(C)

MATH 311

SUBSPACES

Defn. (12): A subspace.

Let V be a vector space and W be a non-empty subset of V. Then W is called a subspace of V if W is itself a vector space under the addition and scalar multiplication defined on V.

Theorem (5): Test for a subspace

Let W be a non-empty subset of a vector space V. Then W is a subspace of V if and only if the following two conditions are met

(i) If $\vec{u}$, and $\vec{v}$ are vectors in W, then $\vec{u}+\vec{v}$ is in W

(ii) If K is any scalar and $\vec{u}$ is any vector in W, then $K\vec{u}$ is in W

Note: Conditions (i), and (ii) are equivalent to: W is closed under addition and scalar multiplication respectively!

proof.: This is a two-way proof.

First: Assume W is a subspace of V. Then vectors in W satisfy all ten axioms of vector space. In particular axioms A1, and S1 hold. But these are precisely conditions (i), and (ii) above!

Conversely: Assume conditions (i), (ii) hold. Since these conditions are axioms A1, and S1 respectively, we need only to show that vectors in W satisfy the remaining Eight axioms: A2, A3, A4, A5; S2, S3, S4, and S5.

However axioms A2, A3, S2, S3, S4, and S5 are automatically satisfied by all vectors in W since they are satisfied by all vectors in V (that is to say: they are inherited from V !!). Therefore there remains to show that axioms A4, and A5 are satisfied!

Indeed, for any scalar K and any vector $\vec{u}$ in W, Condition (ii) tells us that $K\vec{u}$ is in W

In particular if $K=0$, $0\vec{u}=\vec{0}$ is in W. "this proves A4."

and if $K=-1$, $-1\vec{u}=-\vec{u}$ is in W. "this proves A5."

This completes the proof.

<u>Remark</u>: let V be a vector space. V has two obvious subspaces, namely V itself and the zero vector space $\{\vec{0}\}$.

<u>EX</u>: Let $W=\{A \text{ in } M_{22} \mid \det A = 1\}$. Is W a subspace of M_{22}?

<u>solution</u>: No! we shall show that W is not closed under addition.

Indeed, pick $A_1=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $A_2=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$.

clearly $\det A_1=\det A_2=1$, hence $A_1, A_2 \in W$. However,

$A_1+A_2=\begin{bmatrix}0&0\\0&0\end{bmatrix}\notin W$ since $\det(A_1+A_2)=0\neq 1$

<u>EX</u>: let $W=\{p \text{ in } \mathcal{P}_2 \mid p(4)=0\}$. Show that W is a subspace of $\mathcal{P}_2$.

<u>Solution</u>: Clearly W is non-empty. In fact $0\in W$ "why?"

Now, let $p, q\in W$, then $p(4)=0$, $q(4)=0$

$\therefore (p+q)(4) = p(4)+q(4) = 0+0=0 \Rightarrow p+q\in W$

Next, for any scalar K, and any p in W,

$$(Kp)(4) = Kp(4) = K\cdot 0 = 0 \Rightarrow Kp\in W$$

$\therefore$ W is closed under addition and scalar multiplication

$\therefore$ W is a subspace of $\mathcal{P}_2$.

<u>EX</u>: let $W=\{(x,y,z) \mid 2x+y-3z=0\}$. Is W a subspace of $\mathbb{R}^3$?

<u>solution</u>: yes indeed!

Note 1st. that W is a plane through the origin in $\mathbb{R}^3$ with a normal vector $\bar{N} = (2,1,-3)$. We can express W in the simplified form

$$W = \{ \bar{r} \text{ in } \mathbb{R}^3 \mid \bar{r} \cdot \bar{N} = 0 \}$$

let $\bar{r}_1, \bar{r}_2$ be vectors in W. Then $\bar{r}_1 \cdot \bar{N} = 0$, $\bar{r}_2 \cdot \bar{N} = 0$

$$\therefore (\bar{r}_1 + \bar{r}_2) \cdot \bar{N} = \bar{r}_1 \cdot \bar{N} + \bar{r}_2 \cdot \bar{N} = 0 + 0 = 0$$

$\therefore$ $\bar{r}_1 + \bar{r}_2$ is in W $\Rightarrow$ W is closed under $(+)$.

Next, for any scalar K and any $\bar{r}$ in W,

$$(K\bar{r}) \cdot \bar{N} = K(\bar{r} \cdot \bar{N}) = K(0) = 0$$

$\therefore$ $K\bar{r}$ is in W $\Rightarrow$ W is closed under $(\cdot)$

Hence W is a subspace of $\mathbb{R}^3$.

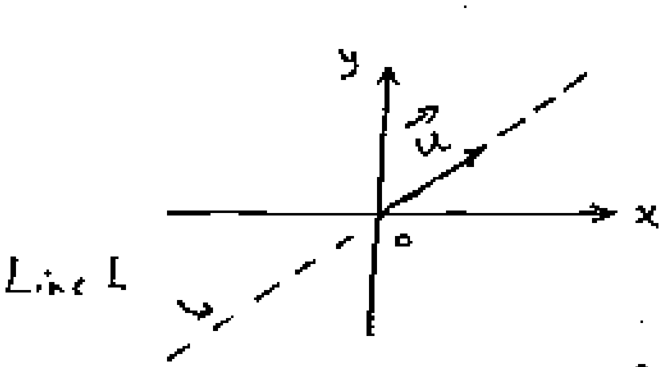
Ex: show that a plane through the origin in $\mathbb{R}^3$:

$$ax + by + cz = 0$$

is a subspace of $\mathbb{R}^3$. Do at Home! (Identical to Ex. above!)

Ex: Use geometric argument to prove that a straight line through the origin of $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$.

proof: let W be a st. line through origin and $\bar{u}, \bar{v}$ be vectors in W. Clearly the sum $\bar{u} + \bar{v}$ lies on this line and for any scalar K, the vector $K\bar{u}$ lies on the line as well. Refere to figures (1), (2)



$W = \{ \vec{u} \mid \vec{u} \text{ lies on } L \}$

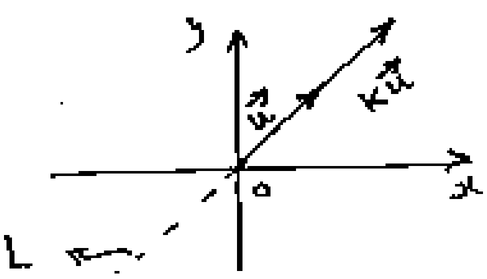


Fig. (2)

W is closed under $(\cdot)$

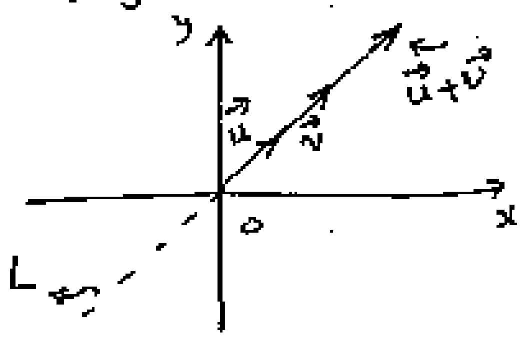


Fig. (1)

W is closed under $(+)$

EX: Show (algebraically) that a straight line through the origin of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

proof: Very Easy!

Hint: A st. line through the origin of $\mathbb{R}^3$ is given by the set

$$W = \{ \bar{r} \text{ in } \mathbb{R}^3 \mid \bar{r} = t\vec{v},\ t \in \mathbb{R} \},$$

for some fixed non-zero vector $\vec{v}$. Called a direction vector of the st. line.

Let us summarize some possible subspace of the Euclidean space $\mathbb{R}^2$, and $\mathbb{R}^3$:

(i) Some possible subspaces of $\mathbb{R}^2$: $\{\vec{0}\}$, $\mathbb{R}^2$, lines through origin.

(ii) Some possible subspaces of $\mathbb{R}^3$: $\{\vec{0}\}$, $\mathbb{R}^3$, lines through origin, planes through origin.

Defn. (13): Solution space of a homogeneous linear system:

Consider the homogeneous linear system

$$A\vec{x} = \vec{0}$$

where "A" is an $m \times n$ matrix with real entries. A vector $\vec{x}$ in $\mathbb{R}^n$ that satisfies the equation is called "a solution vector" of the system. Let W be the set of all solution vectors of the system. We shall show that W is a subspace of $\mathbb{R}^n$ which we shall call "The solution space" of the system.

Indeed $W = \{\vec{x} \mid A\vec{x} = \vec{0}\}$ is non-empty (why?).

Let $\bar{x}_1, \bar{x}_2$ be vectors in W. Hence $A\bar{x}_1 = \vec{0}$, and $A\vec{x}_2 = \vec{0}$

Now, if $\bar{y} = \bar{x}_1 + \bar{x}_2$, $A\vec{y} = A(\bar{x}_1 + \bar{x}_2) = A\bar{x}_1 + A\bar{x}_2 = \vec{0} + \vec{0} = \vec{0}$

$\Rightarrow$ $\bar{y}$ is a solution vector of the system

$\Rightarrow$ $\vec{y} \in W$, that is $\bar{x}_1 + \bar{x}_2 \in W$ $\Rightarrow$ W is closed under $(+)$

Next, for any scalar K and any vector $\vec{x}$ in W, it is clear that

$$A(K\vec{x}) = K(A\vec{x}) = K(\vec{0}) = \vec{0}$$

$\Rightarrow K\vec{x}$ is in W -- hence W is closed under $(\cdot)$.

$\therefore$ W is a subspace of $\mathbb{R}^n$.

Ex: Find the solution space of the homogeneous linear system,

$$\begin{bmatrix} 1 & -2 & 3 \\ -5 & 11 & -14 \\ 1 & -3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution: We need to bring augmented matrix to reduced row-echelon form!

$$[A \,\vdots\, \vec{b}] = \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ -5 & 11 & -14 & 0 \\ 1 & -3 & 2 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 + 5R_1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 + R_1]{R_1 \to R_1 + 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

.. Done! $n - r = 3 - 2$ = one-parameter family.

system reduces to: $x + 5z = 0$,
$y + z = 0$

Since leading one of z is missing, $z = t$, $t \in \mathbb{R}$

$\therefore$ $x = -5t$, $y = -t$

$$\therefore \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} -5 \\ -1 \\ 1 \end{bmatrix} \quad \text{or} \quad \vec{x} = t(-5, -1, 1)$$

Note: a column vector may be viewed as an ordered triple!

$\therefore$ solution space $W = \{\vec{x} \mid \vec{x} = t\vec{v}\}$, where $\vec{v} = (-5, -1, 1)$.

This subspace in $\mathbb{R}^3$ is a st. line through the origin!

Def. (14): A Linear Combination

Let V be a vector space and let $S = \{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_r\}$ be a set of vector in V. A vector $\vec{w}$ in V is a linear Combination of vectors in S if it is expressible in the form

$$\vec{w} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_r \vec{v}_r \quad --- (*)$$

for some scalars $c_1, c_2, \ldots,$ and c_r.

Note that if $\vec{w}$ can not be expressed in the form $(*)$, we say $\vec{w}$ is not a linear Combination of vectors in S.

From now-on we may use the abbreviation l.c for linear Combination

Ex: show that $\vec{w} = (1, 15, -10)$ is a linear Combination of $\bar{v}_1 = (1, 0, -1)$, and $\bar{v}_2 = (1, 5, -4)$.

solution: Assume $\bar{w} = c_1 \bar{v}_1 + c_2 \bar{v}_2$ for some $c_1, c_2 \in \mathbb{R}$

$$\therefore (1, 15, -10) = c_1 (1, 0, -1) + c_2 (1, 5, -4)$$

Equating corresponding components we get the linear system,

$$c_1 + c_2 = 1 \quad --- (1)$$

$$5c_2 = 15 \quad --- (2)$$

$$-c_1 - 4c_2 = -10 \quad --- (3)$$

we can find c_1, c_2 using only 1st. two equations:

From (2) : $c_2 = 3$

From (1) : $c_1 + 3 = 1 \Rightarrow c_1 = -2$

However for the system to be consistent $c_1 = -2$, and $c_2 = 3$ must satisfy the unused equation (3)!

Indeed
$$-c_1 - 4c_2 = -10$$
$$-(-2) - 4(3) = -10$$
$$2 - 12 = -10$$
$$-10 = -10 \quad \text{(True statement)}$$

$\therefore \vec{w}$ is l.c of $\bar{v}_1, \bar{v}_2$.

Ex: show whether $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a linear Combination of

$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

solution: let $K_1, K_2,$ and K_3 be scalars such that,

$$X = K_1 A + K_2 B + K_3 C$$

$$\therefore \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = K_1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + K_2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + K_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating both sides, we get:

$$0 = K_1 \quad , \quad 1 = K_1 \quad , \quad 0 = K_2 \, , \text{ and } 0 = K_2 + K_3$$

However the 1st. two equations $K_1 = 0$, $K_1 = 1$ are contradictory!

$\therefore$ system has no solution meaning no such scalars $K_1, K_2,$ and K_3 exist. Hence X is not a l.C of $A, B,$ and C.

Theorem (6):

Let V be a vector space and $S = \{ \bar{v}_1, \bar{v}_2, \dots, \bar{v}_r \}$ be a set of vectors in V. Then

(i) The set W of all linear Combinations of vectors in S is a subspace of V.

(ii) W is the Smallest subspace of V that Contains $\bar{v}_1, \bar{v}_2, \dots$ and $\bar{v}_r$, in the sense that every other subspace W' of V Containing $\bar{v}_1, \bar{v}_2, \dots,$ and $\bar{v}_r$ must Contain W.

proof. (i) $W = \{ \vec{\omega} \mid \vec{\omega} = c_1 \bar{v}_1 + c_2 \bar{v}_2 + \dots + c_r \bar{v}_r \}$, $c_i \in \mathbb{R}$, $i = 1, 2, \dots r$

Indeed W is non-empty: $\vec{0} = 0 \cdot \bar{v}_1 + 0 \bar{v}_2 + \dots + 0 \bar{v}_r$ is in W.

Now, let $\bar{w}_1, \bar{w}_2$ be vectors in W, then $\bar{w}_1, \bar{w}_2$ are of the form

$$\bar{w}_1 = \alpha_1 \bar{v}_1 + \alpha_2 \bar{v}_2 + \dots + \alpha_r \bar{v}_r \quad , \quad \alpha_i \in \mathbb{R} \, , \, i = 1, 2, \dots r$$

$$\bar{w}_2 = \beta_1 \bar{v}_1 + \beta_2 \bar{v}_2 + \dots + \beta_r \bar{v}_r \quad , \quad \beta_i \in \mathbb{R} \, , \, i = 1, 2, \dots r$$

$\therefore \quad \bar{w}_1 + \bar{w}_2 = (\alpha_1+\beta_1)\bar{v}_1 + (\alpha_2+\beta_2)\bar{v}_2 + \cdots + (\alpha_r+\beta_r)\bar{v}_r$

$\overset{or}{=} \; c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_r\bar{v}_r$

where $c_i = \alpha_i + \beta_i \;, \; i = 1, 2, \ldots r$

It follows that $\bar{w}_1 + \bar{w}_2$ is in W, hence W is closed under $(+)$.

Next, for any scalar K and any $\bar{w} = a_1\bar{v}_1 + a_2\bar{v}_2 + \cdots + a_r\bar{v}_r$ in W, the vector

$$K\vec{w} = K(a_1\bar{v}_1 + a_2\bar{v}_2 + \cdots + a_r\bar{v}_r)$$
$$= (Ka_1)\bar{v}_1 + (Ka_2)\bar{v}_2 + \cdots + (Ka_r)\bar{v}_r$$
$= $ l. c of vectors in S

It follows that $K\vec{w}$ is in W, hence W is closed under $(\cdot)$.
This proves that W is a subspace of V. This subspace will be given a special name and special notation in definition (15) to follow the proof of the theorem!

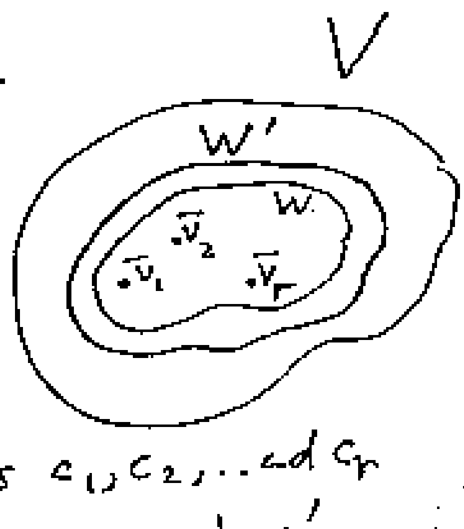
<u>To prove part (ii)</u> :

1st. observe that every vector $\vec{v}_i$ is S is expressible as a l.C of vectors in S, namely

$$\bar{v}_i = 0\bar{v}_1 + 0\bar{v}_2 + \cdots + 1\vec{v}_i + \cdots + 0\bar{v}_r$$

$\therefore \quad \bar{v}_i \;, \; i = 1, 2, \ldots r$ is a vector in W

(In other words : W contains each vector in S).



let W' be a subspace of V that contains the vectors $\bar{v}_1, \bar{v}_2, \ldots$ and $\bar{v}_r$. We need to show that W' must contains W, that is $W \subseteq W'$.
let $\vec{w}$ be a vector in W, then, there exists scalars $c_1, c_2, \ldots$ and c_r such that $\bar{w} = c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_r\bar{v}_r$. But then since W' is a subspace of V, it is closed under $(+)$ and $(\cdot)$.. hence it must contain all l.C. of $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_r$. In particular $\vec{w} \in W'$. This completes the proof.

Def. (15) : spanning set

Let V be a vector space and $S = \{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_r\}$ be a set of vectors in V. Let W be the set of all linear combinations of vectors in S.

In theorem (6), we have proved that W is a subspace of V. This subspace is referred to as the subspace spanned by S and we write

$$W = \text{span}(S)$$

we may also say that S is a "spanning set" of the subspace W.

Important remark : Since V is a subspace of itself, it is possible to find a set S which spans entire vector space V, in which case

$$V = \text{span}(S)$$

Note also that : Theorem (6) tells us that

$$\text{span}(S) \subseteq V \quad --- \quad (1)$$

Therefore to prove $V = \text{span}(S)$, one need only to show

$$V \subseteq \text{span}(S) \quad --- \quad (2)$$

Equality follows from (1), (2) !

EX : spanning set is not unique !

let $S = \{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_r\}$ and $S' = \{\bar{v}_1, \bar{v}_2, \dots \bar{v}_r\}$ be sets of vectors in a vector space V. prove that $\text{span}(S) = \text{span}(S')$ if and only if : Every vector in S is a linear combination of vectors in S' and vice-versa. Home Work!

EX: show that the set $S = \{2-x, 1+x^2, 3x-1\}$ spans $\mathcal{P}_2$.

solution: want to show

$$\mathcal{P}_2 = \text{span}(S)$$

By theorem (6), $\quad \text{span}(S) \subseteq \mathcal{P}_2 \quad --- \quad (1)$

There fore we need only show

$$\mathcal{P}_2 \subseteq \text{span}(S) \quad --- \quad (2)$$

pick an arbitrary element p in $\mathcal{P}_2$, say

$$p(x) = a_0 + a_1 x + a_2 x^2$$

If $p(x) \in \text{span}(S)$, $p(x)$ must be a l.c of members of S

$$\therefore \quad p(x) = c_1(2-x) + c_2(1+x^2) + c_3(3x-1)$$

$$\therefore \quad a_0 + a_1 x + a_2 x^2 = (2c_1 + c_2 - c_3) + (-c_1 + 3c_3)x + c_2 x^2$$

Comparing Coefficents of Various powers of x:

$$\begin{aligned} c_1 + c_2 - c_3 &= a_0 \\ -c_1 \quad\quad + 3c_3 &= a_1 \\ c_2 \quad\quad &= a_2 \end{aligned}$$

or $\quad A\vec{x} = \vec{b}$ where $A = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$.

This system has a solution no matter what $\vec{b}$ is if A is invertible, that is if $\det A \neq 0$ .. let us find out!

Indeed, $\quad \det A = \begin{vmatrix} 2 & 1 & -1 \\ -1 & 0 & 3 \\ 0 & 1 & 0 \end{vmatrix} \leftarrow$ expanding, using 3rd. row

$$= -1 \begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} \neq 0$$

There fore c_1, c_2, and c_3 do exist!

$$\Rightarrow \quad \mathcal{P}_2 \subseteq \text{span}(S)$$

This prove our assertion : $\quad \mathcal{P}_2 = \text{span}(S)$. #

Special spanning sets in $\mathbb{R}^2$, and $\mathbb{R}^3$

1. Let $\vec{v}$ be a non-zero vector in $\mathbb{R}^2$ (or $\mathbb{R}^3$). Then the subspace $W = \text{Span}\{\vec{v}\}$ is a st. line through the origin in $\mathbb{R}^2$ (or $\mathbb{R}^3$).

proof. : $W = \text{span}\{\vec{v}\} \underset{\text{defn.}}{\overset{\text{By}}{=\!=}} \{\bar{r} \mid \bar{r} = t\vec{v}\}$ which represents a st. through origin in $\mathbb{R}^2$ (or $\mathbb{R}^3$)

2. let $\vec{u}, \vec{v}$ be non-collinear position vectors in $\mathbb{R}^3$. Then the subspace $W = \text{Span}\{\bar{u}, \bar{v}\}$ is a plane through the origin in $\mathbb{R}^3$ determined by $\bar{u}, \bar{v}$.

proof. $W = \text{span}\{\bar{u}, \bar{v}\}$ = set of all linear Combinations of $\bar{u}, \bar{v}$

$$= \{\bar{r} \mid \bar{r} = c_1\bar{u} + c_2\bar{v}\}$$

Now $\bar{r} = c_1\bar{u} + c_2\bar{v}$. let $\bar{r} = (x, y, z)$, $\bar{u} = (a, b, c)$, and $\bar{v} = (a', b', c')$

$$(x, y, z) = c_1(a, b, c) + c_2(a', b', c')$$

$\therefore$

$$\Rightarrow \quad \begin{aligned} ac_1 + a'c_2 &= x \\ bc_1 + b'c_2 &= y \\ cc_1 + c'c_2 &= z \end{aligned}$$

Eliminating c_1, c_2 among three equations, we get,

$$\begin{vmatrix} a & a' & x \\ b & b' & y \\ c & c' & z \end{vmatrix} = 0$$

expanding using 3rd. Column, we get

$$(bc' - b'c)x - (ac' - a'c)y + (ab' - a'b)z = 0$$

$$\text{or} \quad Ax + By + Cz = 0$$

where $(A, B, C) = \bar{u} \times \bar{v}$ " verify! "

This is an eq. of a plane through origin in $\mathbb{R}^3$.

EX: Find the equation of the plane in $\mathbb{R}^3$ spanned by $\bar{u} = (1,1,-2)$, and $\bar{v} = (0,1,4)$.

Solution: A normal vector to plane is given by

$$\bar{N} = \bar{u} \times \bar{v} = (6, -4, 1)$$

$\therefore$ Its equation is given by $\bar{r} \cdot \bar{N} = 0$

or $(x,y,z)\cdot(6,-4,1) = 0$

$\Rightarrow$ $6x - 4y + z = 0$.

EX: Show whether the set $\left\{ \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix} \right\} = S$ spans M_{22}.

Solution: Know: $\text{span}(S) \subseteq M_{22}$ --- (1)

Need to show whether $M_{22} \subseteq \text{span}(S)$?

If this is the case: an arbitrary element in M_{22} must be expressible as a l.c. of members of S

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an arbitrary member of M_{22}

$$\therefore \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = K_1 \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + K_2 \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} + K_3 \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} + K_4 \begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}$$

equating corresponding entries of both sides,

$$\begin{aligned} K_1 \qquad\quad + 2K_3 + 3K_4 &= a \\ K_1 \qquad\quad + K_3 + 2K_4 &= b \\ -K_1 - K_2 \qquad\quad - 2K_4 &= c \\ K_2 \qquad\quad + K_4 &= d \end{aligned}$$

or $A\bar{x} = \vec{b}$ where $A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ -1 & -1 & 0 & -2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, $\bar{x} = \begin{bmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

This system is consistent no matter what $\vec{b}$ is provided $\det A \neq$

However $\det A = \begin{vmatrix} 1 & 0 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ -1 & -1 & 0 & -2 \\ 0 & 1 & 0 & 1 \end{vmatrix} \xrightarrow{C_4 \to C_4 - (C_1 + C_2 + C_3)} \begin{vmatrix} 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} = 0$!

$\therefore$ No! $M_{22} \neq \text{span}(S)$.