Inner Products / Inner Product Spaces

P1 $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$ — symmetry axiom

P2 $\langle \vec{u}+\vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ — additivity axiom

P3 $\langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle,\ k \in \mathbb{R}$ — homogenity axiom

P4 $\langle \vec{v}, \vec{v} \rangle \geq 0,\ \langle \vec{v}, \vec{v} \rangle = 0 \Leftrightarrow \vec{v} = \vec{0}$ — positivity axiom

$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$ — Norm

$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$ — Distance between vectors

$\langle \vec{u}, \vec{v} \rangle \leq \|\vec{u}\|^2 \|\vec{v}\|^2$ — Cauchy - Schwartz inequality

$\Leftrightarrow \langle \vec{u}, \vec{u} \rangle \langle \vec{v}, \vec{v} \rangle$

unit vector, unit sphere/circle

$\cos\theta = \dfrac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\|\|\vec{v}\|}$ — angle between vectors

$\vec{u} \perp \vec{v} \Leftrightarrow \langle \vec{u}, \vec{v} \rangle = 0$ — orthogonal vectors

N1 $\|\vec{u}\| \geq 0,\ \|\vec{u}\| = 0 \Leftrightarrow \vec{u} = \vec{0}$ — Properties of norms

N2 $\|k\vec{u}\| = |k|\|\vec{u}\|,\ k \in \mathbb{R}$

N3 $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ — triangular inequality

$\langle \vec{u}, \vec{v} \rangle = \vec{v}^{\,T}(A^T A)\vec{u} = (A\vec{u}) \cdot (A\vec{v})$, A = inv. matrix, $\vec{u}, \vec{v}$ = col. vectors

$\vec{u} \perp \vec{v} \Leftrightarrow \|\vec{u}\|^2 + \|\vec{v}\|^2 = \|\vec{u}+\vec{v}\|^2$ — generalized Pythagoras Theorem

$W^{\perp} = \{\vec{v} \in V \mid \langle \vec{v}, \vec{w} \rangle = 0\ \forall\ \vec{w} \in W\}$ — orthogonal complement of subspace

↳ 1) W is subspace of V

2) $W \cap W^{\perp} = \{\vec{0}\}$

3) $(W^{\perp})^{\perp} = W$

if orthogonal set of non-zero vectors $\Rightarrow$ linearly independent

Existence of orthogonal or orthonormal basis: the Gram-Schmidt Orthogonalization Process

$B = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ basis!

B' = orthogonal set $= \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$

$$\vec{v}_1 = \vec{u}_1$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle}\vec{v}_1$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle}\vec{v}_2 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle}\vec{v}_1$$

B'' = orthonormal set $= \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\rangle$, $\vec{w}_i = \dfrac{\vec{v}_i}{\|\vec{v}_i\|}$, $i = 1, 2, \ldots, n$

$\vec{p} = \text{proj}_W \vec{v} = t_1\vec{v}_1 + t_2\vec{v}_2 + \dots + t_r\vec{v}_r$, $t_j = \dfrac{\langle \vec{v}, \vec{v}_j \rangle}{\|\vec{v}_j\|^2}$, $j \in [1, r]$, $j \in \mathbb{Z}$, orthogonal projection

V = i.p.s., W = non-zero subspace of V, $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ = orthogonal basis,

$\vec{v}$ = any vector in V

$\vec{v} = \vec{w}_1 + \vec{w}_2$, $\vec{w}_1 \in W$, $\vec{w}_2 \in W^\perp$, $\vec{w}_1 = \text{proj}_W \vec{v} = \vec{p}$, $\vec{w}_2 = \vec{v} - \vec{p}$

$\dim V = \dim W + \dim W^\perp$