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STUDENT'S I.D #

SOLUTIONS

MATH 311 NAME:

MID TERM TEST Your Mark: /50

June 12, 2002

Attempt ALL Questions

(1) Let V be a finite-dimensional real vector space, W be a non-empty subset of V, and $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$, $r < \infty$ be a set of vectors in V.

(10 Marks)

Define carefully what is meant by:

(a) W is a subspace of V.

W is a subspace of V if W is itself a vector space under the same operations defined on V.

(b) The vector $\vec{v}$ is a linear combination of vectors in S

$\vec{v}$ is a l.c of vectors in S if $\vec{v}$ is expressible in the form
$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_r\vec{v}_r$, $c_i \in \mathbb{R}$ $i=1,2,\ldots r$

(c) The set S is Linearly Dependent.

S is linearly Dependent if there exists scalars $c_1, c_2, \ldots c_r$ not all zeros such that $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$

(d) The set S is a basis for V

S is a basis for V if

(1) S spans V (2) S is linearly independent

(e) The Dimension of V.

The number of vectors in a basis for V is called "The dimension of V". In addition the zero vector space is considered to be of zero dimension

Given $A = \begin{bmatrix} 2 & 8 & 3 & -3 \\ 1 & 4 & 5 & 2 \\ -1 & -4 & 2 & 5 \end{bmatrix}$

(a) Find a basis for the rowspace of A, the column space of A, and the null space of A.

(b) Find the rank and nullity of A. (10 Marks)

(a) $A = \begin{bmatrix} 2 & 8 & 3 & -3 \\ 1 & 4 & 5 & 2 \\ -1 & -4 & 2 & 5 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 8 & 3 & -3 \\ -1 & -4 & 2 & 5 \end{bmatrix}$

$\xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 0 & -7 & -7 \\ 0 & 0 & 7 & 7 \end{bmatrix} \xrightarrow{R_2 \to \frac{R_2}{-7}} \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 7 & 7 \end{bmatrix}$

$\xrightarrow[R_1 \to R_1 - 5R_2]{R_3 \to R_3 - 7R_2} \begin{bmatrix} 1 & 4 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore$ A basis for row(A) is $\{(1,4,0,-3), (0,0,1,1)\}$

A basis for col(A) is $\{(2,1,-1), (3,5,2)\}$

The solution to $A\vec{x} = \vec{0}$ is thus given by

$$x + 4y - 3w = 0 \Rightarrow x = -4y + 3w$$
$$z + w = 0 \Rightarrow z = -w$$

Let $y = t$, $w = s$

$\therefore (x, y, z, w) = (-4t + 3s, t, -s, s)$

$= t(-4, 1, 0, 0) + s(3, 0, -1, 1)$

$\therefore$ A basis for null(A) $= \{(-4,1,0,0), (3,0,-1,1)\}$

(b) rank(A) $= 2$, nullity(A) $= 4 - 2 = 2$

3) (a) Find a basis and the dimension of the subspace of P_2 given by $W = \{ p \mid p(1) = 0 \}$. Show all work!

(6 Marks)

Let $p(x) = a_0 + a_1 x + a_2 x^2$ be an arbitrary element in W. Then $p(1) = 0 \Rightarrow$

$$0 = a_0 + a_1 + a_2 \Rightarrow \text{say } a_0 = -a_1 - a_2 = -(a_1 + a_2)$$

$$\therefore \; p(x) = -(a_1 + a_2) + a_1 x + a_2 x^2 = a_1(x-1) + a_2(x^2-1)$$

$$= \text{l.c. of } (x-1), (x^2-1)$$

$\therefore$ $W = \text{span}(S)$, where $S = \{ x-1, x^2-1 \}$

Further S is linearly independent since $x-1$, x^2-1 are not scalar multiple of one another.

It follows that $S = \{ x-1, x^2-1 \}$ is a basis for W and that $\dim W = 2$

(b) Let $S = \{ (-\lambda, 20), (-5, \lambda) \}$ be a set of vectors in $\mathbb{R}^2$. Find the values of λ so that S is linearly dependent.

(4 Marks)

let c_1, c_2 be scalars such that

$$c_1(-\lambda, 20) + c_2(-5, \lambda) = \vec{0} = (0,0)$$

$$\Rightarrow \quad -\lambda c_1 - 5 c_2 = 0$$

$$20 c_1 + \lambda c_2 = 0$$

The homogeneous system has a non-trivial solution provided

$$\det A = \begin{vmatrix} -\lambda & -5 \\ 20 & \lambda \end{vmatrix} = 0 \Rightarrow -\lambda^2 + 100 = 0 \Rightarrow \lambda = \pm 10$$

$\therefore$ $\lambda = \pm 10$ are values in which set S is l. dependent!

4) Which of the following sets of vectors in P_2 are bases for P_2. Justify your answer:

(a) $S_1 = \{3+x,\ 2-x+x^2\}$

(b) $S_2 = \{1-x,\ x-1,\ x^2\}$

(c) $S_3 = \{1, x, x^2, 4-2x+5x^2\}$

(d) $S_4 = \{1+x,\ 0,\ x^2\}$

(e) $S_6 = \{1,\ 1+x,\ 1+x+x^2\}$

(7 Marks)

1st. $\dim P_2 = 3 \Rightarrow$ # of vectors in a basis is 3.

S_1 : Not a basis ... it has only two vectors

S_3 : " " " " " four vectors

S_2 : " " " it is linearly dependent since $1-x$ is a scalar multiple of $x-1$

S_4 " " it contains the zero vector .. hence dependent.

S_6 : It may be!

let c_1, c_2, c_3 be scalars s.t. $c_1 + c_2(1+x) + c_3(1+x+x^2) = 0$

$$\Rightarrow \begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_2 + c_3 &= 0 \\ c_3 &= 0 \end{aligned}$$

solve: $c_1 = c_2 = c_3 = 0 \Rightarrow S_6$ is indep and has 3 vectors $\Rightarrow S_6$ is a basis for P_2

5) Let V be a vector space and S be a finite set of vectors in V which contain the zero vector. prove that S is linearly dependent.

(3 Marks)

let $S = \{\bar{0}, \bar{v}_2, \ldots \bar{v}_n\}$

The vector eq: $1.\bar{0} + 0.\bar{v}_2 + \ldots + 0.\bar{v}_n = \vec{0}$

has a non-Trivial solution $c_1 = 1,\ c_2 = \cdots = c_n = 0$

$\Rightarrow S$ is dependent.

(6) Let A be a 3×8 matrix. Prove that the column vectors of A are linearly dependent!
<u>Hint</u>: what is the largest possible value of rank(A)? (4 Marks).

Largest value of rank A is Three which equals to the maximum number of linearly independent rows or columns of A.
There fore "A" can have at most 3 linearly independent column vectors.
Since "A" has 8 columns, they must form a linearly dependent set in $\mathbb{R}^3$.

(7) Let A be a matrix of order 5×9 and let nullity$(A^t) = 3$. Find rank(A), and nullity(A). (4 Marks)

$$\text{rank}(A^t) = m - r = 5 - 3 = 2 = \text{rank}(A)$$

$$\text{nullity}(A) = 9 - 2 = 7$$

(8) If A is an $m\times n$ matrix and if the homogeneous linear system $A\vec{x} = \vec{0}$ has only the Trivial solution. Determine rank(A). Justify your answer! (2 Mark

$A\vec{x} = \vec{0}$ has only Trivial solution

$\Rightarrow$ Nullity(A) = 0

$\Rightarrow$ rank(A) = $n - r = n - 0 = n$.