SECTION(I)

MATH 311

List of Definitions

Definition (1): Vector space

Let V be an arbitrary non-empty set of objects on which two operations are defined, addition and scalar multiplication

- By addition we mean a rule for associating with each pair of objects $\bar{u}$, and $\bar{v}$ in V an object $\bar{u}+\bar{v}$ called the sum of $\bar{u}, \bar{v}$.
- By scalar multiplication we mean a rule for associating with each scalar (number) K and each object $\bar{u}$ in V an object $K\bar{u}$ called the scalar multiple of $\bar{u}$ by K. If the following ten axioms are satisfied by all objects $\bar{u}, \bar{v}$, and $\bar{w}$ in V and all scalars K, and ℓ, then we call V a <u>vector space</u> and we call the objects in V vectors:

<u>addition axioms A1 — A5</u>

<u>A1</u> If $\bar{u}$ and $\bar{v}$ are objects in V, then $\bar{u}+\bar{v}$ is in V

(In words: V is closed under addition)

<u>A2</u> $\bar{u}+\bar{v} = \bar{v}+\bar{u}$

(In words: addition is Commutative)

<u>A3</u> $\bar{u}+(\bar{v}+\bar{w}) = (\bar{u}+\bar{v})+\bar{w}$

(In words: addition is associative)

<u>A4</u> There is an object $\bar{0}$ in V, Called the zero vector for V such that $\bar{0}+\bar{u} = \bar{u}+\bar{0} = \bar{u}$ for all $\bar{u}$ in V

(In words: The Existence of zero)

<u>A5</u> For each $\bar{u}$ in V, there is an object $-\bar{u}$ in V, called the negative of $\bar{u}$ with the property

$$\bar{u}+(-\bar{u}) = (-\bar{u})+\bar{u} = \vec{0}$$

(In words: existence of a negative)

<u>scalar Multiplication axioms S1 — S5</u>

S1: If K is any scalar and $\bar{u}$ is any object in V, then $K\bar{u}$ is in V

(In words: V is closed under scalar multiplication)

S2: $K(\bar{u}+\bar{v}) = K\bar{u} + K\bar{v}$

S3 $(K+l)\bar{u} = K\bar{u} + l\bar{u}$

$\Big\}$ Distributive properties

S4 $K(l\bar{u}) = (Kl)\bar{u}$

S5 $1\,\bar{u} = \bar{u}$

If the scalars used are Complex numbers the vector space is called a Complex vector space.. Likewise if the scalars used are real numbers, we refere to V as a "Real vector space".
In this Course we shall Consider only Real vector spaces.

Definition (2): A subspace

Let V be a vector space and W be a non-empty subset of V. Then W is called a subspace of V if W is itself a vector space under the same operations defined on V.

Definition (3): A Linear Combination

Let V be a vector space and $S = \{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_r\}$ be a set of vectors in V. A vector $\bar{w}$ in V is called a linear combination of vectors in S if $\bar{w}$ can be expressed in the form

$$\bar{w} = K_1\bar{v}_1 + K_2\bar{v}_2 + \dots + K_r\bar{v}_r$$

for some scalars $K_1, K_2, \dots, K_r \in \mathbb{R}$.

Definition (4): Span of a subset

Let V be a vector space and $S = \{\bar{v}_1, \bar{v}_2, \dots \bar{v}_r\}$ be a set of vector in V. The set W of all linear combinations of vectors in S is called "The span of S" and we write

$$W = \text{span}(S)$$. In fact W is a subspace!

$$\therefore \quad W = \{\bar{w} \mid \bar{w} = K_1\bar{v}_1 + K_2\bar{v}_2 + \dots + K_r\bar{v}_r,\ K_1, K_2, \dots K_r \in \mathbb{R}\}$$

We may also say that S is a spanning set for subspace W

Definition (5): Linear Independence

Let V be a vector space and $S=\{\bar{v}_1, \bar{v}_2, \ldots \bar{v}_r\}$ be a set of vectors in V. The set S is said to be <u>Linearly independent</u> if the vector equation $K_1\bar{v}_1 + K_2\bar{v}_2 + \cdots + K_r\bar{v}_r = \vec{0}$ has <u>only</u> the trivial solution $K_1 = K_2 = \cdots = K_r = 0$.

Definition (6): Linear Dependence

Let V be a vector space and $S=\{\bar{w}_1, \bar{w}_2, \ldots \bar{w}_n\}$ be a set of vectors in V. The set S is said to be <u>Linearly Dependent</u> if there exists scalars $K_1, K_2, \ldots,$ and K_n <u>not all</u> zeros such that $K_1\bar{w}_1 + K_2\bar{w}_2 + \cdots + K_n\bar{w}_n = \vec{0}$

Definition (7): A basis for a vector space

let V be a vector space and $S=\{\bar{v}_1, \bar{v}_2, \ldots \bar{v}_n\}$ be a set of vectors in V. The set S is called a <u>Basis for V</u> if

(1) S spans V; that is $V = \text{span}(S)$

(2) S is Linearly independent.

Definition (8): A finite - Dimensional vector space

let V be a non-zero vector space. Then V is called <u>finite-dimensional vector space</u> if it contains a finite set of vectors $B=\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n\}$, $n<\infty$ that forms a <u>Basis</u> for V. If no such set exists, one says that V is an infinite-dimensional vector space.

In addition we shall regard the <u>zero vector space</u>* to be finite-dimensional.

In this course we shall consider only finite-dimensional vector space.

* For the definition of zero vector space see definition (13).

Definition (9): The Dimension of a vector space

Let V be a finite-dimensional vector space. The number of vectors in a basis for V is called "The dimension of V" and is denoted by dim. (V). In addition we define the dimension of the zero vector space to be zero.

Definition (10): Row vectors / Column vectors of a matrix

Let $A = (a_{ij})$ be an $m \times n$ matrix.

The vectors $\bar{r}_1 = (a_{11}, a_{12}, \dots, a_{1n})$, $\bar{r}_2 = (a_{21}, a_{22}, \dots, a_{2n})$, ... $\bar{r}_m = (a_{m1}, a_{m2}, \dots, a_{mn})$ in $\mathbb{R}^n$ formed from the rows of A are called "The row vectors of A". Similarly the vectors

$$\bar{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \text{ or } (a_{11}, a_{21}, \dots, a_{m1}), \quad \bar{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \dots, \bar{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

in $\mathbb{R}^m$ formed from the columns of A are called "The column vectors of A".

Note that the matrix "A" can be express in the form

$$A = \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \vdots \\ \bar{r}_m \end{bmatrix} \quad \text{or in the form } A = [\bar{c}_1 \ \bar{c}_2 \ \dots \ \bar{c}_n].$$

Definition (11): Row space, column space, and null space of a matrix

Let A be an $m \times n$ matrix.

(1) The subspace of $\mathbb{R}^n$ spanned by the row vectors of A is called the row space of A and is denoted by Row(A).

$$\therefore \text{Row}(A) = \text{Span}\{\bar{r}_1, \bar{r}_2, \dots \bar{r}_m\}$$

(2) The subspace of $\mathbb{R}^m$ spanned by the column vectors of A is called the column space of A and is denoted by Col(A).

$$\therefore \text{Col}(A) = \text{Span}\{\bar{c}_1, \bar{c}_2, \dots \bar{c}_n\}$$

Before we define the null space of A, let us define: "Solution space"

The set of all solution vectors of the homogeneous linear system $A\vec{x}=\vec{o}$ (which is a subspace of $\mathbb{R}^n$) is called "The solution space of the homogeneous system".

(3) The solution space of the homogeneous system $A\vec{x}=\vec{o}$ is called "The <u>null space of A</u>" and is denoted by $null(A)$.

<u>Definition (12)</u>: Rank and nullity of a matrix

Let A be an $m\times n$ matrix. The common dimension of the row space and column space of A is called the <u>rank</u> of A and is denoted by $rank(A)$.

It follows that

$$rank(A) = \dim[row(A)] = \dim[Col(A)]$$

The dimension of the null space of A is called "The nullity of A", and is denoted by $nullity(A)$.

It follows that

$$nullity(A) = \dim[null(A)]$$

<u>Definition (13)</u>: The Zero Vector Space

Let V be a set containing a single object which we denote by $\vec{o}$. Define addition and scalar multiplication on V by

$$\bar{o}+\bar{o}=\bar{o} \quad , \quad K\bar{o}=\bar{o} \quad , \quad K\in\mathbb{R}$$

Clearly V is a vector space which we call "The zero vector space". The single object $\vec{o}$ is called "The zero vector". We may write the zero vector space as $\{\vec{o}\}$.

<u>Definition (14)</u>: Coordinate vector

Let V be a vector space and $B=\{\bar{v}_1,\bar{v}_2,\bar{v}_3,..\bar{v}_n\}$ be an ordered basis for V. Then every vector $\bar{v}$ in V is expressible in the form

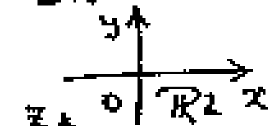
$$\bar{v}=c_1\bar{v}_1+c_2\bar{v}_2+\cdots+c_n\bar{v}_n.$$

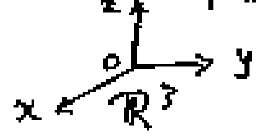
The vector $(c_1,c_2,...,c_n)$ in $\mathbb{R}^n$ is called "The coordinate vector of $\bar{v}$ relative to B; it is denoted by $(\vec{v})_B=(c_1,c_2,...,c_n)$

Definition (15): The Euclidean n-space $\mathbb{R}^n$

Let n be a positive integer. The set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$, $a_i \in \mathbb{R}$ is called "The Euclidean n-space" and is denoted by $\mathbb{R}^n$. In particular:

If $n=1$, we get $\mathbb{R}^1$ or $\mathbb{R}$, the set of all real numbers 

If $n=2$, we get $\mathbb{R}^2$, the set of all points in the xy-plane 

If $n=3$, we get $\mathbb{R}^3$, the set of all points in the xyz-space 

Definition (16): Standard addition / scalar multiplication on $\mathbb{R}^n$

we define addition and scalar multiplication on $\mathbb{R}^n$ as follows

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1+b_1, a_2+b_2, \ldots, a_n+b_n)$$

$$k(a_1, a_2, \ldots, a_n) = (ka_1, ka_2, \ldots ka_n)$$

These are the so called standard operations on $\mathbb{R}^n$.

Definition (17): The set M_{mn} of all $m \times n$ matrices

The set of all matrices of order $m \times n$ with real entries is denoted by M_{mn}. That is $M_{mn} = \{ A = (a_{ij})_{\substack{i=1,2,\ldots m \\ j=1,2,\ldots n}} \mid a_{ij} \in \mathbb{R} \}$

Definition (18): Standard operations on M_{mn}:

we define addition and scalar multiplication on M_{mn} to be the usual matrix addition and scalar multiplication. That is

$$A + B = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij}) \quad i=1,2,\ldots m; j=1,2,\ldots,n$$

$$kA = k(a_{ij}) = (ka_{ij}) \quad i=1,2,\ldots m; j=1,2,\ldots n$$

Definition (19): The set P_n of polynomials of degree n or less

The set of all polynomials $p(x) = a_0 + a_1 x + \cdots + a_m x^m$ of degree $m \leq n$ together with the zero function is denoted by P_n. we assume $a_0, a_1, \ldots, a_m \in \mathbb{R}$.

Definition (20): Standard operations on P_n:

we define addition and multiplication by a scalar on P_n as follows

$$(p+q)(x) = p(x) + q(x), \quad (kp)(x) = kp(x), \quad k \in \mathbb{R}$$

where $p, q \in P_n$

SECTION (I)

MATH 311
List of Theorems

Theorem (1): special vector spaces

(a) The set $\mathbb{R}^n$ together with the standard operations on $\mathbb{R}^n$ is a vector space.

(b) The set M_{mn} together with the standard operations on M_{mn} is a vector space.

(c) The Set P_n together with the standard operations on P_n is a vector space.

Theorem (2): The Cancellation Law

Let V be a vector space and let $\bar{u}, \bar{v}$, and $\bar{w}$ be vectors in V.

If $\bar{u}+\bar{v}=\bar{u}+\bar{w}$, then $\bar{v}=\bar{w}$

In words: The vector $\bar{u}$ may be cancelled from both sides!

Theorem (3): vector Equations

Let V be a vector space and $\bar{u}, \bar{v}$ be vectors in V. The vector Equation $\bar{x}+\bar{u}=\bar{v}$ has the unique solution $\bar{x}=\bar{v}-\bar{u}$.

Theorem (4): Properties of vectors

Let V be a vector space, $\bar{u}$ be a vector in V, and K a scalar. Then

(a) $0\vec{u}=\vec{0}$ (b) $K\vec{0}=\vec{0}$ (c) $(-1)\vec{u}=-\vec{u}$

(d) If $K\vec{u}=\vec{0}$, then Either $K=0$ or $\vec{u}=\vec{0}$.

Theorem (5): subspace Test

Let W be a <u>non-empty</u> subset of vectors from a vector space V, then W is a subspace of V if and only if A1 and S1 are satisfied, that is: (a) If $\bar{u}, \bar{v}$ are vectors in W, then $\bar{u}+\bar{v}$ is in W.

(b) If K is any scalar and $\bar{u}$ is any vector in W, then $K\bar{u}$ is in W

Theorem (6): solution space

The set of all solution vectors of the homogeneous linear system $A\vec{x}=\vec{0}$ (of m equations in n unknowns) form a subspace of $\mathbb{R}^n$ called "The solution space of the system".

Theorem (7): properties of spanning set

Let V be a vector space and $S=\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_r\}$ be a set of vectors in V.

(a) The set W of all l.c. of vectors in S, that is $W=\mathrm{Span}(S)$, is a subspace of W which we call "The space spanned by S"

(b) W is the smallest subspace of V containing S in the sense that every other subspace of V that contain S must contain W.

Theorem (8): Spanning sets are not unique

If $S=\{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n\}$, $S'=\{\bar{w}_1, \bar{w}_2, \dots, \bar{w}_m\}$ are sets of vectors in a vector space V, then $\text{Span}(S)=\text{Span}(S')$ if and only if every vector in S is a l.c. of those in S', and conversely each vector in S' is a l.c. of those in S.

Theorem (9): Linear Independence / Dependence Test:

Let S be a set of two or more vectors in a vector space V. Then

(a) S is linearly dependent if and only if at least one of the vectors in S is expressible as a linear combination of the other vectors in S.

(b) S is linearly independent if and only if no vector in S is expressible as a linear combination of the other vectors in S.

Theorem (10): special cases of theorem (9)

(a) A finite set of vectors that contains the zero vector is linearly Dependent

(b) A set with a single non-zero vector $\bar{v}$ is linearly independent.

(c) A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other

(d) A set with exactly two vectors is linearly Dependent if and only if one vector is a scalar multiple of the other.

Theorem (11): A unique representation:

Let $S=\{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n\}$ be a basis for a vector space V, then every vector $\bar{v}$ in V can be expressed in the form $\bar{v}=c_1\bar{v}_1+c_2\bar{v}_2+\dots+c_n\bar{v}_n$ in exactly one way. In other words the coordinate vector $(c_1, c_2, \dots c_n)$ is unique

Theorem (12): Standard Basis for $\mathbb{R}^n$, M_{mn}, and P_n

(1) The set $B=\{\bar{e}_1=(1,0,0,\dots 0), \bar{e}_2=(0,1,\dots,0), \dots, \bar{e}_n=(0,0,\dots,1)\}$ in $\mathbb{R}^n$ is both linearly independent and spans $\mathbb{R}^n$ and hence is a basis for $\mathbb{R}^n$ called "The standard basis for $\mathbb{R}^n$".

(2) The set $\{M_{ij} \;\; {\scriptstyle i=1,2,\dots m \atop j=1,2,\dots n} \mid$ the ij entry of the matrix M_{ij} is 1 and remaining are zero$\}$ in M_{mn} is both linearly independent and spans M_{mn} and hence is a basis for M_{mn} called "The standard basis for M_{mn}".

For example the set $\left\{ \begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\end{bmatrix} \right\}$ is the standard basis for M_{23}.

(3) The set $\{1, x, x^2, \ldots, x^n\}$ in P_n is both linearly independent and spans P_n and hence is a basis for P_n called "The standard basis for P_n".

Theorem (13): Test for a basis

If V is a vector space and $B = \{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n\}$ is a basis for V, then

(a) Every set with more than n vectors is linearly dependent

(b) No set with fewer than n vectors spans V

Theorem (14): Property of a Basis:

All bases for a finite-dimensional vector space have the same number of vectors.

Theorem (15): Dimensions of special vector spaces:

(a) $\dim(\mathbb{R}^n) = n$ (b) $\dim(M_{mn}) = mn$ (c) $\dim(P_n) = n+1$

Theorem (16): plus / minus Theorem:

Let V be a vector space and S be a non-empty set of vectors in V.

(a) If S is linearly independent and if $\bar{v}$ is a vector in V that is outside of span(S), then the set S' that results by inserting $\bar{v}$ into S is still linearly independent.

(b) If $\bar{v}$ is a vector in S that is expressible as a l.c of other vectors in S and if S' is the set obtained by removing $\bar{v}$ from S then span(S) = span(S') (that is S, S' span same space!).

Theorem (17): An efficient Test for a basis:

Let V be a finite-dimensional vector space and let $\dim(V) = n$.

If B is a set in V with exactly n vectors, then B is a basis for V if Either B is Linearly Independent

OR B spans V

In words: If we know the right # of vectors in a Basis, our work is cut into Half!!

Theorem (18) Constructing a basis by reducing or enlarging a set

Let S be a set of vectors in a finite-dimensional vector space V

(a) If S spans V but is not a basis for V, then S can be <u>reduced</u> to a basis for V by <u>removing appropriate vectors from S</u>

(b) If S is a linearly independent set that is not already a basis for V, then S can <u>enlarged</u> to a basis for V by <u>inserting appropriate vectors into S</u>

Theorem (19): Dimension of a subspace

Let V be a finite-dimensional vector space and W be a subspace of V, then $\dim(W) \leq \dim(V)$. Moreover if $\dim(W) = \dim(V)$, then $W = V$.

Theorem (20): Consistency Test for a non-homogeneous Linear system

A non-homogeneous system of linear equations $A\vec{x} = \vec{b}$ is consistent if and only if $\vec{b}$ lies in the column space of A

(This means $\vec{b}$ = l.c of column vectors of A).

Theorem (21): Rowspace / column space / nullspace and Elementary row operations

(a) Elementary row operations <u>do not</u> change the rowspace of a matrix A.

(b) Elementary row operations <u>do not</u> change the nullspace of a matrix A.

(c) Elementary row operations <u>Can change</u> the column space of a matrix A.

Theorem (22): Bases for rowspace and column space of a row-echelon matrix

Let R be a row-echelon matrix. Then

(a) The r non-zero rows of R form a basis for $Row(R)$

(b) The column vectors of R containing the leading ones of the rows form a basis for the $Col(R)$.

Theorem (23): Bases for the rowspace and column space of a matrix.

suppose a matrix "A" is carried to a row-echelon matrix R by means of a sequence of elementary row operations. Then

(a) The r non-zero rows of R form a basis for row(A).

(b) If S is a set of column vectors of R that form a basis for row(R) then the <u>corresponding</u> column vectors of A form a basis for col(A).

Theorem (24): properties of rowspace and column space

Let A be an $m\times n$ matrix, U be an $p\times m$ matrix and V be an $n\times q$ matrix. Then

(a) $\text{row}(UA) \subseteq \text{row}(A)$

(b) If U is an $m\times m$ invertible matrix, $\text{row}(UA) = \text{row}(A)$

(c) $\text{col}(AV) \subseteq \text{col}(A)$

(d) If V is an $n\times n$ invertible matrix, $\text{col}(AV) = \text{col}(A)$.

Theorem (25): Rowspace / Column space have same dimension

Let A be any matrix, then

$$\dim[\text{row}(A)] = \dim[\text{col}(A)] = \text{say } r$$

This common number is called "The rank of A". Note that the rank of A is: The # of leading ones of a row-echelon matrix equivalent to A.

Theorem (26): Test for invertibility of a square matrix

Let A be a square matrix of order n. Then

The row vectors of A are linearly independent in $\mathbb{R}^n$ if and only if A is invertible.

Similar result holds for the column vectors of A.

Theorem (27): Properties of Rank

For any $m\times n$ matrix A:

(a) $\text{rank}(A) \leq m$, $\text{rank}(A) \leq n$ (b) $\text{rank}(A) = \text{rank}(A^t)$

(c) $\text{rank}(UA) = \text{rank}(A)$ provided U is an $m\times m$ invertible matrix

(d) $\text{rank}(AV) = \text{rank}(A)$ provided V is an $n\times n$ invertible matrix

Theorem (28) : Full rank of a matrix

Let A be a square matrix of order n.
The matrix A is invertible if and only if A has full rank (that is $\text{rank}(A) = n$).

Theorem (29) : Dimension Theorem for matrices

Let A be an $m \times n$ matrix, then

$$\text{rank}(A) + \text{nullity}(A) = n$$

In words : # of leading ones + # of free variables = # of columns.

Theorem (30) : Another property of Rank!

Let A be an $m \times n$ matrix. The matrix "A" has rank r if and only if there exists invertible matrices U, and V (of order $m \times m$ and $n \times n$ respectively) such that

$$UAV = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$, I_r is the unit matrix of order $r \times r$

Note: The matrices U, and V can be determined as follows

(1) Carry A to a reduced row-echelon matrix R by means of a sequence of elementary row operations $P_1, P_2, \dots P_k$. This operations will carry I_m to U :

$$[A \mid I_m] \xrightarrow{P_1, P_2, \dots, P_k} [R \mid U]$$

(2) Carry R^t to a reduced row-echelon matrix (which is $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$) by means of a sequence of elementary row-operations $q_1, q_2, \dots q_s$. This operations will carry I_n to V^t

$$[R^t \mid I_n] \xrightarrow{q_1, q_2, \dots, q_s} \left[\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \mid V^t \right]$$

once V^t is found, we can Find V!!

$\boxed{\text{END}}$

MATH 311

SECTIONS (II), (III)

Support Material

DEFINITIONS

Definition (1): An inner Product / An inner product space

An inner product on a real vector space V is a function that associates a real number $<\bar{u},\bar{v}>$ with each pair of vectors $\bar{u}$ and $\bar{v}$ in V in such a way that the following four axioms are satisfied for all $\bar{u},\bar{v}$, and $\bar{w}$ in V and all scalars K in $\mathbb{R}$

P1 : $<\bar{u},\bar{v}> = <\bar{v},\bar{u}>$ " symmetry axiom "

P2 : $<\bar{u}+\bar{v},\bar{w}> = <\bar{u},\bar{w}> + <\bar{v},\bar{w}>$ " Additivity axiom "

P3 : $<K\bar{u},\bar{v}> = K<\bar{u},\bar{v}>$ " homogenity axiom "

P4 : $<\bar{v},\bar{v}> \geq 0$ and that $<\bar{v},\bar{v}> = 0$ if and only if $\bar{v}=\bar{0}$ " The positivity axiom "

A real vector space with an inner product is called a real inner product space.

Definition (2): Norm / Distance

Let V be an i.p.s and $\bar{u},\bar{v}$ be vectors in V

(a) The norm (or length) of a vector $\vec{u}$ is denoted and defined by

$$\|\vec{u}\| = \sqrt{<\bar{u},\bar{u}>}$$

(b) The distance between two vectors $\bar{u},\bar{v}$ is denoted and defined by

$$d(\bar{u},\bar{v}) = \|\bar{u}-\bar{v}\|$$

Definition (3): A unit vector

a vector $\bar{v}$ in an i.p.s V is said to be a unit vector if

$$\|\bar{v}\| = 1$$

Definition (4): The unit sphere (or unit circle) in an i.p.s:

let V be an i.p.s. The set of all vectors (or points) in V that satisfy $\|\bar{v}\|=1$ is called the unit sphere or sometimes the unit circle in V.

Definition (5): Angle between vectors

Let V be an i.p.s and $\bar{u}, \bar{v}$ be non-zero vectors in V. We define the angle θ between $\bar{u}, \bar{v}$ as the unique angle determined by

$$\cos\theta = \frac{\langle \bar{u}, \bar{v} \rangle}{\|\bar{u}\| \, \|\bar{v}\|}, \quad \text{and} \quad 0 \leq \theta \leq \pi$$

Definition (6): Orthogonal vectors

Two vectors $\bar{u}, \bar{v}$ in an i.p.s are said to be orthogonal and we write $\bar{u} \perp \bar{v}$ if $\langle \bar{u}, \bar{v} \rangle = 0$ "that is if $\theta = \frac{\pi}{2}$"

Definition (7): Orthogonal Complement of a subspace

Let W be a subspace of an i.p.s V. The set of all vectors in V that are orthogonal to every vector in W is called "The orthogonal Complement of W and is denoted by $W^{\perp}$ (read "W" perp).

It follows that $W^{\perp} = \{ \bar{v} \text{ in } V \mid \langle \bar{v}, \bar{w} \rangle = 0 \text{ for all } \bar{w} \in W \}$

Definition (8): orthogonal set / orthonormal set

Let V be an i.p.s and $S = \{ \bar{v}_1, \bar{v}_2, \dots, \bar{v}_r \}$ be a set of vectors in V. The set S is said to be an orthogonal set if

$$\langle \bar{v}_i, \bar{v}_j \rangle = 0, \quad i \neq j \quad i,j = 1,2,\dots r$$

If in addition every vector in S is a unit vector; that is $\|\bar{v}_i\| = 1$, $i = 1,2,\dots r$, one says that S is an orthonormal set.

Definition (9): orthogonal / orthonormal Basis

(1) In an i.p.s V, a basis consisting of orthogonal vectors is called an orthogonal basis for V

(2) In an i.p.s V, a basis consisting of orthonormal vectors is called an orthonormal basis for V

Definition (10): orthogonal projection

Let W be a subspace of an i.p.s V and let $B = \{ \bar{v}_1, \bar{v}_2, \dots \bar{v}_r \}$ be an orthogonal (or orthonormal) basis for W. If $\vec{x}$ is a vector in V, the orthogonal projection of $\vec{x}$ on W is denoted and defined by

$$\vec{p} \overset{\text{or}}{=} \operatorname{proj}_W \vec{x} = t_1 \vec{v}_1 + t_2 \vec{v}_2 + \dots + t_r \vec{v}_r \quad \text{where}$$

$$t_j = \frac{\langle \vec{x}, \vec{v}_j \rangle}{\| \vec{v}_j \|^2}, \quad j = 1, 2, \dots r$$

In addition we define $\operatorname{proj}_{\{\vec{0}\}} \vec{x} = \vec{0}$

<u>Definition (11)</u>: <u>Least square solution of Linear systems</u>

Given a linear system $A\vec{x} = \vec{b}$ of m equations in n unknowns. If there exists a vector $\vec{x}$ that minimizes $\| A\vec{x} - \vec{b} \|$ with respect to the Euclidean i.p in $\mathbb{R}^m$, such a vector is referred to as a least squares solution of system.

Note that a least square solution $\vec{x}$ may be regarded as an approximate solution to the generally inconsistent system $A\vec{x} = \vec{b}$, and the quantity $\| A\vec{x} - \vec{b} \|$ may be viewed as a measure of the error that results from that approximation!

<u>Definition (12)</u>: <u>Normal Equations / Normal system</u>

Let $A\vec{x} = \vec{b}$ be a system of linear equations. The linear system

$$M\vec{x} = A^t \vec{b}, \quad M = A^t A$$

is called "The normal system associated with $A\vec{x} = \vec{b}$, and the individual equations are called the normal equations associated with $A\vec{x} = \vec{b}$.

Observe that a least square solution of $A\vec{x} = \vec{b}$ is in fact the <u>exact</u> solution of the associated normal system.

<u>Definition (13)</u>: <u>orthogonal matrix</u>

Let A be a square matrix with real entries. Then "A" is said to be an orthogonal matrix if it satisfies

$$A^{-1} = A^t$$

or equivalently: a square matrix A is orthogonal if and only if

$$AA^t = A^t A = I$$