MATH 311

(D)

LINEAR INDEPENDENCE

Defn.(16) : Linear Dependence / Linear Independence

let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ be a set of vectors in a vector space V. The set S is said to be linearly Dependent if there exists scalars $c_1, c_2, \ldots,$ and c_r not all zeros such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_r\vec{v}_r = \vec{0}$$

However if the vector equation above is satisfied by only the trivial solution $c_1 = c_2 = \cdots = c_r = 0$, one says the set S is linearly dependent.

EX1: show that the set $S = \{\bar{e}_1 = (1,0,0), \bar{e}_2 = (0,1,0), \bar{e}_3 = (0,0,1)\}$ is linearly independent in $\mathbb{R}^3$.

solution: let $c_1, c_2,$ and c_3 be scalars such that

$$c_1\bar{e}_1 + c_2\bar{e}_2 + c_3\bar{e}_3 = \vec{0} \quad \text{--- } (*)$$

that is $\quad c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1) = (0,0,0)$

$$\therefore \quad c_1 = 0, \quad c_2 = 0, \quad c_3 = 0$$

The vector equation $(*)$ is satisfied by only the trivial solution $c_1 = c_2 = c_3 = 0$. Hence S is linearly Independent.

EX2: show that the set $S = \{1+2x, \; x^2+x-5, \; 4x^2-2x-23\}$ is linearly dependent!

solution: we need to show that there are scalars c_1, c_2, c_3 not all zeros such that

$$c_1(1+2x) + c_2(x^2+x-5) + c_3(4x^2-2x-23) = 0$$

expanding and rearranging:

$$(c_1 - 5c_2 - 23c_3) + (2c_1 + c_2 - 2c_3)x + (c_2 + 4c_3)x^2 = 0 + 0x + 0x^2$$

Comparing, we get:

$$c_1 - 5c_2 - 23c_3 = 0$$
$$2c_1 + c_2 - 2c_3 = 0$$
$$c_2 + 4c_3 = 0$$

or $A\vec{x} = \vec{0}$, where $A = \begin{bmatrix} 1 & -5 & -23 \\ 2 & 1 & -2 \\ 0 & 1 & 4 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

Now, $\det A = \begin{vmatrix} 1 & -5 & -23 \\ 2 & 1 & -2 \\ 0 & 1 & 4 \end{vmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{vmatrix} 1 & -5 & -23 \\ 0 & 11 & 44 \\ 0 & 1 & 4 \end{vmatrix}$

$\therefore \det A = +1 \begin{vmatrix} 11 & 44 \\ 1 & 4 \end{vmatrix} = 0$

Hence homogeneous system has a non-trivial solution!

set S is linearly Dependent!

EX: show whether the set $S = \{(2,5), (61,-20), (41,103)\}$ is linearly independent in $\mathbb{R}^2$.

Solution: let c_1, c_2, c_3 be scalars such that

$$c_1(2,5) + c_2(61,-20) + c_3(41,103) = \vec{0} = (0,0)$$

$$\therefore \begin{cases} 2c_1 + 61c_2 + 41c_3 = 0 \\ 5c_1 - 20c_2 + 103c_3 = 0 \end{cases}$$

This is a homogeneous system having fewer Equations $(m = 2)$ than unknowns $(n = 3)$.. Hence it must have a non-trivial solution (that is not all of c_1, c_2, c_3 are zeros!)

$\therefore$ set S is linearly Dependent.

Theorem (7): Test for Linear Dependence:

let $S = \{\bar{v}_1, \bar{v}_2, \dots \bar{v}_r\}$, $r \geq 2$ be a set of vectors in a vector space V. Then S is linearly dependent if and only if at least one of the vectors in S is expressible as a linear combination of the other vectors in S.

proof.: This is a two-way proof.

First: Assume that $S = \{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_r\}$, $r \geq 2$ is linearly dependent. Then by definition, there exist scalars $c_1, c_2, \ldots, c_r$ not all zeros such that

$$c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_r\bar{v}_r = \vec{0}$$

To be specific, assume $c_1 \neq 0$. Dividing by c_1 and rearranging, we get:

$$\vec{v}_1 = \left(-\frac{c_2}{c_1}\right)\bar{v}_2 + \left(-\frac{c_3}{c_1}\right)\bar{v}_3 + \cdots + \left(-\frac{c_r}{c_1}\right)\vec{v}_r$$

$$\Rightarrow \bar{v}_1 = \text{l.c of } \bar{v}_2, \bar{v}_3, \ldots, \bar{v}_r$$

$$\therefore \bar{v}_1 = \text{l.c of the } \underline{\text{other}} \text{ vectors in } S.$$

Conversely: Assume that a vector in S, say $\bar{v}_1$ is a linear combination of the other vectors in S, namely $\bar{v}_2, \bar{v}_3, \ldots,$ and $\bar{v}_r$. Hence, there are scalars $\alpha_2, \alpha_3, \ldots, \alpha_r$ such that

$$\vec{v}_1 = \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 + \cdots + \alpha_r\vec{v}_r$$

$$\Rightarrow \quad -1\bar{v}_1 + \alpha_2\bar{v}_2 + \alpha_3\bar{v}_3 + \cdots + \alpha_r\bar{v}_r = \vec{0} \quad \cdots (*)$$

$\therefore$ There exists scalars $\alpha_1 = -1, \alpha_2, \alpha_3, \ldots, \alpha_r$ not all zeros that satisfy vector eq. $(*)$.

$\therefore$ By defn. the set $S = \{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_r\}$ is linearly dependent.

proof is Complete!

Theorem (7) may be restated in the following equivalent form:

Theorem (7)': Test for Independence

Let $S = \{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_r\}$, $r \geq 2$ be a set of vectors in a vector space V. Then S is linearly Independent if and only if no vector in S is expressible as a linear Combination of the other vectors in S

The proof of theorem (7)' is very Similar to the proof of theorem (7).

Theorem (8) :

A finite set of vectors that contains the zero vector is linearly dependent.

proof: let $S = \{ \bar{v}_1, \bar{v}_2, \dots, \bar{v}_r, \vec{0} \}$ be a finite set of vectors in a vector space V. clearly

$$0\bar{v}_1 + 0\bar{v}_2 + \dots + 0\bar{v}_r + 1\vec{0} = \vec{0} \quad \dots (*)$$

There fore, there exists scalars

$$c_1 = 0,\ c_2 = 0, \dots, c_r = 0, \text{ and } c_{r+1} = 1$$

not all zero which satisfy vector eq. (*). Hence S is linearly dependent!

Theorem (9):

let $\bar{v}$ be a non-zero vector in a vector space $\bar{v}$. Then the set $S = \{\vec{v}\}$ is linearly independent!

proof.: let c be a scalar such that

$$c\vec{v} = \vec{0} \quad \dots (*)$$

Hence either $c = 0$ or $\vec{v} = \vec{0}$. But but assumption, $\vec{v} \neq \vec{0}$, hence $c = 0$. Therefore vector eq. (*) has only the trivial solution $c = 0$, hence $\{\vec{v}_1\}$ is linearly Independent.

Remark : Quick Test for linear dependence / Independence for a set of exactly two non-zero vectors :

let $S = \{ \bar{v}_1, \bar{v}_2 \}$ be a set of non-zero vectors in a vector space V. Then by theorem (7) " with $r = 2$ ", S is linearly dependent if and only if $\vec{v}_1, \vec{v}_2$ are scalar multiple of one another. Consequently by theorem (7)', S is linearly independent if neither vector is a scalar multiple of the other!

EX : Determine whether the given set of vectors in linearly dependent or linearly independent.

1. $S_1 = \{2-x,\ x-2\}$ in $\mathcal{P}_1$
2. $S_2 = \{(0,0,0,0), (4,1,-1,2), (10,9,-1,40), (0,1,5,-3)\}$ in $\mathbb{R}^4$
3. $S_3 = \{(6,-3,2),\ (1,-\frac{1}{2},\frac{1}{3})\}$ in $\mathbb{R}^3$
4. $S_4 = \left\{ \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix} \right\}$ in M_{22}
5. $S_5 = \{x+4,\ -2x-8,\ x^2+9x-1\}$ in $\mathcal{P}_2$

Solution : 1. S_1 is linearly dependent since $P(x) = 2-x$ is a scalar multiple of $q(x) = x-2$.

In fact $\quad p = 2-x = -(x-2)$

$$\therefore \quad p = -q$$

2. S_2 is linearly dependent since it contains the zero vector $(0,0,0,0) = \vec{0}$ in $\mathbb{R}^4$!

3. S_3 is linearly dependent!

<u>Note</u>: $\bar{v}_1 = (6,-3,2)$, $\bar{v}_2 = (1,-\frac{1}{2},\frac{1}{3})$

clearly $\quad \vec{v}_1 = 6\vec{v}_2$

4. S_4 is linearly Independent since $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix}$ are obviously not a scalar multiple of one another!

5. S_5 is linearly dependent!

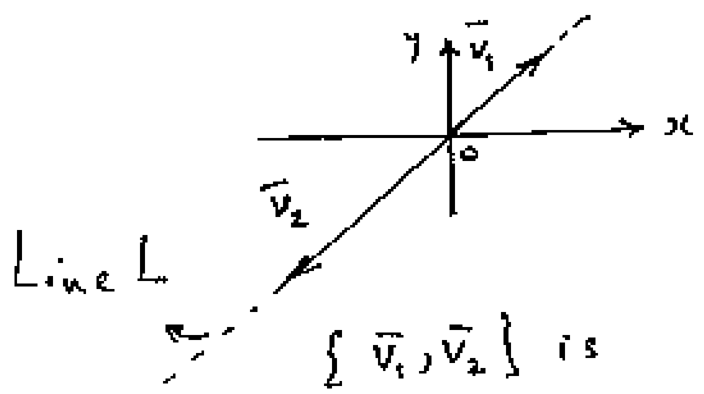
By theorem (7): Since an element in S_5, $-2x-8$ is a scalar multiple of others, namely

$$-2x-8 = -2(x+4) + 0(x^2+9x-1).$$

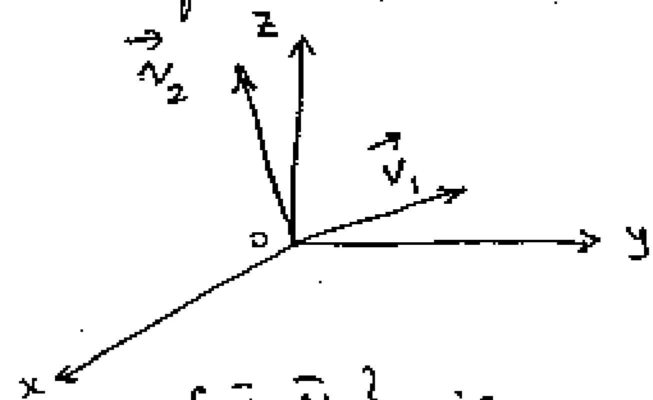
— — — — — —

Geometric Interpretation : Dependent and Independent sets in $\mathbb{R}^2$, and $\mathbb{R}^3$:

1. Let $S = \{\vec{v}_1, \vec{v}_2\}$ be set of "position" vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$.
Then S is linearly Independent if and only if $\vec{v}_1, \vec{v}_2$ do not lie on the same line.

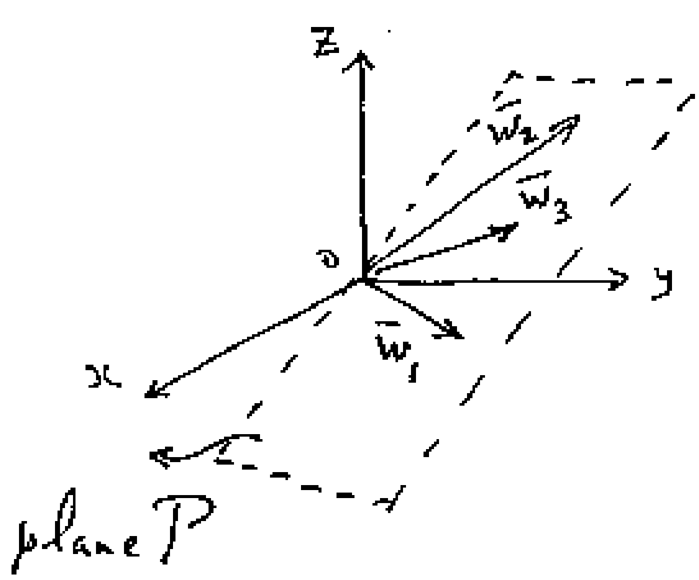


$\{\vec{v}_1, \vec{v}_2\}$ is Linearly Dependent in $\mathbb{R}^2$

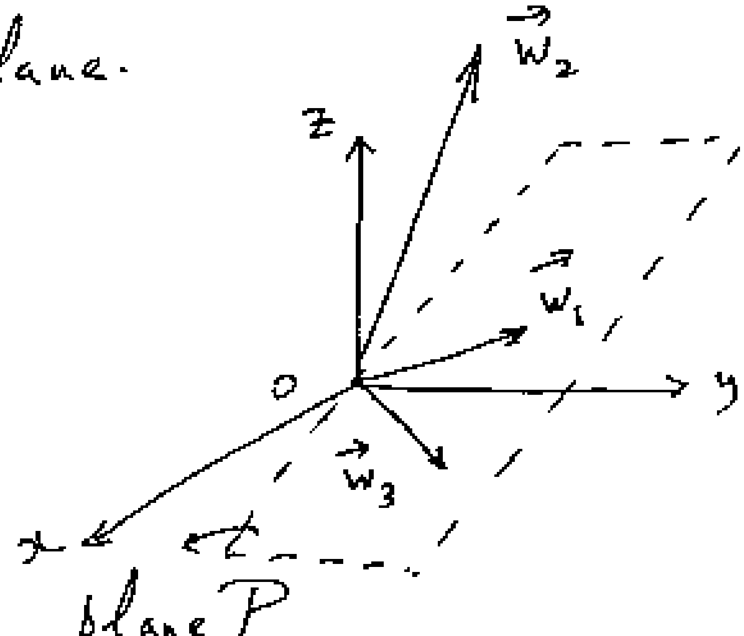


$\{\vec{v}_1, \vec{v}_2\}$ is linearly Independent in $\mathbb{R}^3$

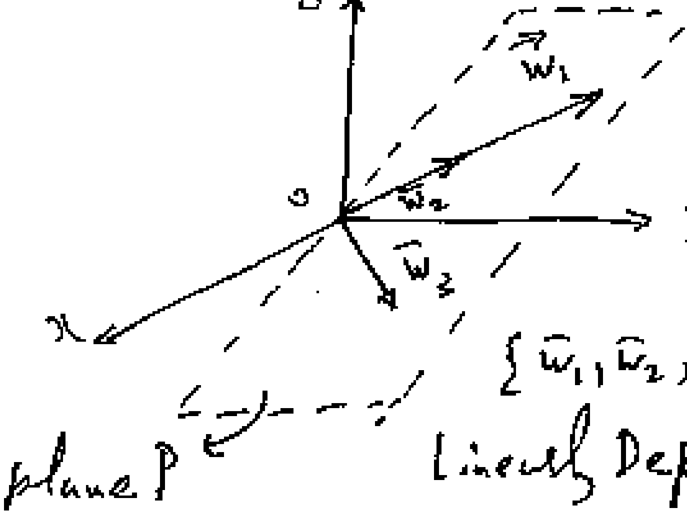
2. let $S = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ be set of "position" vectors in $\mathbb{R}^3$.
Then S is linearly Independent if and only if $\vec{w}_1, \vec{w}_2$, and $\vec{w}_3$ do not lie on the same plane.



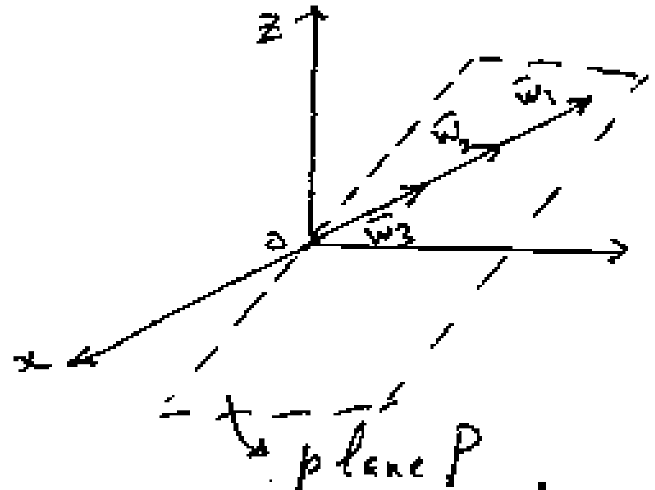
$\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is linearly Dependent in $\mathbb{R}^3$



$\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is Linearly Independent in $\mathbb{R}^3$.



$\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is Linearly Dependent in $\mathbb{R}^3$



$\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is dependent.

EX: prove that if $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent set of vectors in a vector space V, then so are $\{\vec{v}_1\}$, $\{\vec{v}_1, \vec{v}_2\}$.

proof.: Since $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent, it can not contain the zero vector. Hence the set $\{\vec{v}_1\}$, where $\vec{v}_1 \neq \vec{0}$ is linearly Independent!

Next, to show that the set $\{\vec{v}_1, \vec{v}_2\}$ is linearly Independent, we assume on the contrary that the set is linearly dependent. It follows that $\vec{v}_1$ must be a scalar multiple of $\vec{v}_2$:

$$\therefore \quad \vec{v}_1 = k\vec{v}_2$$

$$\text{or} \quad \vec{v}_1 = k\vec{v}_2 + 0\,\vec{v}_3$$

Therefore a vector $\vec{v}_1$ in S is expressible as a linear combination of the other vectors in S. This would mean that S is linearly dependent.. a contradiction which implies that $\{\vec{v}_1, \vec{v}_2\}$ can't be linearly dependent and hence it must be linearly Independent!

EX: Let $\{\vec{x}, \vec{y}\}$ be a linearly independent set of vectors in a vector space V. Show that the set $\{2\vec{x}+3\vec{y}, \vec{x}-2\vec{y}\}$ is also linearly independent.

solution: let $S = \{\vec{x}, \vec{y}\}$, $S' = \{2\vec{x}+3\vec{y}, \vec{x}-2\vec{y}\}$

To show that S' is linearly Independent, let c_1, c_2 be scalars such that:

$$c_1(2\bar{x}+3\bar{y})+c_2(\bar{x}-2\bar{y})=\vec{0} \quad \text{---} (*)$$

$$\Rightarrow (2c_1+c_2)\bar{x}+(3c_1-2c_2)\bar{y}=\vec{0}$$

let $K_1 = 2c_1+c_2$, and $K_2 = 3c_1-2c_2$

$$\therefore \quad K_1\bar{x}+K_2\bar{y}=\vec{0} \quad \text{----} (**)$$

But then since $S=\{\bar{x},\bar{y}\}$ is linearly independent, the vector equation $(**)$ is satisfies by only the trivial solution $K_1=0$, $K_2=0$

That is
$$\begin{cases} 2c_1+c_2=0 \\ 3c_1-2c_2=0 \end{cases}$$

solving we get $c_1=c_2=0$. Hence S' is linearly independent since $(*)$ has only the trivial solution $c_1=c_2=0$.

Ex: prove that for any vectors $\bar{u}, \bar{v}$, and $\bar{w}$ in a vector space V, the set $S=\{\bar{u}-\bar{v}, \bar{v}-\bar{w}, \bar{w}-\bar{u}\}$ is linearly dependent.

Solution: The idea is to try to express a vector in S as a linear combination of the other two.

Indeed, by inspection:

$$\bar{u}-\bar{v}=(-1)(\bar{v}-\bar{w})+(-1)(\bar{w}-\bar{u})$$

Hence by theorem (7), the set S is linearly dependent!
