MATH 311

SHEET (0)

REVIEW

(1) Which of the following matrices are in row-echelon form, reduced row-echelon form or neither. Justify your answers

(a) $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & -7 & 5 & 5 \\ 0 & 1 & 3 & 2 \end{bmatrix}$

(2) Without solving the homogeneous system given, determine whether it has a non trivial solution or only the Trivial Solution

(a) $\begin{cases} 2x_1 - 3x_2 + 4x_3 - x_4 = 0 \\ 7x_1 + x_2 - 8x_3 + 9x_4 = 0 \\ 2x_1 + 8x_2 + x_3 - x_4 = 0 \end{cases}$

(b) $\begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 + 2x_2 = 0 \\ x_2 + x_3 = 0 \end{cases}$

(c) $\begin{array}{l} 3x_1 + x_2 + x_3 + x_4 = 0 \\ 5x_1 - x_2 + x_3 - x_4 = 0 \end{array}$

(d) $\begin{cases} 2x_1 + 2x_2 - x_3 + x_5 = 0 \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0 \\ x_1 + x_2 - 2x_3 - x_5 = 0 \\ x_3 + x_4 + x_5 = 0 \end{cases}$

(e) $\begin{cases} 2x + 5y = 0 \\ x - 2y = 0 \\ x + y = 0 \\ 7x - 9y = 0 \end{cases}$

(f) $\begin{array}{l} 2u - v - 3w = 0 \\ -u + 2v - 3w = 0 \\ u + v + 4w = 0 \end{array}$

(3) Find the solution to systems (c), (d), (e), and (f) of exercise (2) above.

(4) Give an example of 2×2 matrices A and B so that $\det A = 0$, $\det B = 0$ but $\det(A+B) \neq 0$.

(5) Show by an example that: The sum of two invertible matrices need not be invertible.

(6) If A is an $n\times n$ skew-symmetric matrix, prove that all the main-diagonal entries of A are zeros (that is $a_{ii} = 0 \quad i = 1, 2, \dots n$).

(7) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$. Show that A is invertible and determine A^{-1}. Use the result to solve the non-homogeneous system $A\vec{x} = \vec{b}$ where $\vec{b} = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$.

(8) What conditions must b_1, b_2, and b_3 satisfy in order for the system

(a) $\begin{aligned} x_1 + x_2 + 2x_3 &= b_1 \\ x_1 + x_3 &= b_2 \\ 2x_1 + x_2 + 3x_3 &= b_3 \end{aligned}$ (b) $\begin{cases} x_1 + 2x_2 + 3x_3 = b_1 \\ 2x_1 + 5x_2 + 3x_3 = b_2 \\ x_1 + 8x_3 = b_3 \end{cases}$

to be consistent?

(9) Find all values of λ for which the given matrix A
(i) is invertible (ii) Not invertible

(a) $A = \begin{bmatrix} \lambda-2 & 1 \\ -5 & \lambda+4 \end{bmatrix}$ (b) $A = \begin{bmatrix} \lambda-4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda-1 \end{bmatrix}$ (c) $A = \begin{bmatrix} \lambda & 1 \\ 1 & -\lambda \end{bmatrix}$

(10) Find $\det A$ if $A = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta-\cos\theta & \sin\theta+\cos\theta & 1 \end{bmatrix}$

(11) If $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, find A^{10}, $\det(A^{-3})$, A^{-1}, $(A^t)^{-1}$.

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Support Material

Sheet (1)

(1) Find scalars c_1, c_2, and c_3 such that

$$c_1(1,0,1,0) + c_2(1,0,-2,1) + c_3(2,0,1,2) = (1,-2,2,3)$$

Ans. scalars do not exist!

(2) Find scalars c_1, c_2, c_3, and c_4 such that

(a) $c_1\bar{u}_1 + c_2\bar{u}_2 + c_3\bar{u}_3 + c_4\bar{u}_4 = (0,5,6,-3)$ where $\bar{u}_1 = (-1,3,2,0)$, $\bar{u}_2 = (2,0,4,-1)$, $\bar{u}_3 = (7,1,1,4)$, and $\bar{u}_4 = (6,3,1,2)$

Ans. $c_1 = 1$, $c_2 = 1$, $c_3 = -1$, and $c_4 = 1$

(3) Find $\vec{x}$ if $5\vec{x} - 2\vec{v} = 2(\vec{w} - 5\vec{x})$ where $\vec{v} = (4,7,-3,2)$, and $\vec{w} = (5,-2,8,1)$. Ans. $\vec{x} = (\frac{6}{5}, \frac{2}{3}, \frac{2}{3}, \frac{2}{5})$

(4) In each case a set of objects is given together with operations of addition and scalar multiplication. Determine which sets are vector spaces under the given operations. For those that are not, list all axioms that Fail to hold!

(a) The set of all ordered pairs of real number (x,y) where $x \geq 0$ with standard operations in $\mathbb{R}^2$

(b) The set of all 2×2 matrices of the form $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$ with matrix addition and matrix scalar multiplication

(c) $\{(x,y) \mid x,y \in \mathbb{R}\}$ with operations:

$$(x_1,y_1) + (x_2,y_2) = (x_1+x_2, y_1+y_2), \quad K(x,y) = (2Kx, 2Ky)$$

(d) $\{(x,y) \mid x,y \in \mathbb{R}\}$ with operations

$$(x,y) + (x',y') = (x+x'+1, y+y'+1), \quad K(x,y) = (Kx, Ky)$$

(e) $\{(x,y,z) \mid x,y,z \in \mathbb{R}\}$ with standard addition in $\mathbb{R}^3$ and scalar multiplication defined by $K(x,y,z) = (0,0,0)$

(f) $\{(x,0) \mid x \in \mathbb{R}\}$ with standard operations in $\mathbb{R}^2$

(g) The set of all 2×2 matrices of the form $\begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix}$ with standard matrix addition and scalar multiplication.

Ans. (a) Not a vector space .. Axioms A5, S1 Fail
(b) Not a vector space .. Axioms A1, A4, A5, and S1 fail
(c) Not a vector space .. Axioms S4, S5 fail
(d) Not a vector space .. Axioms S2, S3 fail
(e) Not a vector space .. Axioms S5 fails
(f) A vector space! (g) a vector space!

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(5) show that the set $\mathbb{R}^+$ of all positive real numbers with the operations

$$x + y = xy \quad , \quad kx = x^k \qquad x, y \in \mathbb{R}^+, \ k \in \mathbb{R}$$

is a vector space.

(6) show that the set $V = \{ (x,y) \mid x, y \in \mathbb{R} \}$ with the standard scalar multiplication in $\mathbb{R}^2$ but addition defined by

$$(x_1, y_1) + (x_2, y_2) = \left(\sqrt[3]{x_1^3 + x_2^3} \, , \ \sqrt[3]{y_1^3 + y_2^3} \right)$$

is a vector space.

(7) Let the set V be a straight line through the origin in $\mathbb{R}^3$; that is $V = \{ (x,y,z) \mid x = at, y = bt, z = ct \}$. prove that V is a vector space under standard operations in $\mathbb{R}^3$.

(8) Let $V = \{ (x,y,z) : ax + by + cz = 0 \}$ " that is V is a plane through origin in $\mathbb{R}^3$ ". Prove that V is a vector space under standard operations in $\mathbb{R}^3$.

(9) Find scalars c_1, c_2, and c_3 such that $c_1 \bar{u} + c_2 \bar{v} + c_3 \bar{w} = 0$ where $\bar{u} = x^3 + x$, $\bar{v} = x^2 + 1$, and $\bar{w} = x^3 - x^2 + x + 1$

(10) Let V be a vector space and let $\bar{u}, \bar{v}$ be vectors in V. prove that
(1) If $k\bar{u} = k\bar{v}$, where $k \neq 0$, then $\bar{u} = \bar{v}$ (Cancellation Law
(2) If $k\bar{u} = l\bar{u}$ and if $\bar{u} \neq \vec{0}$, then $k = l$
(3) $\vec{0} = -\vec{0}$, where $\vec{0}$ is the zero vector in V

(11) prove that a vector space can not have more than one zero vector

(12) prove that every vector in a vector space has exactly one negative

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MATH 311

Support Material

Sheet (2)

(1) Which of the following are subspaces of $\mathbb{R}^3$?

(a) $W = \{ (x, y, 1) \mid x, y \in \mathbb{R} \}$ (b) $W = \{ (a, b, c) \mid b = a + c \}$

(c) $W = \{ (0, 0, t) \mid t \in \mathbb{R} \}$ (d) $W = \{ (x, y, z) \mid x^2 + y^2 = z^2, \ x, y, z \in \mathbb{R} \}$

Answer: (a), (d) are not subspaces; (b), and (c) are subspaces.

(2) Which of the following are subspaces of P_3?

(a) $U = \{ p(x) \mid p(x)$ is of degree $3 \}$ (b) $U = \{ q(x) \mid q(0) = 0 \}$

(c) $U = \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_0 + a_1 + a_2 + a_3 = 0 \}$

(d) $U = \{ p(x) \mid p(2) = 1 \}$

Answer: (a), (d) are not subspaces; (b), (c) are subspaces

(3) Which of the following a subspaces of M_{22}?

(a) The set of all 2×2 matrices A such that $tr(A) = 0$

Note: $tr(A)$ = sum of main-diagonal entries.

(b) $W = \{ A \mid A = A^t \}$ (c) $W = \{ A \mid A^t = -A \}$

(d) $W = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a + b + c + d = 0 \}$ (e) $\{ A \mid \det A = 0 \}$

(f) $W = \{ A \mid$ Linear system $A\vec{x} = \vec{0}$ has only trivial solution $\}$

(g) $W = \{ A \mid AB = 0 \}$, B is a fixed 2×2 matrix

Answer: (a), (b), (c), (d), and (g) are subspaces; (e), and (f) are not.

(4) Determine which the solution space of the system $A\vec{x} = \vec{0}$ is a line through the origin, a plane through the origin, or the origin itself! If it is a plane, find an equation for it; if it is a line, find parametric equations for it.

(a) $A = \begin{bmatrix} 1 & -2 & 3 \\ -3 & 6 & 9 \\ -2 & 4 & -6 \end{bmatrix}$ Ans: st. line: $x = 2t$, $y = t$, $z = 0$

(b) $A = \begin{bmatrix} 1 & 2 & -6 \\ 1 & 4 & 4 \\ 3 & 10 & 6 \end{bmatrix}$ Ans: The origin $(0, 0, 0) = \vec{0}$

(c) $A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -6 & 2 \\ 3 & -9 & 3 \end{bmatrix}$ Ans. A plane: $x - 3y + z = 0$

line - 1 parameter

↳ reduce $\begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{0}$; $x - 3y + z = 0$

(5) which of the following vectors in $\mathbb{R}^3$ are linear combinations of $\vec{u} = (0, -2, 2)$, $\vec{v} = (1, 3, -1)$?

(a) $\vec{x} = (2, 2, 2)$ (b) $\vec{x} = (0, 0, 0)$ (c) $\vec{x} = (0, 4, 5)$

Ans. (a), (b) are linear combinations ; (c) is not.

(6) let $\vec{u} = (2, 1, 4)$, $\vec{v} = (1, -1, 3)$, and $\vec{w} = (3, 2, 5)$.

Express (a) $\vec{x} = (-9, -7, -15)$, (b) $\vec{y} = (0, 0, 0)$

as linear combinations of $\vec{u}$, $\vec{v}$, and $\vec{w}$.

Ans (a) $\vec{x} = -2\vec{u} + 1.\vec{v} - 2\vec{w}$, (b) $\vec{y} = 0\vec{u} + 0\vec{v} + 0\vec{w}$

(7) let $p_1(x) = 2 + x + 4x^2$, $p_2(x) = 1 - x + 3x^2$, $p_3(x) = 3 + 2x + 5x^2$.

express (a) $p(x) = 6 + 11x + 6x^2$ (b) $p(x) = 0$

as linear combinations of p_1, p_2, and p_3.

Ans. (a) $= 4p_1 - 5p_2 + p_3$ (b) $p = 0p_1 + 0p_2 + 0p_3$

(8) let $A = \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}$

which of the following are linear combinations of A, B, and C?

(a) $X = \begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ (b) $X = \begin{bmatrix} 11 & -1 \\ 3 & 9 \end{bmatrix}$

Ans. (a) Not a linear combination (b) $X = 2A + 3B + 1.C$

(9) Show whether the set S spans $\mathbb{R}^3$:

(a) $S = \{ (2,2,2), (0,0,3), (0,1,1) \}$ (b) $S = \{ (1,2,6), (3,4,1), (4,3,1), (3,3,1) \}$

Ans. (a) yes! (b) No!

(10) Show whether the set spans P_2 :

(a) $S = \{ x+2, x^2+1, -2x^2+x \}$ (b) $S = \{ 1-x+2x^2, 3+x, 5-x+4x^2, -2-2x+2x^2 \}$

(c) $S = \{ 1+2x^2, 3x, 1+x \}$

Ans. (a), (b) : No! (c) : yes!

(11) show that M_{22} is spanned by $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$

(12) Let V be a vector space and let $\vec{u}, \vec{v}$, and $\vec{w}$ be vectors in V. Prove that

$$\operatorname{Span}\{\vec{u}, \vec{v}, \vec{w}\} = \operatorname{Span}\{\vec{u}+\vec{v}, \vec{u}+\vec{w}, \vec{v}+\vec{w}\}$$

(13) Let "A" be an $m \times n$ matrix and $\vec{b}$ be a column vector of order m. Let $U = \{\vec{x} \text{ in } \mathbb{R}^n \mid A\vec{x} = \vec{b}\}$, is U a subspace of $\mathbb{R}^n$? Ans. No! unless $\vec{b} = \vec{0}$.

(14) Describe $\operatorname{span}\{\vec{0}\}$. Ans. $\{\vec{0}\}$ as well!

(15) Show that the sets $S_1 = \{(1,6,4), (2,4,-1), (-1,2,5)\}$, $S_2 = \{(1,-2,-5), (0,8,9)\}$ span the <u>Same</u> subspace U of $\mathbb{R}^3$.

(16) If X, and Y are non-empty subsets of vectors in a vector space V such that $X \subseteq Y$. Prove that

$$\operatorname{span} X \subseteq \operatorname{span} Y$$

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Support Material

sheet (3)

(1) In each case S is a set of vectors in a vector space V. show whether the set S is Linearly independent or linearly dependent

(a) $S = \{ \bar{u}_1 = (-1, 2, 4), \bar{u}_2 = (5, -10, -20) \}$ in $V = \mathbb{R}^3$

(b) $S = \{ A = \begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 3 & -4 \\ -2 & 0 \end{bmatrix} \}$ in $V = M_{22}$

(c) $S = \{ P_1 = 3 - 2x + x^2, P_2 = 6 - 4x + 2x^2 \}$ in $V = P_2$

(d) $S = \{ \bar{u}_1 = (3, -1), \bar{u}_2 = (4, 5), \bar{u}_3 = (-4, 7) \}$ in $V = \mathbb{R}^2$

(e) $S = \{ (1, -1, 1, 0), (2, 1, 7, 13), (0, 0, 0, 0) \}$ in $V = \mathbb{R}^4$

(f) $S = \{ \bar{u} = (4, -1, 2), \bar{v} = (-4, 10, 2) \}$ in $V = \mathbb{R}^3$

(g) $S = \{ \bar{w}_1 = (0, 0, 2, 2), \bar{w}_2 = (3, 3, 0, 0), \bar{w}_3 = (1, 1, 0, -1) \}$ in $V = \mathbb{R}^4$

(h) $S = \{ P_1 = 6 - x^2, P_2 = 1 + x + 4x^2 \}$ in $V = P_2$

(i) $S = \{ P_1 = 1 + 3x + 3x^2, P_2 = x + 4x^2, P_3 = 5 + 6x + 3x^2, P_4 = 7 + 2x - x^2 \}$ in $V = P_2$

(2) Assume that $\bar{v}_1, \bar{v}_2$, and $\bar{v}_3$ are position vectors in $\mathbb{R}^3$. show that $\bar{v}_1 = (-6, 7, 2)$, $\bar{v}_2 = (3, 2, 4)$, and $\bar{v}_3 = (4, -1, 2)$ lie in the same plane. what do you conclude about the set $S = \{ \bar{v}_1, \bar{v}_2, \bar{v}_3 \}$?

(3) prove that for any vectors $\bar{u}, \bar{v}$, and $\bar{w}$ in a vector space V, the set $\{ \bar{u} - \bar{v}, \bar{v} - \bar{w}, \bar{w} - \bar{u} \}$ is Linearly dependent! (Very Simple!)

Hint: Express a vector in S as L.C of the other two !!

(4) show that if $\{ \bar{v}_1, \bar{v}_2 \}$ is a linearly independent set of vectors in a vector space V and if $\bar{v}_3$ is a vector in V such that $\bar{v}_3 \notin \text{span}\{ \bar{v}_1, \bar{v}_2 \}$, then the set $\{ \bar{v}_1, \bar{v}_2, \bar{v}_3 \}$ is Linearly indep.

(5) show that if $\{ \bar{v}_1, \bar{v}_2, \bar{v}_3 \}$ is linearly independent set of vectors in a vector space V, then so are $\{ \bar{v}_1, \bar{v}_2 \}$, $\{ \vec{v}_1 \}$.

(6) Let U, W be subspaces of a vector space V. Define their intersection $U \cap W$ by $U \cap W = \{ \bar{x} \mid \bar{x} \in U \text{ and } \bar{x} \in W \}$.

(a) show that $U \cap W$ is a subspace of both U, and W.

(b) show that $U \cap W = \{ \vec{0} \}$ iff the set $\{ \bar{u}, \bar{w} \}$ is Linearly independent for any non-zero vectors $\vec{u} \in U$, $\vec{w} \in W$.

Answers :

(1) (a) S is Linearly dependent since $\bar{u}_2$ is a scalar multiple of $\bar{u}_1$.

(b) S ~ ~ ~ ~ A ~ ~ ~ ~ ~ B.

(c) S ~ ~ ~ ~ P_1 ~ ~ ~ ~ ~ P_2.

(d) S ~ ~ ~ ~ it contains three vectors.

(e) ~ ~ ~ ~ ~ it contains the zero vector.

(f) S is Linearly independent since neither $\bar{u}$ nor $\bar{v}$ is a scalar multiple of one another.

(g) S is Linearly independent since $c_1\bar{w}_1 + c_2\bar{w}_2 + c_3\bar{w}_3 = \vec{0}$ has only the trivial solution $c_1 = c_2 = c_3 = 0$

(h) S is Linearly independent (i) S is Linearly dependent.

(2) show that say $\bar{v}_1$ is a l.c. of $\bar{v}_2$ and $\bar{v}_3$ (In fact $\bar{v}_1 = 2\bar{v}_2 - 3\bar{v}_3$)
Hence by theorem, the set $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is Linearly dependent!
Now, $\bar{v}_1$ may be removed and we get the linearly independent set $\{\bar{v}_2, \bar{v}_3\}$ which spans the plane " with Eq. $8x + 10y - 11z = 0$).

(3) Clearly $\bar{v} - \bar{w} = -1.(\bar{u} - \bar{v}) - 1.(\bar{w} - \bar{u})$
$\therefore$ $\bar{v} - \bar{w} =$ l.c. of $(\bar{u} - v), (\bar{w} - \bar{u})$ $\Rightarrow$ set is dependent.

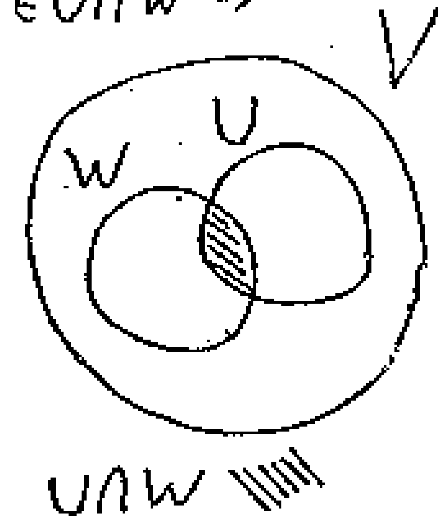
(4) Assume on the contrary that the set $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is dependent and establish a contradiction.

(5) Same as (4)!

(6) 1st. note that $U \cap W$ is a subset of U, a subset of W, as well as a subset of V! Further $\bar{o} \in U$, $\bar{o} \in W \Rightarrow \vec{o} \in U \cap W \Rightarrow$ non-empty

(a) Let $\bar{x}, \bar{y} \in U \cap W$
$\therefore$ $\bar{x} \in U, \bar{y} \in U \Rightarrow \bar{x} + \bar{y} \in U$, $k\bar{x} \in U$
$\bar{x} \in W, \bar{y} \in W \Rightarrow \bar{x} + \bar{y} \in W$, $k\bar{x} \in W$

hence $\bar{x} + \bar{y} \in U \cap W$, $k\bar{x} \in U \cap W$
$\therefore$ A1, S1 are true!
$\Rightarrow U \cap W$ is a subspace of V



accordingly $U \cap W$ is a subspace of U as well as a subspace of W!

(b) Easy!!

MATH 311

Support Material

Sheet (4)

(1) In each case show whether the set $\mathcal{B}$ is a basis for the vector space V.

(a) $\mathcal{B} = \{(1,2), (0,3), (2,7)\}$, $V = \mathbb{R}^2$

(b) $\mathcal{B} = \{p_1 = 1 + x + x^2,\ p_2 = x - 1\}$, $V = \mathcal{P}_2$

(c) $\mathcal{B} = \left\{ A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 6 & 0 \\ -1 & 4 \end{bmatrix}, C = \begin{bmatrix} 3 & 0 \\ 1 & 7 \end{bmatrix}, D = \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}, E = \begin{bmatrix} 7 & 1 \\ 2 & 9 \end{bmatrix} \right\}$, $V = M_{22}$

(d) $\mathcal{B} = \{(1,6,4), (2,4,-1), (-1,2,5)\}$, $V = \mathbb{R}^3$

(e) $\mathcal{B} = \{(1,0,0), (2,2,0), (3,3,3)\}$, $V = \mathbb{R}^3$

(f) $\mathcal{B} = \{1 - 3x + 2x^2,\ 1 + x + 4x^2,\ 1 - 7x\}$, $V = \mathcal{P}_2$

(g) $\mathcal{B} = \{1 + x + x^2,\ x + x^2,\ x^2\}$, $V = \mathcal{P}_2$

(h) $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right\}$, $V = M_{22}$

(i) $\mathcal{B} = \left\{ \begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \right\}$, $V = M_{22}$

(2) Find the coordinates of the vector $\vec{v}$ relative to the ordered basis S in each case

(a) $\bar{v} = (2,-1,3)$; $S = \{\bar{v}_1 = (1,0,0), \bar{v}_2 = (2,2,0), \bar{v}_3 = (3,3,3)\}$

(b) $\bar{v} = (5,-12,3)$; $S = \{\bar{v}_1 = (1,2,3), \bar{v}_2 = (-4,5,6), \bar{v}_3 = (7,-8,9)\}$

(3) Find the coordinates of the vector P relative to the ordered basis S in each case

(a) $p = 4 - 3x + x^2$; $S = \{p_1 = 1, p_2 = x, p_3 = x^2\}$

(b) $p = 2 - x + x^2$; $S = \{p_1 = 1 + x, p_2 = 1 + x^2, p_3 = x + x^2\}$

(4) Find the coordinates of the vector $A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$ relative to the ordered basis $S = \left\{ A_1 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

(5) In each case : Find a basis and determine the dimension of the solution space of the system given:

(a)
$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ -2x_1 - x_2 + 2x_3 &= 0 \\ -x_1 \qquad + x_3 &= 0 \end{aligned}$$

(b) $\begin{bmatrix} 1 & -4 & 3 & -1 \\ 2 & -8 & 6 & -2 \end{bmatrix} \bar{x} = \vec{0}$

(c)
$$\begin{aligned} 2x + y + 3z &= 0 \\ x \qquad + 5z &= 0 \\ y + z &= 0 \end{aligned}$$

(d)
$$\begin{cases} x + y + z = 0 \\ 3x + 2y - 2z = 0 \\ 4x + 3y - z = 0 \\ 6x + 5y + z = 0 \end{cases}$$

(6) Find a basis and the dimension of each of the following subspaces of $\mathbb{R}^3$

(a) The plane $3x - 2y + 5z = 0$ (b) The line $x = 2t,\ y = -t,\ z = 4t$

(7) Let U be the subspace of $\mathbb{R}^4$ defined by

$$U = \{(a, b, c, 0) \mid a, b, c \in \mathbb{R}\}$$

Find a basis and determine the dimension of U

(8) Repeat Ex. (7) for $U = \{(a, b, c, d) \mid d = a + b,\ c = a - b\}$

(9) Find a basis and determine the dimension of each of the following subspaces of P_2

(a) $U = \{p(x) \mid p(1) = 0\}$ (b) $\{p = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_0 = 0\}$

(c) $U = \{p(x) \mid p(x) = p(-x)\}$

(10) Find a basis and determine the dimension of each of the following subspaces of M_{22}

(a) $U = \{A \mid A^t = A\}$ (b) $U = \{A \mid A^t = -A\}$

(c) $W = \{A \mid \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} A = A\}$ (d) $W = \{A \mid A \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\}$

(11) Find a basis for the subspace U of $\mathbb{R}^4$ given by

$$U = \text{span}\{(1, -1, 3, 2), (2, 1, 1, 3), (1, 5, -7, 0), (4, -1, 7, 7)\}$$

(12) Find a basis for $\mathbb{R}^3$ by extending the set $S = \{\bar{v}_1 = (-1, 2, 3), (1, -2, -2)\}$

Hint: you may insert a standard basis vector to enlarge S to a basis B

(13) Let $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ be a basis for vector space V. Show that

$\{\bar{v}_1, \bar{v}_1 + \bar{v}_2, \bar{v}_1 + \bar{v}_2 + \bar{v}_3\}$ is also a basis for V.

Hint: Just prove that the set is linearly independent!

(14) <u>A Difficult one!</u>

Let U, W be subspaces of a vector space V. If $\dim V = 3$, $\dim U = 2$, $\dim W = 2$, $U \neq W$, show that $\dim(U \cap W) = 1$.

(15) Let A be a $n \times n$ matrix and let $\{\bar{v}_1, \bar{v}_2, \ldots \bar{v}_n\}$ be a linearly independent set of vectors in $\mathbb{R}^n$. If A is invertible, prove that the set $\{A\bar{v}_1, A\bar{v}_2, \ldots, A\bar{v}_n\}$ is linearly independent. where $\vec{v}_i$ is viewed as $n \times 1$ matrix

Answers To

MATH 311 – Sheet (4)

(1) For parts (a), (b), (c), (d), (f), and (h) the set $\mathcal{B}$ is not a basis for indicated vector space.
For parts (e), (g), and (i), the set $\mathcal{B}$ is a basis for the indicated vector space.

(2) (a) $(\vec{v})_S = (3,-2,1)$, (b) $(\vec{v})_S = (-2,0,1)$

(3) (a) $(P)_S = (4,-3,1)$, (b) $(P)_S = (0,2,-1)$

(4) $(A)_S = (-1,1,-1,3)$

(5) (a) A basis $\mathcal{B} = \{(1,0,1)\}$, dimension $= 1$
(b) A basis $\mathcal{B} = \{(4,1,0,0),(-3,0,1,0),(1,0,0,1)\}$, dimension $= 3$
(c) No basis, dimension $= 0$
(d) A basis $\mathcal{B} = \{(4,-5,1)\}$, dimension $= 1$

(6) (a) A basis $\mathcal{B} = \{(2,3,0),(-5,0,3)\}$, dimension $= 2$
(b) A basis $\mathcal{B} = \{(2,-1,4)\}$, dimension $= 1$

(7) A basis $\mathcal{B} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, dimension $= 3$

(8) Assignment problem!

(9) (b) a Basis $\mathcal{B} = \{x, x^2, x^3\}$, dimension $= 3$
(a) a Basis $\mathcal{B} = \{1-x, 1-x^2\}$, dimension $= 2$
(c) a basis $\mathcal{B} = \{1, x^2\}$, dimension $= 2$

(10) (a) a basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$, dimension $= 3$
(b) a basis $\mathcal{B} = \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$, dimension $= 1$
(c) a basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$, dimension $= 2$
(d) a basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$, dimension $= 2$

(11) a basis $\mathcal{B} = \{(1,-1,3,2),(2,1,1,3)\}$, dimension $= 2$

(12) $S' = \{\vec{v}_1, \vec{v}_2, \vec{e}_1\}$ or $\{\vec{v}_1, \vec{v}_2, \vec{e}_2\}$

(13) show that $c_1\vec{v}_1 + c_2(\vec{v}_1+\vec{v}_2) + c_3(\vec{v}_1+\vec{v}_2+\vec{v}_3)$ has only the Trivial solution $c_1 = c_2 = c_3 = 0$!

(14) Hint: possible dimensions of $U \cap W$ are 0, 1, or 2. show it can't be 0.

(15) Show that $c_1(A\vec{v}_1) + c_2(A\vec{v}_2) + \dots + c_n(A\vec{v}_n) = \vec{0}$ has only the Trivial solution $c_1 = c_2 = \dots = c_n = 0$!

MATH 311

Support Material

Sheet (5)

(1) Let $A = \begin{bmatrix} 2 & -1 & 0 & 1 \\ 3 & 5 & 7 & -1 \\ 1 & 4 & 2 & 7 \end{bmatrix}$. List the row vectors and column vectors of A.

(2) Determine whether $\vec{b} = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$ is in the column space of $A = \begin{bmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and if so express $\vec{b}$ as a linear combination of the column vectors of A.

(3) Find (a) A basis for the row space (b) A basis for the column space (c) A basis for the null space (d) Rank and nullity for each of the following matrices

(i) $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$ (iv) $A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$

(4) In each case find a basis for row space of A consisting entirely of row vectors of A

(i) $\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 0 & -2 \\ 0 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix}$

(5) Find a basis for the subspace of $\mathbb{R}^4$ spanned by the given vectors

(a) $\vec{v}_1 = (1,1,-4,-3)$, $\vec{v}_2 = (2,0,2,-2)$, $\vec{v}_3 = (2,-1,3,2)$

(b) $\vec{v}_1 = (1,1,0,0)$, $\vec{v}_2 = (0,0,1,1)$, $\vec{v}_3 = (-2,0,2,2)$, $\vec{v}_4 = (0,-3,0,3)$

(6) Find a basis for the row space and column space of the matrix "A" by <u>inspection</u>:

(a) $A = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(7) Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Show that the null space of A consists of all points on the z-axis and the column space consists of all points in the xy-plane (Assuming a rectangular xyz-system in 3-space)

(8.) Find a sub set of vectors that form a basis for the subspace of $\mathbb{R}^4$ spanned by the given vectors; then express each vector that is not in the basis as a linear combination of the basis vectors

(a) $\bar{v}_1 = (1,0,1,1)$, $\bar{v}_2 = (-3,3,7,1)$; $\bar{v}_3 = (-1,3,9,3)$, $\bar{v}_4 = (-5,3,5,-1)$

(b) $\bar{v}_1 = (1,-1,5,2)$, $\bar{v}_2 = (-2,3,1,0)$, $\bar{v}_3 = (4,-5,9,4)$, $\bar{v}_4 = (0,4,2,-3)$; $\bar{v}_5 = (-7,18,2,-8)$

(9.) Find invertible matrices U, and V such that $UAV = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, $r = \operatorname{rank}(A)$

(a) $A = \begin{bmatrix} 1 & 1 & -1 \\ -2 & -2 & 4 \end{bmatrix}$ (b) $A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & 0 & 3 \\ 0 & 1 & -4 & 1 \end{bmatrix}$ (d) $A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 3 & 2 & 1 & 1 \\ 1 & 0 & 1 & 3 \end{bmatrix}$

(10) In each part use the information in the table to find the dimension of row space of A, the column space of A, the null space of A, and the null space of A^t

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
Size of A	3x3	3x3	3x3	9x5	5x9	4x4	6x2
Rank(A)	3	2	1	2	2	0	2

(11) suppose that A is a 3x3 matrix whose column space is a plane through the origin in 3-space. Can the null space be a plane through the origin? Can the row space be a line through the origin?

(12) Let A be an $m \times n$, $m \neq n$. show that either the row vectors of A or the column vectors of A are linearly dependent!

(13) Discuss how the rank of $A = \begin{bmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{bmatrix}$ varies with t.

(14) (a) Can a 3x4 matrix have independent rows? independent columns?

(b) Can a 4x3 ~ ~ ~ ~ ? ~ ~ ?

(15) prove that null $A = \{\vec{0}\}$ iff the column vectors of A are linearly independent. Hint: use dimension theorem for matrices.

(16) show $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$, $\operatorname{rank}(AB) \leq \operatorname{rank}(B)$

MATH 311

Answers to sheet (5)

(1) $\bar{r}_1 = (2,-1,0,1)$, $\bar{r}_2 = (3,5,7,-1)$, $\bar{r}_3 = (1,4,2,7)$;

$\bar{c}_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $\bar{c}_2 = \begin{bmatrix} -1 \\ 5 \\ 4 \end{bmatrix}$, $\bar{c}_3 = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$, and $\bar{c}_4 = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}$

(2) yes! The system $A\bar{x} = \bar{b}$ has the solution $\bar{x} = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$ and that

$$1\begin{bmatrix} 1 \\ 9 \\ 1 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} + 1\cdot\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$$

(3) (i) (a) $\{(1,-1,3), (0,1,-19)\}$ (b) $\{(1,5,7), (-1,-4,-6)\}$ (c) $\{(16,19,1)\}$ (d) $2;1$

(ii) (a) $\{(1,0,-\frac{1}{2})\}$ (b) $\{(2,4,0)\}$, (c) $\{(1,0,2), (0,1,0)\}$; (d) $1;2$

(iii) (a) $\{(1,4,5,6,9), (0,1,1,1,2)\}$ (b) $\left\{\begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 0 \\ 3 \end{bmatrix}\right\}$ (c) $\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$,

(d) $2;3$

(iv) (a) $\{(1,-3,2,2,1), (0,1,2,0,-1), (0,0,1,0,-\frac{5}{12})\}$

(b) $\left\{\begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} -3 \\ 3 \\ -3 \\ -6 \\ 9 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ -2 \\ 0 \\ 2 \end{pmatrix}\right\}$ (c) $\left\{\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -16 \\ 2 \\ 5 \\ 0 \\ 12 \end{bmatrix}\right\}$, (d) $3;2$

(4) (i) $\{(2,0,-1)\}$ or $\{(4,0,-2)\}$ (ii) $\{(1,4,5,2), (2,1,3,0)\}$

(5) This problem is equivalent to: Find a basis for row(A) where the row vectors of A are the vectors given!

(a) $\{(1,1,-4,-3), (0,1,-5,-2), (0,0,1,-\frac{1}{2})\}$

(b) $\{(1,1,0,0), (0,1,1,1), (0,0,1,1), (0,0,0,1)\}$

(6) Note: Matrices are already in Row-echelon form!

(a) Row(A) = span $\{(1,-2,5,0,3), (0,1,3,0,0), (0,0,0,1,0)\}$

Col(A) = span $\{(1,0,0,0), (-2,1,0,0), (0,0,1,0)\}$

(b) a basis for row(A) = $\{(1,2,4,5), (0,1,-3,0), (0,0,1,-3), (0,0,0,1)\}$

a basis for col(A) = $\{(1,0,0,0,0), (2,1,0,0,0), (4,-3,1,0,0), (5,0,-3,1,0)\}$

(7) A basis for null(A) is $\{(0,0,1)\}$. $\Rightarrow$ null space $= \{(x,y,z) \mid x=0, y=0, z=t\}$

(a line through 0 .. namely z-axis.

A basis for Col(A) is $\{(1,0,0), (0,1,0)\}$ .. a plane through 0.

eq. of plane is $z = 0$! or The xy-plane.

(8) (a) a basis for the subspace is $\{\bar{v}_1, \bar{v}_2\}$; $\bar{v}_3 = 2\bar{v}_1 + \bar{v}_2$, $\bar{v}_4 = -2\bar{v}_1 + \bar{v}_2$

(b) ~ ~ ~ ~ ~ ~ $\{\bar{v}_1, \bar{v}_2, \bar{v}_4\}$; $\bar{v}_3 = 2\bar{v}_1 - \bar{v}_2$, $\bar{v}_5 = -\bar{v}_1 + 3\bar{v}_2 + 2\bar{v}_4$

(9) (a) Done in class (b) $U = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$, $V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $r = 2$

(c) $U = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$, $V = \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $r = 2$ (d) Assignment problem

(10)

	$\dim[\text{row}(A)]$	$\dim[\text{col}(A)]$	$\dim[\text{null}(A)]$	$\dim[\text{null}(A^T)]$
(a)	3	3	0	0
(b)	2	2	1	1
(c)	1	1	2	2
(d)	2	2	3	7
(e)	2	2	7	3
(f)	0	0	4	4
(g)	2	2	0	4

(11) If $\text{col}(A)$ is a plane through 0, $\dim \text{col}(A) = 2$. Therefore, $\dim(\text{null}(A)) = 1$ and hence can't be a plane through 0.
Since $\dim[\text{row}(A)] = 2$ as well .. it can't be a line through 0!

(12) Examine two cases: (1) $m > n$ (2) $m < n$

Case 1: $m > n$ .. we shall show: Row vectors of A form a linearly dependent set. Assume on the contrary that the row vectors of A are linearly independent. It follows that

$$\#\text{ of rows of } A = \underline{\underline{m}} = \dim[\text{row}(A)] = \text{rank}(A) \leq \min(m,n) = \underline{\underline{n}}$$

$\therefore\ m \leq n$ .. a contradiction to $m > n$

$\therefore$ rows of A form a linearly dependent set.

Case 2: $m < n$ .. show column vectors of A must be dependent!
The proof is identical in idea to one above!

(13) $$A = \begin{bmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - tR_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & t \\ 0 & t-1 & 1-t \\ 0 & 1-t & 1-t^2 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2}$$

$$\begin{bmatrix} 1 & 1 & t \\ 0 & t-1 & 1-t \\ 0 & 0 & -t^2-t+2 \end{bmatrix}$$ significant values $t-1=0 \Rightarrow t=1,\ t \neq 1$
$-t^2-t+2=0,\ t=1, t=-2,\ t\neq -2$

If $t=1$, $\text{rank}(A) = 1$
$t=-2$, $\text{rank}(A) = 2$, $t \neq 1, -2$ $\text{rank}(A) = 3$

(14) (a) yes! it can have independent rows; but not indep. column.
(b) No! it can not have indep. rows; but it can have indep. col.
Refer to EX. (12) above!

(15) Very Easy!

(16) To show $\text{rank}(AB) \leq \text{rank}(A)$. Recall: $\text{col}(AB) \subseteq \text{col}(A)$
$\therefore \dim \text{col}(AB) \leq \dim \text{col}(A)$
$\text{Rank}(AB) \leq \text{Rank}(A)$. The other part is similar

MATH 311

Support Material

REVIEW SHEET

FOR TERM TEST

(1) Know how to state Definitions 2, 3, 4, 5, 6, 7, 8, 9, and 12 (in the handouts provided) very carefully.

(2) Know how to write the proofs of theorems 9-(a), 10-(a), (b), 11, and 27 precisely (These theorems are in the handout provided and their proofs are in class notes).

(3) Given $A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$

(a) Find a basis for the row space of A, the column space of A, and the null space of A.

(b) Find the rank and nullity of A, and A^t.

(4) Find a subset of vectors of $S = \{\bar{v}_1 = (1,-2,0,0,3), \bar{v}_2 = (2,-5,-3,-2,6), \bar{v}_3 = (0,5,15,10,0), \bar{v}_4 = (2,6,18,8,6)\}$ in $\mathbb{R}^5$ that forms a basis for the subspace spanned by S.

Express the vectors not in the basis as a linear combination of the basis vectors.

(5) Repeat Ex. (4) if $S = \{\bar{v}_1 = (1,-2,0,3), \bar{v}_2 = (2,-5,-3,6), \bar{v}_3 = (0,1,3,0), \bar{v}_4 = (2,-1,4,-7), \bar{v}_5 = (5,-8,1,2)\}$ in $\mathbb{R}^4$.

(6) Let W be the subset of M_{22} defined by

$$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; b+2c=0,\ a+d=0,\ a,b,c,d \in \mathbb{R} \right\}$$

(a) Show that W is a subspace of M_{22}.

(b) Find a basis and the dimension of W.

(7) Find invertible matrices U, and V such that $UAV = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, $r = \text{rank}(A)$ if $A = \begin{bmatrix} 2 & 5 & 3 \\ -1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(8) Let U be the subset of P_3 defined by

$$U = \{ p(x) \mid p(0) = 0,\ p(2) = 0 \}$$

(a) Show that U is a subspace of U.

(b) Find a basis and the dimension of U

(9) Which of the following sets of vectors in P_2 are bases for P_2:

(i) $S_1 = \{ 4+x,\ x^2+2x-1 \}$ (ii) $S_2 = \{ 7-x,\ x^2,\ 2x-14 \}$

(iii) $S_3 = \{ x,\ 0,\ x^2+x-4 \}$ (iv) $S_4 = \{ 1, x, x^2, 9-4x+2x^2 \}$

(v) $S_5 = \{ 1-x+x^2,\ x^2-x,\ x^2 \}$ (vi) $S_6 = \{ x-3x^2,\ x+4x^2,\ 3x-2x^2 \}$

(10) which of the following sets of vectors in $\mathbb{R}^3$ are bases for $\mathbb{R}^3$

(i) $S_1 = \{ (0,0,0), (2,1,5), (3,-1,7) \}$ (ii) $S_2 = \{ (1,0,-1), (2,1,9) \}$

(iii) $S_3 = \{ (1,1,1), (2,1,9), (-4,-2,-18) \}$ (iv) $S_4 = \{ (1,1,1), (1,1,0), (1,0,0) \}$

(v) $S_5 = \{ (0,1,5), (2,1,-1), (4,17,27) \}$ (vi) $S_6 = \{ (2,1,3), (-1,3,5), (0,7,13) \}$

(11) If A is a 7×3 matrix, prove that the row vectors of "A" are linearly independent. Hint: what is the largest value of rank(A)?

(12) If A is a 5×9 matrix and if nullity$(A^t) = 2$. Find rank(A), nullity(A)

(13) If A is a 7×5 matrix, find the largest possible rank of A and the smallest possible nullity of A.

(14) Repeat Ex. (13) if A is of order 5×7.

(15) If A is a 8×5 matrix and if $\dim[\text{row}(A)] = 5$, show that the homogeneous linear system $A\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$.

Hint: Find nullity of A!

(16) Let A be an $n\times n$ invertible matrix and let $\{ \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_k \}$ be a linearly independent set of vectors in $\mathbb{R}^n$. Prove that the set $\{ A\bar{w}_1, A\bar{w}_2, \ldots, A\bar{w}_k \}$, "$\bar{w}_i$ are viewed as column matrices" is also linearly independent.

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MATH 311

Support Material

Sheet (6)

(1) Let $\mathbb{R}^2$ have the Euclidean inner product $<\bar{u},\bar{v}> = \bar{u}\cdot\bar{v}$.
If $\bar{u} = (3,-2)$, $\bar{v} = (4,5)$, and $\bar{w} = (-1,6)$, find

(a) $<\bar{u},\bar{v}>$ (b) $<\bar{u}+\bar{v},\bar{w}>$ (c) $<-4\bar{u},\bar{v}>$

(d) $\|\bar{w}\|$ (e) $d(\bar{v},\bar{w})$ (f) $\|\bar{v}-\bar{u}\|$

(2) Repeat Ex. (1) if $\mathbb{R}^2$ have the weighted Euclidean i.p

$$<(u_1,u_2),(v_1,v_2)> = 4u_1v_1 + 5u_2v_2$$

(3) Let M_{22} have the i.p $<\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}> = aa' + bb' + cc' + dd'$.
If $A = \begin{bmatrix} 3 & -2 \\ 4 & 8 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$, find $<A,B>$, $\|A\|$, and $d(B,A)$.

(4) Let P_2 have the i.p $<a_0+a_1x+a_2x^2, b_0+b_1x+b_2x^2> = a_0b_0+a_1b_1+a_2b_2$
Find $<p,q>$, $\|q\|$, and $d(p,q)$ if $p = -2+x+3x^2$, $q = 4-7x^2$

(5) Find the i.p on $\mathbb{R}^2$ generated by the matrix $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$.
<u>Hint</u>: The i.p is given by $<\bar{u},\bar{v}> = (A\bar{v})^t A\bar{u}$, $\bar{u} = (u_1,u_2)$, $\bar{v} = (v_1,v_2)$

(6) Determine which of the following are inner products on $\mathbb{R}^3$.

(a) $<\bar{u},\bar{v}> = u_1v_1 + u_3v_3$, (b) $<\bar{u},\bar{v}> = u_1v_1 - u_2v_2 + u_3v_3$

where $\bar{u} = (u_1,u_2,u_3)$, $\bar{v} = (v_1,v_2,v_3)$

(7) If $\bar{u},\bar{v}$, and $\bar{w}$ are vectors in an i.p.s such that
$<\bar{u},\bar{v}> = 2$, $<\bar{v},\bar{w}> = -3$, $<\bar{u},\bar{w}> = 5$, $\|\bar{u}\| = 1$, $\|\bar{v}\| = 2$,
$\|\bar{w}\| = 7$, find
$<2\bar{v}-\bar{w}, 3\bar{u}+2\bar{w}>$, $d(\bar{u},\bar{w})$, and $\|\bar{u}-2\bar{v}+4\bar{w}\|$

(8) Sketch the unit circle in $\mathbb{R}^2$ using the given i.p

(a) $<\bar{u},\bar{v}> = u_1v_1 + u_2v_2$

(b) $<\bar{u},\bar{v}> = \frac{1}{4}u_1v_1 + \frac{1}{16}u_2v_2$

(c) $<\bar{u},\bar{v}> = 2u_1v_1 + u_2v_2$

where $\bar{u} = (u_1,u_2)$, $\bar{v} = (v_1,v_2)$ are vectors in $\mathbb{R}^2$

(9) Let $\bar{u}, \bar{v}$ be vectors in an i.p.s. Prove that
$$\|\bar{u}+\bar{v}\|^2 + \|\bar{u}-\bar{v}\|^2 = 2\|\bar{u}\|^2 + 2\|\bar{v}\|^2,$$
$$\|\bar{u}+\bar{v}\|^2 - \|\bar{u}-\bar{v}\|^2 = 4\langle\bar{u},\bar{v}\rangle$$
(10) Let M_{22} have the i.p $\langle A,B\rangle = tr(AB^t)$. Find cosine of the angle between A, and B in each case

(a) $A = \begin{bmatrix} 2 & 6 \\ 1 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 1 \\ 4 & 2 \end{bmatrix}$

are A and B orthogonal w.r.t. i.p.?

(11) Let P_2 have the i.p of Ex. (4), and let θ be the angle between p, and q. Find $\cos\theta$ if

(a) $p = -1+5x+2x^2$, $q = 2+4x-9x^2$

(b) $p = -1+3x-x^2$, $q = 1+x$

are p, and q orthogonal w.r.to i.p?

(12) Let $\mathbb{R}^4$ have the Euclidean i.p and let θ be angle between $\bar{u} = (1,0,1,0)$, $\bar{v} = (-3,-3,-3,-3)$. Find θ

(13) Let M_{22} have the i.p of Ex. (3). Verify Cauchy-Schwarz inequality $|\langle\bar{u},\bar{v}\rangle| \leq \|\bar{u}\|\|\bar{v}\|$ for $\bar{u} = \begin{bmatrix} -1 & 2 \\ 6 & 1 \end{bmatrix}$, $\bar{v} = \begin{bmatrix} 1 & 0 \\ 3 & 3 \end{bmatrix}$

(14) Let W be the line in $\mathbb{R}^2$ with Eq. $y = 2x$. Find an Eq. for $W^\perp$

(15) Let W be the plane in $\mathbb{R}^3$ with Eq. $x-2y-3z=0$. Find parametric Equations for $W^\perp$

(16) Let W be the line in $\mathbb{R}^3$ with parametric Eqs. $x=2t$, $y=-5t$, $z=4t$, $t\in\mathbb{R}$. Find an Eq. for $W^\perp$

(17) Find a unit vector orthogonal to the set $\{\bar{u}=(2,1,-4,0), \bar{v}=(-1,-1,2,2), \bar{w}=(3,2,5,4)\}$

(18) Let V be an i.p. space. Show that if $\bar{u}$, and $\bar{v}$ are orthogonal unit vectors, then $\|\bar{u}-\bar{v}\| = \sqrt{2}$

(19) Find a basis and the dimension for the orthogonal complement of the subspace of $\mathbb{R}^n$ spanned by the given sets:

(a) $S = \{(1,-1,3), (5,-4,-4), (7,-6,2)\}$ in $\mathbb{R}^3$

(b) $S = \{(2,0,-1), (4,0,-2)\}$ in $\mathbb{R}^3$

(c) $S = \{(1,4,5,2), (2,1,3,0), (-1,3,2,2)\}$ in $\mathbb{R}^4$

(20) Let M_{22} have i.p of Ex. (3) and let $U = \text{span}\left\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\right\}$. Find a basis and the dimension of $U^\perp$.

ANSWERS
TO
SHEET (6)

(1) (a) 2 (b) 11 (c) $<-4\bar{u},\bar{v}> = -4<\bar{u},\bar{v}> = -8$. (d) $\sqrt{37}$

(e) $d(\bar{v},\bar{w}) = \|\bar{v}-\bar{w}\| = \sqrt{26}$ (f) $\sqrt{50}$ or $5\sqrt{2}$

(2) (a) -2 (b) 62 (c) 8 (d) $\sqrt{184}$ (e) $\sqrt{105}$ (f) $\sqrt{249}$

(3) $<A,B> = 3$; $\|A\| = \sqrt{93}$, $d(B,A) = \|B-A\| = \left\| \begin{bmatrix} -4 & 5 \\ -3 & -7 \end{bmatrix} \right\| = \sqrt{99}$

(4) $<p,q> = -29$, $\|q\| = \sqrt{<q,q>} = \sqrt{65}$, $d(p,q) = \|p-q\| = \sqrt{137}$

(5) let $\bar{u} = (u_1,u_2)$, $\bar{v} = (v_1,v_2)$. Hence $<\bar{u},\bar{v}> = A\bar{u} \cdot A\bar{v}$

$= (A\bar{v})^t A\bar{u} = \bar{v}^t (A^t A)\bar{u}$

Ans. $5u_1v_1 - u_1v_2 - u_2v_1 + 10u_2v_2$

(6) (a) Not an i.p .. axiom P4 fails. Let $\bar{u} = (0,1,0)$ .. hence
$<\bar{u},\bar{u}> = (0)(0) + (0)(0) = 0$ .. However $\vec{u} \neq \vec{0}$.

(b) Not an i.p .. axiom P4 fails .. try $\bar{u} = (0,1,0)$!

(7) $<2\bar{v}-\bar{w}, 3\bar{u}+2\bar{w}> = 6<\bar{v},\bar{u}> + 4<\bar{v},\bar{w}> - 3<\bar{w},\bar{u}> - 2<\bar{w},\bar{w}>$ ← This is $\|\bar{w}\|$

$= 6(2) + 4(-3) - 3(5) - 2(7)^2$

$= -113$

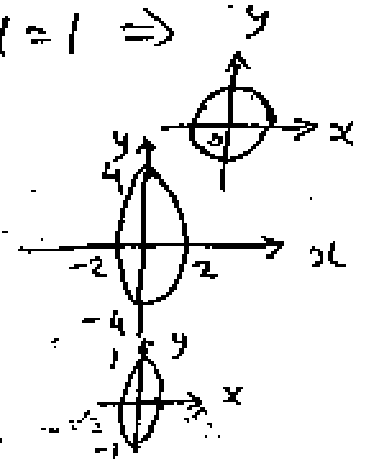
$d(\bar{u},\bar{w}) = \|\bar{u}-\bar{w}\| = \{<\bar{u}-\bar{w},\bar{u}-\bar{w}>\}^{\frac{1}{2}} = \sqrt{40}$

$\|u - 2\bar{v} + 4\bar{w}\| = \sqrt{<\bar{u}-2\bar{v}+4\bar{w}, \bar{u}-2\bar{v}+4\bar{w}>} = 881$

(8) (a) let $\bar{u} = (x,y) \in \mathbb{R}$ .. The unit circle is given by $\|\bar{u}\| = 1 \Rightarrow$
$\|\bar{u}\|^2 = 1 \Rightarrow <\bar{u},\bar{u}> = 1 \Rightarrow x^2 + y^2 = 1$

(b) $\|\bar{u}\| = 1 \Rightarrow \frac{1}{4}x^2 + \frac{1}{16}y^2 = 1$ .. an eq. of an ellipse

(c) $\|\bar{u}\| = 1 \Rightarrow 2x^2 + y^2 = 1$ .. an eq. of an ellipse



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(9) show $\|\bar{u}+\bar{v}\|^2 + \|\bar{u}-\bar{v}\|^2 = 2\|\bar{u}\|^2 + 2\|\bar{v}\|^2$

L.H.S $= \|\bar{u}+\bar{v}\|^2 + \|\bar{u}-\bar{v}\|^2 = <\bar{u}+\bar{v},\bar{u}+\bar{v}> + <\bar{u}-\bar{v},\bar{u}-\bar{v}>$

$= <\bar{u},\bar{u}> + \cancel{2<\bar{u},\bar{v}>} + <\bar{v},\bar{v}>$

$+ <\bar{u},\bar{u}> - \cancel{2<\bar{u},\bar{v}>} + <\bar{v},\bar{v}>$

$= 2<\bar{u},\bar{u}> + 2<\bar{v},\bar{v}> = 2\|\bar{u}\|^2 + 2\|\bar{v}\|^2$

$=$ R.H.S

The other part of this problem is identical!!

(10) (a) $\cos\theta = \dfrac{\langle A,B\rangle}{\|A\|\|B\|} = \dfrac{6+12+1-0}{\sqrt{50}\sqrt{14}} = \dfrac{19}{10\sqrt{7}}$

(b) $\cos\theta = 0 \Rightarrow \theta = 90^\circ \Rightarrow$ matrix $A \perp B$.

(11) (a) $\cos\theta = \dfrac{\langle p,q\rangle}{\|p\|\|q\|} = \dfrac{-2+20-18}{\sqrt{30}\sqrt{101}} = 0 \Rightarrow p \perp q$

(b) $\cos\theta = \dfrac{2}{\sqrt{22}}$

(12) $\cos\theta = \dfrac{-6}{\sqrt{2}\sqrt{36}} = -\dfrac{1}{\sqrt{2}} \Rightarrow \theta = 135^\circ$

(13) $\langle \bar{u},\bar{v}\rangle = 20$, $\|\bar{u}\| = \sqrt{42}$, $\|\bar{v}\| = \sqrt{19}$ .. clearly $|\langle \bar{u},\bar{v}\rangle| = 20$
$< \|\bar{u}\|\|\bar{v}\| \cong 28.25$

(14) Eq. for $W^\perp$ is $y = -\frac{1}{2}x$ .. The line $\perp$ to $y = 2x$ ..

(15) This is the line normal to plane! Ans: $x = t,\ y = -2t,\ z = -3t,\ t \in \mathbb{R}$

(16) $2x - 5y + 4z = 0$ .. This is a plane through origin with normal vector the line given!

(17) Assume $\mathbb{R}^4$ has Euclidean i.p Ans: $\pm \frac{1}{57}(-34, 44, -6, 11)$

(18) If $\bar{u} \perp \bar{v}$, then $\langle \bar{u},\bar{v}\rangle = 0$ .. Also: Given $\|\bar{u}\| = \|\bar{v}\| = 1$.

Now
$$\begin{aligned}\|\bar{u}-\bar{v}\|^2 &= \langle \bar{u}-\bar{v}, \bar{u}-\bar{v}\rangle \\ &= \langle \bar{u},\bar{u}\rangle - 2\langle \bar{u},\bar{v}\rangle + \langle \bar{v},\bar{v}\rangle \\ &= \|u\|^2 - 2\langle \bar{u},\bar{v}\rangle + \|\bar{v}\|^2 \\ &= 1 - 2(0) + 1 = 2 \\ \Rightarrow \|\bar{u}-\bar{v}\| &= \sqrt{2}\end{aligned}$$

(19) (a) $\{(16,19,1)\}$, $\dim U^\perp = 1$ — Note: This is a basis for null (A) when A has given vectors as row.

See note

(b) $\{(0,1,0), (\frac{1}{2},0,1)\}$, $\dim U^\perp = 2$

(c) $\{(-1,-1,1,0), (2,-4,0,7)\}$, $\dim U^\perp = 2$

(20) The coordinate vectors relative to standard basis of M_{22} of spanning set is $\{(1,1,0,0), (1,0,1,0), (1,0,1,1)\}$.
Let A be the matrix whose rows are the vectors given!

$$\therefore A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Now, find a basis for the null space of A! Ans: $\{(-1,1,1,0)\}$

$\therefore$ A basis for $U^\perp$, written back in matrix form! .. is $\left\{\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\right\}$

$\dim U^\perp = 1$ .. hence $\dim U = 3$.

MATH 311
Support Material
Sheet (7)

(1) Let $\mathbb{R}^3$ has the Euclidean i.p. Determine which of the following sets of vectors are orthogonal and which are orthonormal?

(a) $S = \{(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}), (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})\}$

(b) $S = \{(2,-2,1), (2,1,-2), (1,2,2)\}$

(c) $S = \{(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}), (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0), (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})\}$

(2) Let P_2 has the i.p $\langle a_0+a_1x+a_2x^2, b_0+b_1x+b_2x^2\rangle = a_0b_0+a_1b_1+a_2b_2$. Determine which of the following sets are orthogonal and which are orthonormal

(a) $S = \{\frac{2}{3} - \frac{2}{3}x + \frac{1}{3}x^2, \frac{2}{3} + \frac{1}{3}x - \frac{2}{3}x^2, \frac{1}{3} + \frac{2}{3}x + \frac{2}{3}x^2\}$

(b) $S = \{1, \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}x^2, x^2\}$

(3) Let M_{22} has the i.p $\langle A, B\rangle = tr(AB^t)$. Determine which of the following sets are orthogonal and which are orthonormal?

(a) $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}, \begin{bmatrix} 0 & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \right\}$

(b) $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}$

(4) Show that the set $\{\bar{x} = (\frac{1}{\sqrt{5}}, -\frac{1}{\sqrt{5}}), \bar{y} = (\frac{2}{\sqrt{30}}, \frac{3}{\sqrt{30}})\}$ is orthonormal if $\mathbb{R}^2$ has the weighted Euclidean i.p $\langle \bar{u}, \bar{v}\rangle = 3u_1v_1 + 2u_2v_2$

(5) Show that the sets $\{\bar{v}_1 = (1,-2,3,-4), \bar{v}_2 = (2,1,-4,-3), \bar{v}_3 = (-3,4,1,-2), \bar{v}_4 = (4,3,2,1)\}$ is an orthogonal basis for $\mathbb{R}^4$.

Use Expansion theorem to express $\bar{x} = (-1,2,3,7)$ as a l.c. of vectors in S

(6) Let $\mathbb{R}^2$ have the Euclidean i.p. Use Gram-schmidt process to transform the basis $\{\bar{u}_1 = (1,-3), \bar{u}_2 = (2,2)\}$ into an <u>orthogonal basis</u>

(7) Let $\mathbb{R}^3$ have the Euclidean i.p. Use Gram-schmidt process to transform the basis $\{(1,1,1), (-1,1,0), (1,2,1)\}$ into an <u>orthonormal basis</u>.

(8) Let $\mathbb{R}^3$ has the i.p $\langle \bar{u}, \bar{v}\rangle = u_1v_1 + 2u_2v_2 + 3u_3v_3$. Use Gram-schmidt to transform $\{\bar{u}_1 = (1,1,1), \bar{u}_2 = (1,1,0), \bar{u}_3 = (1,0,0)\}$ into an orthonormal basis.

(9) Let M_{22} has the i.p $<A,B> = tr(AB^t)$. Use Gram-schmidt process to transform the basis $\mathcal{B}$ into an orthogonal Basis

(a) $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

(b) $\mathcal{B} = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

(10) Let P_2 has the i.p $<p,q> = p(0)q(0) + p(1)q(1) + p(2)q(2)$. Use Gram-schmidt process to transform the basis $\{1, x, x^2\}$ into an orthogonal basis.

(11) Let M_{22} have the i.p $<A,B> = tr(AB^t)$. In each case find

(a) An orthogonal basis of subspace U

(b) $\text{proj}_U A$ (c) The vector in U closest to A.

(a) $U = \text{span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$, $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

(b) $U = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$, $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

(12) Let $\mathbb{R}^4$ has the Euclidean i.p. In each case find

(a) An orthogonal basis for the subspace W.

(b) $\text{proj}_W \vec{x}$ (c) The vector in W closest to $\vec{x}$

(a) $W = \text{span}\{(2,1,3,-4), (1,2,0,1)\}$, $\vec{x} = (1,-2,1,6)$

(b) $W = \text{span}\{(1,0,1,1), (0,1,-1,1), (-2,0,1,1)\}$, $\vec{x} = (3,1,5,9)$

(13) Let P_2 has the i.p $<p,q> = p(0)q(0) + p(1)q(1) + p(2)q(2)$.

(a) Find an orthogonal basis for the subspace

$$U = \text{span}\{1+x, x^2\}$$

(b) Find $\text{proj}_U P$, $P = 1 + x^2$

(c) Find the polynomial in U closest to $p = 1 + x^2$

(d) Express $p = 1 + x^2$ as a sum of two polynomials $P_1 \in U$, $P_2 \in U^\perp$

(14) Let $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be an orthonormal basis for an i.p.s. Prove that if $\vec{w} \in V$, then $\|\vec{w}\|^2 = <\vec{w}, \vec{v}_1>^2 + <\vec{w}, \vec{v}_2>^2 + <\vec{w}, \vec{v}_3>^2$

**(15) Let $\mathbb{R}^n$ has the Euclidean i.p and let A be an $n \times n$ matrix such that $A^t A = I_n$. Prove that $\|A\vec{x}\|^2 = \|\vec{x}\|^2$ for all $\vec{x} \in \mathbb{R}^n$.

** means very Important!!

MATH 311

Support Material

Sheet (8)

(1) Find the normal system associated with the linear system

(a) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 & 0 \\ 3 & 1 & 2 \\ -1 & 4 & 5 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$

(2) Find the Least squares approximation to a solution of each of the following systems of Linear Equations $A\vec{x} = \vec{b}$ and hence find the orthogonal projection of $\vec{b}$ on the column space of A

(a) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

(3) Find a least squares approximating function of the form

$$\bar{f} = r_0 x + r_1 x^2 + r_2 (-1)^x$$

for each of the following sets of data points:

(a) $(-1,1), (0,3), (1,1), (2,0)$

(b) $(0,1), (1,1), (2,5), (3,10)$

(4) Given the data pairs $(-1,0)$, $(0,1)$, and $(1,4)$, find the best fit by a function of the form $\bar{f} = r_0 x + r_1 2^x$

(5) Find the least squares approximating function of the form,

$$\bar{g} = r_0 + r_1 x^2 + r_2 \sin\left(\frac{\pi x}{2}\right)$$

for the set of data points $(0,3)$, $(1,0)$, $(1,-1)$, and $(-1,2)$

Answers

(1) (a) $\begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$, (b) $\begin{bmatrix} 15 & -1 & 5 \\ -1 & 22 & 30 \\ 5 & 30 & 45 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 13 \end{bmatrix}$

(2) (a) $x_1 = 3, x_2 = \frac{9}{2}$; $\left(\frac{15}{2}, \frac{3}{2}, 6\right)$ (b) $x_1 = \frac{3}{7}, x_2 = -\frac{2}{3}$; $\left(\frac{46}{21}, \frac{5}{21}, \frac{13}{21}\right)$

(3) (b) $\bar{f} = \frac{1}{68}\left[11x + 75x^2 + 33(-1)^x\right]$ (a) You Find out!

(4) $\bar{f} = \frac{10}{11} x + \frac{16}{11} \cdot 2^x$

(5) (h) $\bar{g} = 3 - \frac{9}{4} x^2 - \frac{5}{4} \sin\left(\frac{\pi x}{2}\right)$

MATH 311

Support Material

Sheet (9)

(1) For each of the following matrices determine

(a) The characteristic polynomial of A

(b) The characteristic equation of A

(c) a basis and the dimension for the Eigenspaces of A.

(a) $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$ (c) $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(d) $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ (e) $A = \begin{bmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{bmatrix}$ (f) $\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} = A$

(2) Find Eigenvalues of (a) $A = \begin{bmatrix} -1 & 6 \\ 0 & 5 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 3 & 7 & 11 \\ 0 & \frac{1}{2} & 3 & 8 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

(3) Find Eigenvalues and a basis for Eigen spaces of A^{25} if $A = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

(4) prove that: A square matrix A is invertible iff $\lambda = 0$ is <u>not</u> an Eigenvalue of A

(5) prove: If λ is an Eigenvalue of an invertible matrix A and $\vec{x}$ is a Corresponding Eigenvector, then $\frac{1}{\lambda}$ is an Eigenvalue of A^{-1}, and $\vec{x}$ is a Corresponding Eigenvector.

(6) If A is a 3×3 matrix and if $C_A(\lambda) = -\lambda^3 - 2\lambda^2 + \lambda + 5$, find $\det A$.

(7) prove that if A is a square matrix, then A, A^t have <u>same</u> Eigenvalues.

show by an example that A, A^t need not have <u>same</u> Eigenspaces!

MATH 311

Support Material

Sheet (10)

(1) Let $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$

(a) Find the Eigenvalues of A Ans. $\lambda = 3, 3, 5$

(b) Find the dimensions of Eigenspaces of A Ans. $\dim E_3 = 2$, $\dim E_5 = 1$

(c) Is A diagonalizable? Justify your answer. yes!

(2) Determine if the matrix given is diagonalizable

(a) $A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & -1 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

Ans. (a), (b), (c) : No!

(3) In each case, show that A is diagonalizable and find a matrix P that diagonalizes A. What is $P^{-1}AP$?

(a) $A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$ Ans. $P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$, $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ Ans. $P = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $P^{-1}AP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

(c) $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ Ans. $P = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, $P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ Ans. $P = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(4) Refer to matrix "A" of part (d) in Ex. (3). Find A^{1000}, A^{2301}, and A^{-3248}

(5) The Eigenvalues of $A = \begin{bmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{bmatrix}$ are $\lambda = 6$, $\lambda = -3, -3$.

What are the dimensions of E_6 and E_{-3}? Hint: A is symmetric

(6) In each case show that "A" is orthogonally diagonalizable and find a matrix P that orthogonally diagonalizes A. What is P^tAP

(a) $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ Ans. $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, $P^tAP = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$

(b) $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$ .. Done in class

(c) $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$, $C_A(\lambda) = \lambda(\lambda-3)^2$ Ans. $P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$.

MATH 311

Support Material

SHEET (11)

(1) Determine whether the function is a linear transformation

(a) $T: V \longrightarrow \mathbb{R}$, $T(\vec{v}) = \|\vec{v}\|$

(b) $T: M_{22} \rightarrow M_{23}$, $T(A) = AB$, B is a fixed 2×3 matrix

(c) $T: P_2 \longrightarrow P_2$, $T(a_0 + a_1x + a_2x^2) = (a_0+1) + (a_1+1)x + (a_2+1)x^2$

Ans. (a) non-linear (b) linear (c) non-linear

(2) let $\{\bar{v}_1 = (1,1), \bar{v}_2 = (1,0)\}$ be a basis for $\mathbb{R}^2$ and let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T(\bar{v}_1) = (1,-2)$, $T(\bar{v}_2) = (-4,1)$.

Find a formula for $T(x,y)$ and use it to find $T(5,-3)$.

Hint: express (x,y) as a l.c. of $\bar{v}_1, \bar{v}_2$, then use linearity!

Ans $T(x,y) = (-4x+5y, x-3y)$; $T(5,-3) = (-35, 14)$

(3) let $T: V \to \mathbb{R}^3$ be a linear transformation and let $\hat{v}_1, \hat{v}_2$, and $\hat{v}_3$ be vectors in V such that $T(\bar{v}_1) = (1,-1,2)$, $T(\bar{v}_2) = (0,3,2)$, $T(\bar{v}_3) = (-3,1,2)$.

Find $T(2\bar{v}_1 - 3\bar{v}_2 + 4\bar{v}_3)$. Hint $T(2\bar{v}_1 - 3\bar{v}_2 + 4\bar{v}_3) = 2T(\bar{v}_1) - 3T(\bar{v}_2) + 4T(\bar{v}_3)$

Ans. $(-10, -7, 6)$

(4) Let $T_1: M_{22} \to M_{22}$, $T_2: M_{22} \to \mathbb{R}$ be the linear transformations given by $T_1(A) = A^t$, $T_2(A) = tr(A)$. Find $(T_1 \circ T_2)(A)$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Can you find $(T_2 \circ T_1)(A)$? Ans. $(T_1 \circ T_2)(A) = a+b$, $(T_2 \circ T_1)(A)$ D.N.E

(5) let $T_1: \mathbb{R}^2 \to \mathbb{R}^3$, $T_2: \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T_1(x,y) = (x-y, y+z, x-z)$, $T_2(x,y,z) = (0, x+y+z)$. Find $(T_2 \circ T_1)(x,y)$

(6) let $T_1: V \to V$ be the dilation $T_1(\bar{v}) = \bar{v}$. Find a linear operator $T_2: V \to V$ such that $T_1 \circ T_2 = I_V$, $T_2 \circ T_1 = I_V$. Ans. $T_2(\bar{v}) = \frac{1}{4}\bar{v}$

(7) In each case find a basis for $Ker(T)$, $Im(T)$ where T is a linear transformation

(1) $T: \mathbb{R}^4 \to \mathbb{R}^3$, $T(x,y,z,w) = (4x+y-2z-3w, 2x+y+z-4w, 6x-9z+9w)$

(2) $T: P_2 \longrightarrow P_3$, $T(p) = xp$

(3) $T_A: \mathbb{R}^4 \to \mathbb{R}^2$, $A = \begin{bmatrix} 4 & 1 & 5 & 2 \\ 1 & 2 & 3 & 0 \end{bmatrix}$. Find $rank(T_A)$, $nullity(T_A)$.

(8) Let $T: \mathbb{R}^3 \to \mathbb{R}^4$, $T(x,y,z) = (x+y+z, x, -z, x-y)$

$S: \mathbb{R}^4 \to \mathbb{R}^3$, $T(x,y,z,w) = (x+y-w, z+w, x)$

show that T is one to one but not onto, where as S is onto but not one to one. Find $T \circ S$, $S \circ T$.

(9) Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$, $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformations given by $T_1(x,y) = (x+y, x-y)$, $T_2(x,y) = (2x+y, x-2y)$

(a) show that T_1 and T_2 are isomorphism

(b) Find a formula for $T_1^{-1}(x,y)$, $T_2^{-1}(x,y)$

(c) verify that $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$

(10) In each case determine whether

$$T: M_{22} \longrightarrow M_{22}$$

is an isomorphism and find $T^{-1}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$

(a) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ (b) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

(c) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Ans. (a) Not an isomorphism

(b) an isomorphism, $T^{-1}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & c \\ d & b \end{bmatrix}$

(c) ~ ~ , $T^{-1}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

(11) Let $T: P_2 \to P_3$ be the linear transformation defined by

$$T(a_0 + a_1 x + a_2 x^2) = x\left(a_0 + a_1(x-6) + a_2(x-6)^2\right)$$

(a) Find $[T]_{D,B}$, the matrix of T w.r.t. to ordered bases

$B = \{1, x, x^2\}$, $D = \{x^3, x^2, x, 1\}$

(b) Use part (a) to compute $(T(\vec{v}))_D$ and hence $T(\vec{v})$ where $\vec{v} = 3 - 2x + 4x^2$

(c) check the result in part (b) by computing $T(\vec{v})$ directly

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MATH 311

Support Material

REVIEW SHEET

1. Know how to state Definitions 7, 8, 9, 10, 17, 18, 20, 21, 25, and 27 (in the Handout provided) very carefully!

2. Know how to write proofs of theorems 7, 8, 12, 23, 27, 28, and 31 precisely (These theorems are stated in the handout provided. For their proofs refere to class notes.

3. Let W be the subspace of $\mathbb{R}^5$ spanned by
$$S = \{(1,-2,0,0,3), (0,5,15,10,0), (2,-5,-3,-2,6), (2,6,18,8,6)\}$$
(a) Find a subset of S that forms a basis for W and determine $\dim W$
(b) Express the vectors not in the basis as a linear combination of the basis vectors. You may do so by inspection!

4. Find invertible matrices U, and V such that $UAV = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$. $r = \text{rank } A$ if
$$A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & 0 & 3 \\ 0 & 1 & -4 & 1 \end{bmatrix}$$

5. Let
$$A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$$
Find (a) A basis for $\text{row}(A)$, $\text{col}(A)$
(b) A basis for $\text{null}(A^t)$
(c) Rank and nullity of A, and A^t

6. In each case find a basis and the dimension of the orthogonal complement of subspace of $\mathbb{R}^3$ spanned by the set given. Assume $\mathbb{R}^3$ has the Euclidean i.p
(a) $S = \{(2,0,-1), (4,0,-2)\}$
(b) $S = \{(1,-1,3), (5,-4,-4), (7,-6,2)\}$

(7) let $\mathcal{P}_2$ have the i.p $<p,q> = a_0 b_0 + a_1 b_1 + a_2 b_2$ where $p = a_0 + a_1 x + a_2 x^2$, $q = b_0 + b_1 x + b_2 x^2$. let W be the subspace of $\mathcal{P}_2$ spanned by $S = \{5 - 2x + 3x^2, 1 + 7x\}$, find a basis and the dimension of $W^{\perp}$

(8) Let M_{23} have the i.p $<X,Y> = tr(XY^t)$ and let W be the subspace of M_{23} spanned by $S = \left\{ \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \right\}$. Find a basis and the dimension of $W^{\perp}$

(9) Let $\mathbb{R}^3$ has the i.p generated by the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$. Use Gram-Schmidt process to transform the set $S = \{\bar{u}_1 = (1,1,1), \bar{u}_2 = (1,1,0), \bar{u}_3 = (1,0,0)\}$ into an orthonormal basis

(10) Let $\mathcal{P}_2$ has the i.p of Ex. (7) above. Use Gram-Schmidt process to transform the basis $\{1, x, x^2\}$ into an orthogonal basis. Repeat Ex. (10) if i.p is given by

$$<p,q> = p(0)q(0) + p(1)q(1) + p(2)q(2)$$

(11) let M_{22} have the i.p $<A,B> = tr(AB^t)$, and let $U = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$. Find $\text{proj}_U A$ if $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ and express A as a sum of two matrices $A_1 \in U$, $A_2 \in U^{\perp}$. What is the matrix in U closest to A?

(12) let $\mathbb{R}^4$ has the Euclidean i.p and let W be the subspace of $\mathbb{R}^4$ spanned by $S = \{(1,0,1,1), (0,1,-1,1), (-2,0,1,1)\}$.

(a) Find $\text{proj}_W \vec{x}$ if $\bar{x} = (3,1,5,9)$

(b) Express $\bar{x}$ as a sum of a vector in W and a vector in $W^{\perp}$

(c) Determine a basis and the dimension of $W^{\perp}$.

(13) Find the least squares approximating function of the form

$$\bar{g} = r_0 + r_1 x^2 + r_2 \sin\left(\frac{\pi x}{2}\right)$$

for the set of data points $(0,3), (1,0), (1,-1)$, and $(-1,2)$

(14) Find the least squares approximation of

$$A\vec{x} = \vec{b}$$

where $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix}$; $\vec{b} = \begin{bmatrix} 7 \\ 0 \\ 7 \end{bmatrix}$. Determine $\mathrm{proj}_W \vec{b}$ where W is the column space of A.

(14) prove that if A is a square matrix, then A, and A^t have the same characteristic polynomials.

(15) Let $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

Find Eigenvalues and a basis for each of the Eigenspaces of A

16) Given $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$. Determine whether A is diagonalizable and if so find a matrix P that diagonalizes A. what is $P^{-1}AP$?

(17) show that $A = \begin{bmatrix} 2 & -1 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ is <u>not</u> diagonalizable

(18) Let $A = \begin{bmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{bmatrix}$. The Eigen values of A are $\lambda = 6, -3, -3$

(a) state clearly why A is orthoganally diaganalizable.

(b) what are the dimensions of E_6, and E_3 ?

(19) Refer to Ex. (18), find an orthogonal matrix P that diagonalizes A and determine P^tAP.

(20) Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(x,y,z) = (x-2y+z,\ x-2z)$.

Find (a) The standard matrix of T

(b) A basis for Ker(T), Im(T).

To be Continued!