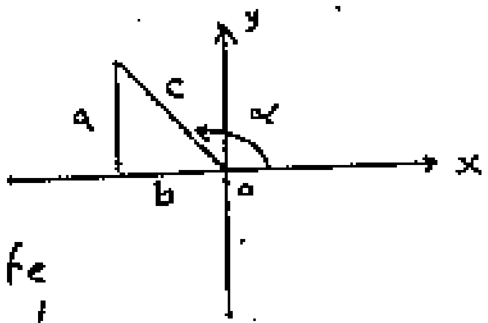
TRIGONOMETRIC AND HYPERBOLIC IDENTITIES

TRIGONOMETRIC	HYBERBOLIC
GROUP(A) $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$, $\cot\alpha = \frac{\cos\alpha}{\sin\alpha}$ $\sec\alpha = \frac{1}{\cos\alpha}$, $\csc\alpha = \frac{1}{\sin\alpha}$	$\tanh\alpha = \frac{\sinh\alpha}{\cosh\alpha}$, $\coth\alpha = \frac{\cosh\alpha}{\sinh\alpha}$ $\operatorname{sech}\alpha = \frac{1}{\cosh\alpha}$, $\operatorname{csch}\alpha = \frac{1}{\sinh\alpha}$
GROUP(B) $\cos^2\alpha + \sin^2\alpha = 1$ $1 + \tan^2\alpha = \sec^2\alpha$ $\cot^2\alpha + 1 = \csc^2\alpha$	$\cosh^2\alpha - \sinh^2\alpha = 1$ $1 - \tanh^2\alpha = \operatorname{sech}^2\alpha$ $\coth^2\alpha - 1 = \operatorname{csch}^2\alpha$
GROUP(C) $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$ $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$ $\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$	$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$ $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$ $\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$
GROUP(D) $\sin 2\alpha = 2\sin\alpha\cos\alpha$ $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$ $= 2\cos^2\alpha - 1$ $= 1 - 2\sin^2\alpha$ Hence $\cos^2\alpha = \frac{1}{2}[1 + \cos 2\alpha]$ $\sin^2\alpha = \frac{1}{2}[1 - \cos 2\alpha]$ $\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}$	$\sinh 2\alpha = 2\sinh\alpha\cosh\alpha$ $\cosh 2\alpha = \cosh^2\alpha + \sinh^2\alpha$ $= 2\cosh^2\alpha - 1$ $= 1 + 2\sinh^2\alpha$ Hence $\cosh^2\alpha = \frac{1}{2}[\cosh 2\alpha + 1]$ $\sinh^2\alpha = \frac{1}{2}[\cosh 2\alpha - 1]$ $\tanh 2\alpha = \frac{2\tanh\alpha}{1 + \tanh^2\alpha}$
GROUP(E) $\sin(-x) = -\sin x$ $\cos(-x) = \cos x$ $\tan(-x) = -\tan x$	$\sinh(-x) = -\sinh x$ $\cosh(-x) = \cosh x$ $\tanh(-x) = -\tanh x$
GROUP(F) $\sin x \cos y = \frac{1}{2}[\sin(x-y) + \sin(x+y)]$ $\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$ $\cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$	$\sinh x \cosh y = \frac{1}{2}[\sinh(x+y) + \sinh(x-y)]$ $\sinh x \sinh y = \frac{1}{2}[\cosh(x+y) - \cosh(x-y)]$ $\cosh x \cosh y = \frac{1}{2}[\cosh(x+y) + \cosh(x-y)]$
special values: $\sin 0 = 0$, $\cos 0 = 1$, $\tan 0 = 0$	$\sinh 0 = 0$, $\cosh 0 = 1$, $\tanh 0 = 0$

TRIGONOMETRIC

Definitions



a : opposite
b : adjacent
c : Hypotenuse

$\sin\alpha = \dfrac{\text{opposite}}{\text{Hyp.}}$, $\cos\alpha = \dfrac{\text{adjacent}}{\text{Hyp.}}$ . .

$\tan\alpha = \dfrac{\text{opposite}}{\text{adjacent}}$.

Note: opposite and adjacent may be positive or negative. However the hypotenuse is always positive.

HYPERBOLIC

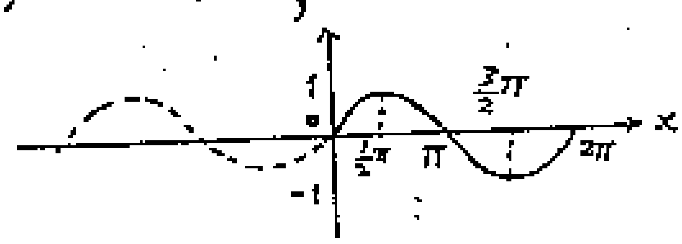
$\sinh x = \dfrac{e^{x} - e^{-x}}{2}$ or $\frac{1}{2}[e^{x} - e^{-x}]$

$\cosh x = \dfrac{e^{x} + e^{-x}}{2}$ or $\frac{1}{2}[e^{x} + e^{-x}]$

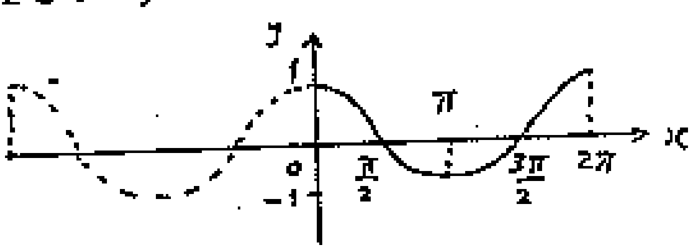
$\tanh x = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$ or $\dfrac{\sinh x}{\cosh x}$.

Standard Graphs:

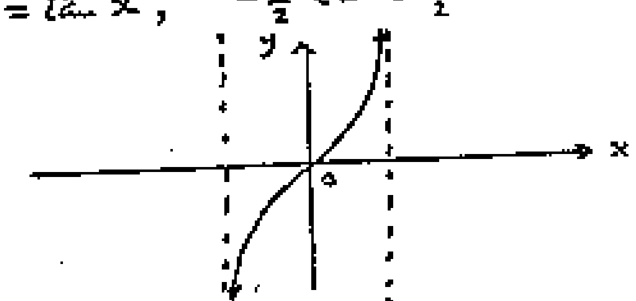
$y = \sin x$, $0 \leq x \leq 2\pi$



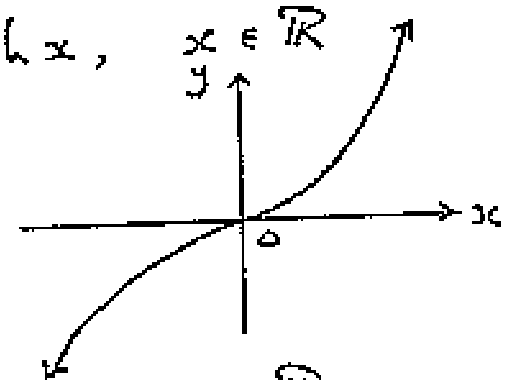
$y = \cos x$, $0 \leq x \leq 2\pi$



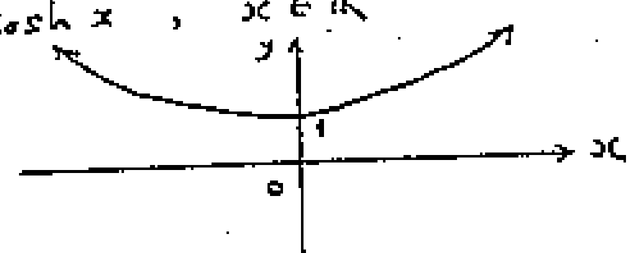
$y = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$



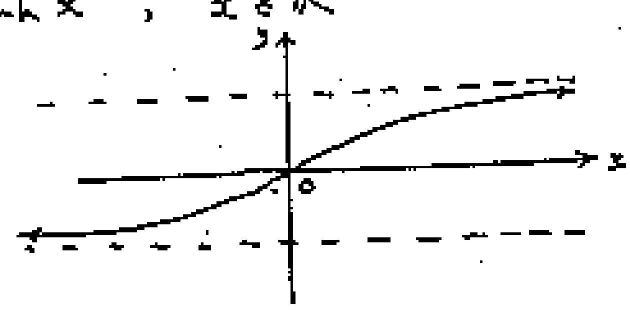
$y = \sinh x$, $x \in \mathbb{R}$



$y = \cosh x$, $x \in \mathbb{R}$

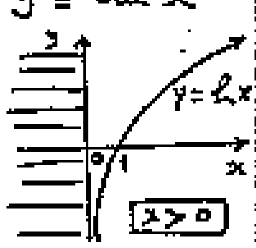


$y = \tanh x$, $x \in \mathbb{R}$



Miscellaneous:

The logarithmic function $y = \ln x$



properties:

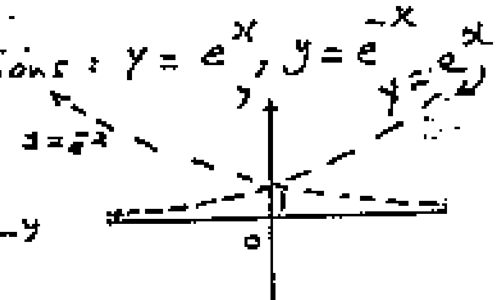
$\ln x + \ln y = \ln(xy)$
$\ln x - \ln y = \ln\left(\frac{x}{y}\right)$
$\ln x^{n} = n \ln x$,
$\ln e^{x} = x$, $e^{\ln x} = x$

special values: $\ln 1 = 0$, $\ln e = 1$,

$\lim_{x \to 0^{+}} \ln x = -\infty$ or $\ln(0^{+}) = -\infty$

$\lim_{x \to \infty} \ln x = \infty$ or $\ln(+\infty) = +\infty$

The exponential functions : $y = e^{x}$, $y = e^{-x}$



properties:

$e^{x} \cdot e^{y} = e^{x+y}$, $\dfrac{e^{x}}{e^{y}} = e^{x-y}$

$(e^{x})^{y} = e^{xy}$, for instance $(e^{x})^{4} = e^{4x}$.

special values: $e^{0} = 1$, $e^{\pm x} > 0$, $x \in \mathbb{R}$

$\lim_{x \to \infty} e^{x} = \infty$ or $e^{\infty} = \infty$

$\lim_{x \to -\infty} e^{x} = 0$ or $e^{-\infty} = 0$.

TABLE OF BASIC DERIVATIVES

Let $u = u(x)$ be a differentiable function of the independent variable x - that is $u'(x)$ exists.

(A) The Power Rule

(1) $\frac{d}{dx}\{u^n\} = n u^{n-1} \cdot u'$, $n \in \mathbb{R}$

(2) $\frac{d}{dx}\{c\} = 0$, c is a constant.

(3) optional: $\frac{d}{dx}\{\sqrt{u}\} = \frac{u'}{2\sqrt{u}}$

EXAMPLES

$\frac{d}{dx}\{(x^3+4x+1)^{\frac{3}{4}}\} = \frac{3}{4}(x^3+4x+1)^{-\frac{1}{4}} \cdot (3x^2+4)$

$\frac{d}{dx}[\pi^6] = 0$ Since π is a Constant.

$\frac{d}{dx}\sqrt{\tan x} = \frac{\sec^2 x}{2\sqrt{\tan x}}$

(B) The Six Trigonometric Rules

(1) $\frac{d}{dx}\{\sin u\} = \cos u \cdot u'$

(2) $\frac{d}{dx}\{\cos u\} = -\sin u \cdot u'$

(3) $\frac{d}{dx}\{\tan u\} = \sec^2 u \cdot u'$

(4) $\frac{d}{dx}\{\cot u\} = -\csc^2 u \cdot u'$

(5) $\frac{d}{dx}\{\sec u\} = \sec u \cdot \tan u \cdot u'$

(6) $\frac{d}{dx}(\csc u) = -\csc u \cot u \cdot u'$

$\frac{d}{dx}\{\sin x^3\} = \cos(x^3) \cdot 3x^2$

$\frac{d}{dx}\{\cos\sqrt{x}\} = -\sin\sqrt{x} \cdot \frac{1}{2\sqrt{x}}$

$\frac{d}{dx}\{\tan(\frac{5}{x^2})\} = \sec^2(\frac{5}{x^2}) \cdot (-\frac{10}{x^3})$

$\frac{d}{dx}\{\cot(\sin 2x)\} = -\csc^2(\sin 2x) \cdot (\sin 2x)'$
$= -\csc^2(\sin 2x) \cdot \cos 2x \cdot 2$

$\frac{d}{dx}\{\sec(e^{2x})\} = \sec(e^{2x}) \tan(e^{2x}) \cdot (e^{2x})'$
$= \sec(e^{2x}) \tan(e^{2x}) \cdot e^{2x} \cdot 2$

$\frac{d}{dx}[\csc(\ln x)] = -\csc(\ln x)\cot(\ln x) \cdot (\ln x)'$
$= -\csc(\ln x)\cot(\ln x) \cdot \frac{1}{x}$

(C) The Six Hyperbolic Rules

(1) $\frac{d}{dx}\{\sinh u\} = \cosh u \cdot u'$

(2) $\frac{d}{dx}\{\cosh u\} = \sinh u \cdot u'$

(3) $\frac{d}{dx}\{\tanh u\} = \operatorname{sech}^2 u \cdot u'$

(4) $\frac{d}{dx}\{\coth u\} = -\operatorname{csch}^2 u \cdot u'$

(5) $\frac{d}{dx}\{\operatorname{sech} u\} = -\operatorname{sech} u \tanh u \cdot u'$

(6) $\frac{d}{dx}\{\operatorname{csch} u\} = -\operatorname{csch} u \coth u \cdot u'$

$\frac{d}{dx}[\sinh(\sqrt[3]{x})] = \cosh(\sqrt[3]{x}) \cdot \frac{1}{3}x^{-2/3}$

$\frac{d}{dx}\cosh(\sec x) = \sinh(\sec x) \cdot (\sec x)'$
$= \sinh(\sec x) \cdot \sec x \tan x$

$\frac{d}{dx}\{\tanh(9x)\} = \operatorname{sech}^2(9x) \cdot 9$

$\frac{d}{dx}\{\coth(\frac{1}{x}+2x)\} = -\operatorname{csch}^2(\frac{1}{x}+2x) \cdot (-\frac{1}{x^2}+2)$

$\frac{d}{dx}\operatorname{sech}\{x^3+\sin(x^2)\}$ This is u !!
$= -\operatorname{sech}(x^3+\sin(x^2)) \tanh(x^3+\sin(x^2))$
$\cdot (3x^2 + 2x\cos(x^2))$

$\frac{d}{dx}\operatorname{csch}(\sinh 3x)$
$= -\operatorname{csch}(\sinh 3x)\coth(\sinh 3x) \cdot$
$3\cosh 3x$

(D) Exponential and Logarithmic Rule

(1) $\frac{d}{dx}\{e^u\} = e^u \cdot u'$

(2) $\frac{d}{dx}\{\ln|u|\} = \frac{u'}{u}$

(3) $\frac{d}{dx}\{a^u\} = a^u \ln a \cdot u'$, $a \in \mathbb{R}$, $a > 0$, $a \neq 1$

(4) $\frac{d}{dx}\{\log_a|u|\} = \frac{1}{\ln a} \cdot \frac{u'}{u}$, $a \in \mathbb{R}$, $a > 0$, $a \neq 1$

EXAMPLES

$$\frac{d}{dx}(e^{-x^3}) = e^{-x^3}(-x^3)' = e^{-x^3} \cdot (-3x^2)$$

$$\frac{d}{dx}\{\ln|x^3+5x+1|\} = \frac{3x^2+5}{x^3+5x+1}$$

$$\frac{d}{dx}(2^{\sec x}) = 2^{\sec x} \ln 2 \cdot \sec x \tan x$$

$$\frac{d}{dx}\{\log_4|\sec x|\} = \frac{1}{\ln 4} \cdot \frac{(\sec x)'}{\sec x} = \frac{1}{\ln 4} \cdot \frac{\sec x \tan x}{\sec x} = \frac{\tan x}{\ln 4}.$$

(E) The Inverse Trigonometric functions

(1) $\frac{d}{dx}\{\sin^{-1} u\} = \frac{u'}{\sqrt{1-u^2}}$

(2) $\frac{d}{dx}\{\cos^{-1} u\} = -\frac{u'}{\sqrt{1-u^2}}$

(3) $\frac{d}{dx}(\tan^{-1} u) = \frac{u'}{1+u^2}$

(4) $\frac{d}{dx}(\cot^{-1} u) = -\frac{u'}{1+u^2}$

(5) $\frac{d}{dx}(\sec^{-1} u) = \frac{u'}{|u|\sqrt{u^2-1}}$

(6) $\frac{d}{dx}(\csc^{-1} u) = -\frac{u'}{|u|\sqrt{u^2-1}}$

$$\frac{d}{dx}\sin^{-1}(4x^2) = \frac{(4x^2)'}{\sqrt{1-(4x^2)^2}} = \frac{8x}{\sqrt{1-16x^4}}$$

$$\frac{d}{dx}\cos^{-1}(3x) = -\frac{3}{\sqrt{1-9x^2}}$$

$$\frac{d}{dx}\{\tan^{-1}(\underbrace{\sqrt{x}}_{\text{This is "u"}})\} = \frac{(\sqrt{x})'}{1+(\sqrt{x})^2} = \frac{\frac{1}{2\sqrt{x}}}{1+x} = \frac{1}{2\sqrt{x}(1+x}$$

$$\frac{d}{dx}\cot^{-1}(e^x) = -\frac{e^x}{1+(e^x)^2} = -\frac{e^x}{1+e^{2x}}$$

$$\frac{d}{dx}\{\sec^{-1}(x^4)\} = \frac{(x^4)'}{|x^4|\sqrt{(x^4)^2-1}} = \frac{4x^3}{x^4\sqrt{x^8-1}} = \frac{4}{x\sqrt{x^8-1}}$$

$$\frac{d}{dx}\{\csc^{-1}(2x)\} = -\frac{2}{|2x|\sqrt{4x^2-1}} = -\frac{1}{|x|\sqrt{4x^2-1}}$$

(F) The Inverse Hyperbolic functions

(1) $\frac{d}{dx}\{\sinh^{-1} u\} = \frac{u'}{\sqrt{1+u^2}}$

(2) $\frac{d}{dx}\{\cosh^{-1} u\} = \frac{u'}{\sqrt{u^2-1}}$

(3) $\frac{d}{dx}\{\tanh^{-1} u\} = \frac{u'}{1-u^2}$

$$\frac{d}{dx}\sinh^{-1}(\ln x) = \frac{(\ln x)'}{\sqrt{1+(\ln x)^2}} = \frac{\frac{1}{x}}{\sqrt{\ln^2 x+1}}$$

$$\frac{d}{dx}\{\cosh^{-1}(5x)\} = \frac{5}{\sqrt{25x^2-1}}$$

$$\frac{d}{dx}\tanh^{-1}\left(\frac{3}{x}\right) = \frac{(\frac{3}{x})'}{1-(\frac{3}{x})^2} = \frac{-\frac{3}{x^2}}{1-\frac{9}{x^2}} = \frac{-3}{x^2-9}$$

(G) Product and Quotient Rules

(1) $\frac{d}{dx}(uv) = uv' + u'v$

(2) $\frac{d}{dx}(Ku) = Ku'$, K is Constant

(3) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - v'u}{v^2}$

$$\frac{d}{dx}\{x^3 \ln(5x+1)\} = x^3 \cdot \frac{5}{5x+1} + 3x^2 \cdot \ln(5x+1)$$

$$\frac{d}{dx}\left\{\frac{x^3}{4}\right\} = \frac{d}{dx}\left\{\frac{1}{4} \cdot x^3\right\} = \frac{1}{4} \cdot 3x^2$$

$$\frac{d}{dx}\left\{\frac{\tan 2x}{x^3}\right\} = \frac{x^3 \cdot 2\sec^2 2x - 3x^2 \tan 2x}{x^6}$$

TABLE OF BASIC INTEGRALS

The table consists of six groups (A) $\rightarrow$ (E). Let us first mention the following two important remarks

\# (1): This table must be strictly applied "as is" without absolutely any alteration or modification except as noted in remark # (2) below.

\# (2): A powerful Generalization to Table:
If the variable "x" is replaced by the LINEAR expression $\alpha x+\beta$, $\alpha \neq 0$, divide the corresponding formula by "α" which is the derivative of $\alpha x+\beta$!!

For example: we know $\int \sin x\,dx = -\cos x + C$

Hence $\int \sin(14x-3)\,dx = -\dfrac{\cos(14x-3)}{14} + C$

(A) The Power Rule

(1) $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$, $n \in \mathbb{R}$, $\boxed{n \neq -1}$

(2) $\int dx = \int 1\,dx = x + C$

(3) optional $\int \dfrac{1}{\sqrt{x}}\,dx = 2\sqrt{x} + C$

EXAMPLES

$\int x^{-5}\,dx = \dfrac{x^{-4}}{-4} + C$

$\int (3x-1)^{-2}\,dx = \dfrac{(3x-1)^{-1}}{(-1)\cdot 3} + C = \dfrac{(3x-1)^{-1}}{-3} + C$

$\int 7\,dx = 7\int dx = 7x + C$

$\int dt = t + C$, $\int dz = z + C$, ...etc.

$\int \dfrac{1}{\sqrt{x+4}}\,dx = \dfrac{2\sqrt{x+4}}{1} + C$

Note: Remark # (2) is used!

(B) The Ten Trigonometric Rules

(1) $\int \sin x\,dx = -\cos x + C$

(2) $\int \cos x\,dx = \sin x + C$

(3) $\int \tan x\,dx = \ln|\sec x| + C$

(4) $\int \cot x\,dx = \ln|\sin x| + C$

(5) $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

(6) $\int \csc x\,dx = \ln|\csc x - \cot x| + C$

(7) $\int \sec^2 x\,dx = \tan x + C$

(8) $\int \csc^2 x\,dx = -\cot x + C$

(9) $\int \sec x \tan x\,dx = \sec x + C$

(10) $\int \csc x \cot x\,dx = -\csc x + C$

EXAMPLES

$\int \sin(9x-2)\,dx = -\dfrac{\cos(9x-2)}{9} + C$

Linear!

$\int \cos 3w\,dw = \dfrac{\sin 3w}{3} + C$

$\int \tan(5x)\,dx = \dfrac{\ln|\sec(5x)|}{5} + C$

$\int \cot(7u)\,du = \dfrac{1}{7}\ln|\sin 7u| + C$

$\int \sec(2-3x)\,dx = -\dfrac{1}{3}\ln|\sec(2-3x) + \tan(2-3x)| + C$

$\int \csc(2x)\,dx = \dfrac{1}{2}\ln|\csc(2x) - \cot(2x)| + C$

$\int \sec^2(\tfrac{2}{3}u)\,du = \dfrac{\tan(\frac{2}{3}u)}{\frac{2}{3}} + C$

$\int \csc^2(\tfrac{x}{2})\,dx = \dfrac{-\cot(\frac{x}{2})}{\frac{1}{2}} + C$

$\int \sec(3w)\tan(3w)\,dw = \dfrac{1}{3}\sec(3w) + C$

$\int \csc 5t \cot 5t\,dt = \dfrac{-1}{5}\csc(5t) + C$

P.T.O $\Rightarrow$

(C) The Eight Hyperbolic Rules

(1) $\int \sinh x\,dx = \cosh x + C$

(2) $\int \cosh x\,dx = \sinh x + C$

(3) $\int \tanh x\,dx = \ln(\cosh x) + C$

(4) $\int \coth x\,dx = \ln|\sinh x| + C$

(5) $\int \operatorname{sech}^2 x\,dx = \tanh x + C$

(6) $\int \operatorname{csch}^2 x\,dx = -\coth x + C$

(7) $\int \operatorname{sech} x \tanh x\,dx = -\operatorname{sech} x + C$

(8) $\int \operatorname{csch} x \coth x\,dx = -\operatorname{csch} x + C$

EXAMPLES

$\int \sinh(3x)\,dx = \frac{1}{3}\cosh(3x) + C$

$\int \cosh\left(\frac{2x}{5}\right)dx = \frac{5}{2}\sinh\left(\frac{2x}{5}\right) + C$

$\int \tanh(2u)\,du = \frac{1}{2}\ln\left(\cosh(2u)\right) + C$

$\int \coth(x+3)\,dx = \frac{\ln|\sinh(x+3)|}{1} + C$

$\int \operatorname{sech}^2(4w)\,dw = \frac{1}{4}\tanh(4w) + C$

$\int \operatorname{csch}^2(2u)\,du = \frac{-1}{2}\coth(2u) + C$

$\int \operatorname{sech} 3x \tanh 3x\,dx = \frac{-1}{3}\operatorname{sech} 3x + C$

$\int \operatorname{csch}\left(\frac{x}{3}\right)\coth\left(\frac{x}{3}\right) = -3\operatorname{csch}\left(\frac{x}{3}\right) + C$

(D) The Exponential and Logarithmic Rules

(1) $\int \frac{1}{x}\,dx = \ln|x| + C$

(2) $\int e^x\,dx = e^x + C$

(3) $\int a^x\,dx = \frac{a^x}{\ln a} + C, \quad a>0, a\neq 1$

$\int \frac{1}{2x+3}\,dx = \frac{\ln|2x+3|}{2} + C$

Linear! $= \frac{1}{2}\ln|2x+3| + C$

$\int e^{7x}\,dx = \frac{e^{7x}}{7} + C, \quad \int e^{2-5x}\,dx = \frac{e^{2-5x}}{-5} + C$

$\int 2^{10x-17}\,dx = \frac{2^{10x-17}}{(\ln 2)\cdot(10)} + C$

(E) Inverse Trigonometric/Hyperbolic Rules

Let "a" be a positive constant real number.

(1) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

(2) $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$

(3) $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C$

(4) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

(5) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C, \quad x>a$

$\int \frac{dx}{\sqrt{16-x^2}} = \sin^{-1}\left(\frac{x}{4}\right) + C$

$\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1}\left(\frac{x}{1}\right) + C$

$\int \frac{dx}{\sqrt{x^2-5}} = \cosh^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$. Here $a=\sqrt{5}$

$\int \frac{dx}{3+x^2} = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$

$\int \frac{dx}{x\sqrt{x^2-4}} = \frac{1}{2}\sec^{-1}\left(\frac{x}{2}\right) + C, \quad x>2$

Fundamental Theorems

Let $f(x)$ be continuous on $[a,b]$.

(1) If $\int f(x)\,dx = g(x) + C$, then $\int_a^b f(x)\,dx = g(x)\Big|_a^b = g(b) - g(a)$

(2) $\frac{d}{dx}\int_a^u f(t)\,dt = f(t)\Big|^u \cdot u'$

$\int_e^{e^3} \frac{1}{x}\,dx = \ln|x|\Big|_e^{e^3} = \ln e^3 - \ln e^2 = 3\ln e - 2\ln e = 3-2 = 1$

$\frac{d}{dx}\int_0^{x^2} \cos(t^2)\,dt = \cos(t^2)\Big|^{t=x^2} \cdot (x^2)' = \cos(x^4)\cdot 2x$