STUDENT I.D. #

Solutions

MATH 311 NAME:

QUIZ #(1) YOUR MARK: /30

June -03, 2002

Attempt ALL Questions:

(I) In each case determine whether W is a subspace of V

(3 marks)

(a) $V = \mathbb{R}^3$, $W = \{(x, y, 1) \mid x, y \in \mathbb{R}\}$

W is not a sub space! of $\mathbb{R}^3$

A Counter Example; The zero vector $(0,0,0) \notin W$

or W is not closed under addition.

pick $(2,5,1)$, $(-3,4,1)$ in W. The sum $(1,9,2) \notin W$.

(4 marks)

(b) $V = M_{22}$, $W = \{A \mid A \text{ in } M_{22},\ A^t = A\}$

1st. W is non-empty. Indeed $0^t = 0 \Rightarrow 0 \in W$

Let $A, B \in W$. Then $A = A^t$, $B = B^t$

Now, $(A+B)^t = A^t + B^t = A + B$

$\Rightarrow$ $A+B$ is symmetric $\Rightarrow A + B \in W$

$\Rightarrow$ W is closed under $(+)$ or A1 is true.

Next, Let $K \in \mathbb{R}$, $A \in W$. Then $A^t = A$ and

$(KA)^t = KA^t = KA \Rightarrow KA \in W \Rightarrow$ W is closed under (
or S1 is true!

$\therefore$ W is a subspace of M_{22}.

(3 marks)

(c) $V = P_2$, $W = \{a + bx \mid a, b \in \mathbb{R}, b \neq 0\}$

W is not a subspace of P_2 Since for instance

$0 \notin W$

OR: W is not closed under addition. For instance

$p = 1 + 2x \in W$, $q = 5 - 2x \in W$. However

$p + q = 6 \notin W$.

1) (a) Show that the solution space of the homogeneous Linear system

$$x - 2y + 3z = 0$$
$$-2x + 5y + 2z = 0$$
$$5x - 11y + 7z = 0$$

is a straight Line through the origin in $\mathbb{R}^3$ and find its parametric Equations (6 marks)

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ -2 & 5 & 2 & 0 \\ 5 & -11 & 7 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 + 2R_1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 1 & 8 & 0 \\ 0 & -1 & -8 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 + R_2]{R_1 \to R_1 + 2R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 19 & 0 \\ 0 & 1 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad n = 3,\ r = 2 \Rightarrow n - r = 1\text{-parameter Family}$$

$\therefore$ $x + 19z = 0$, $y + 8z = 0$, $z = t$, $t \in \mathbb{R}$

$\therefore$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} -19 \\ -8 \\ 1 \end{pmatrix}$ which is an equation of a line through origin in $\mathbb{R}^3$.

(b) Express $p(x) = 5 - 5x + x^2$ as a Linear Combinations of $S = \{1,\ 3 - 2x,\ 4 + x - x^2\}$ (4 marks)

$$5 - 5x + x^2 = c_1 \cdot 1 + c_2(3 - 2x) + c_3(4 + x - x^2)$$

rearrange

$$5 - 5x + x^2 = (c_1 + 3c_2 + 4c_3) + (-2c_2 + c_3)x - c_3 x^2$$

Comparing we get:

$$c_1 + 3c_2 + 4c_3 = 5$$
$$-2c_2 + c_3 = -5$$
$$-c_3 = 1$$

solve: $c_1 = 3$, $c_2 = 2$, $c_3 = -1$

$\therefore$ $p(x) = 5 - 5x + x^2 = 3 \cdot 1 + 2(3 - 2x) - 1 \cdot (4 + x - x^2)$

(III) Let W be the subset of M_{22} defined by

$$W = \left\{ \begin{bmatrix} a & a-b \\ a+b & a \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$$

(a) Show that W is a subspace of M_{22}. (4 Marks)

Clearly W is non-empty! $\begin{bmatrix} 0 & 0-0 \\ 0+0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$

let $A = \begin{bmatrix} a & a-b \\ a+b & a \end{bmatrix}$, $B = \begin{bmatrix} a' & a'-b' \\ a'+b' & a' \end{bmatrix} \in W$

$\therefore A+B = \begin{bmatrix} (a+a') & (a+a')-(b+b') \\ (a+a')+(b+b') & a+a' \end{bmatrix} \in W \Rightarrow$ A1 is true

let $k \in \mathbb{R}$, $kA = \begin{bmatrix} ka & ka-kb \\ ka+kb & ka \end{bmatrix} \in W \Rightarrow$ S1 is true

$\therefore$ W is a subspace of M_{22}

(b) Find a basis and the dimension of W. Justify your answer (6 Marks)

$$\begin{bmatrix} a & a-b \\ a+b & a \end{bmatrix} = a\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$\therefore W = \text{span}\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$

Since the set $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$ is linearly independent (neither matrix is a scalar multiple of the other),

the set $B = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$ is a basis for W

and $\dim W = 2$

STUDENT'S I.D#:

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Solutions

MATH 311

QUIZ(II)

NAME:

Your Mark: /30

June 19, 2002

Attempt All Questions:

(1) Let $\mathbb{R}^3$ have the Euclidean inner product and let W be the subspace of $\mathbb{R}^3$ spanned by $S = \{(1,1,1), (2,1,-3)\}$

(a) Show that S is an orthogonal basis for W (4 Marks)

First S is a basis for W since S is linearly independent (neither vector in S is a scalar multiple of the other)

Now, let $\bar{v}_1 = (1,1,1)$, $\bar{v}_2 = (2,1,-3)$

$\langle \bar{v}_1, \bar{v}_2 \rangle = \bar{v}_1 \cdot \bar{v}_2 = 2+1-3 = 0 \Rightarrow S$ is orthogonal

(b) Find $\vec{p} = \text{proj}_W \vec{x}$ where $\vec{x} = (3,-5,5)$. (3 Marks)

$\vec{p} = t_1 \bar{v}_1 + t_2 \bar{v}_2$ where

$$t_1 = \frac{\langle \bar{x}, \bar{v}_1 \rangle}{\|\bar{v}_1\|^2} = \frac{3-5+5}{1+1+1} = 1, \quad t_2 = \frac{\langle \bar{x}, \bar{v}_2 \rangle}{\|\bar{v}_2\|^2} = \frac{6-5-15}{4+1+9} = -1$$

$$\therefore \bar{p} = 1.\bar{v}_1 - 1.\bar{v}_2 = (1,1,1) - (2,1,-3) = (-1,0,4)$$

(c) express $\vec{x}$ as a sum of a vector in W and a vector in $W^\perp$ (2 Marks)

$\bar{x} - \bar{p} = (3,-5,5) - (-1,0,4) = (4,-5,1)$

$\therefore \vec{x} = \vec{p} + (\bar{x} - \bar{p})$ where $\vec{p} = (-1,0,4) \in W$,

$\vec{x} - \vec{p} = (4,5,1) \in W^\perp$

(d) Find $\dim(W^\perp)$ - Justify your answer! (one Mark)

S is a basis for $W \Rightarrow \dim(W) = 2$

Now, $\dim(V = \mathbb{R}^3) = \dim W + \dim W^\perp$

$3 = 2 + \dim(W^\perp) \Rightarrow \dim(W^\perp) = 1$

(2) Let $\mathbb{R}^3$ have the Euclidean i.p. Use Gram-Schmidt process to transform the basis $B = \{(1,1,1), (-1,1,0), (1,2,1)\}$ into an <u>orthonormal</u> basis.

(10 Marks)

First let $B = \{\bar{u}_1 = (1,1,1), \bar{u}_2 = (-1,1,0), \bar{u}_3 = (1,2,1)\}$

Construct an <u>orthogonal</u> basis $B' = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ as follows

$$\bar{v}_1 = \bar{u}_1 = (1,1,1),$$

$$\bar{v}_2 = \bar{u}_2 - \frac{\langle \bar{u}_2, \bar{v}_1 \rangle}{\|\bar{v}_1\|^2} \bar{v}_1 = (-1,1,0) - \frac{0}{3}\bar{v}_1 = (-1,1,0)$$

$$\bar{v}_3 = \bar{u}_3 - \frac{\langle \bar{u}_3, \bar{v}_1 \rangle}{\|\bar{v}_1\|^2} \bar{v}_1 - \frac{\langle \bar{u}_3, \bar{v}_2 \rangle}{\|\bar{v}_2\|^2} \bar{v}_2$$

$$= (1,2,1) - \frac{1+2+1}{3}(1,1,1) - \frac{-1+2+0}{2}(-1,1,0)$$

$$= (1,2,1) - \frac{4}{3}(1,1,1) - \frac{1}{2}(-1,1,0)$$

$$= \frac{6}{6}(1,2,1) - \frac{8}{6}(1,1,1) - \frac{3}{6}(-1,1,0)$$

$$= \frac{1}{6}\left[6(1,2,1) - 8(1,1,1) - 3(-1,1,0)\right] = \frac{1}{6}(1,1,-2)$$

You may drop $\frac{1}{6}$ and take $\bar{v}_3 = (1,1,-2)$

$\therefore B' = \{\bar{v}_1 = (1,1,1), \bar{v}_2 = (-1,1,0), \bar{v}_3 = (1,1,-2)\}$ is an orthogonal basis

To find orthonormal basis $B'' = \{\bar{w}_1, \bar{w}_2, \bar{w}_3\}$

$$\bar{w}_1 = \frac{\bar{v}_1}{\|\bar{v}_1\|} = \frac{(1,1,1)}{\sqrt{3}}, \quad \bar{w}_2 = \frac{\bar{v}_2}{\|\bar{v}_2\|} = \frac{(-1,1,0)}{\sqrt{2}}, \quad \bar{w}_3 = \frac{\bar{v}_3}{\|\bar{v}_3\|} = \frac{(1,1,-2)}{\sqrt{6}}$$

$$\therefore B'' = \left\{ \frac{1}{\sqrt{3}}(1,1,1), \frac{1}{\sqrt{2}}(-1,1,0), \frac{1}{\sqrt{6}}(1,1,-2) \right\}.$$

is an orthonormal basis for W.

(3) Let M_{22} have the i.p $\langle X, Y \rangle = tr(XY^t)$ and let $U = \text{span}\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$. Find a basis and the dimension of $U^\perp$. (10 Marks)

Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in U^\perp$. Therefore $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is orthogonal to $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, and $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. It follows that

$$\left\langle \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\rangle = 0 \Rightarrow a + b + c + d = 0$$

and $$\left\langle \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle = 0 \Rightarrow b - c + d = 0$$

Consider the homogeneous system $A\vec{x} = \vec{0}$, $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$

let us find a basis for null(A).

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array}\right]$$

system reduces to

$$\begin{aligned} a \quad\quad + 2c + d &= 0 \Rightarrow a = -2c - d \\ b - c \quad\quad &= 0 \Rightarrow b = c \end{aligned}$$

($n - r = 4 - 2 = 2$-parameter family), $c = t$, $d = s$

$\therefore (a, b, c, d) = (-2t - s, t, t, s)$

$= t(-2, 1, 1, 0) + s(-1, 0, 0, 1)$

$\therefore$ A basis for null(A) is $\{(\underset{a}{-2}, \underset{b}{1}, \underset{c}{1}, \underset{d}{0}), (-1, 0, 0, 1)\}$

$\therefore$ A basis for $U^\perp$ is $\left\{ \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$\therefore \dim(U^\perp) = 2$