(B)

# MATH 311

## VECTOR SPACES

**Defn. (1): A vector Space**

let $V$ be an arbitrary non-empty set of objects on which two operations are defined, addition and scalar multiplication.

- By addition we mean a rule for associating with each pair of objects $\vec{u}$, and $\vec{v}$ in $V$ a unique object $\vec{u}+\vec{v}$, called "The sum of $\vec{u}$, $\vec{v}$;"
- By scalar multiplication we mean a rule for associating with each scalar $K$ and each object $\vec{u}$ in $V$ a unique object $K\vec{u}$, called the scalar multiple of $\vec{u}$ by $K$.

If the following ten axioms A1 — A5, S1 → S5 are satisfied by all objects $\vec{u}, \vec{v}$, and $\vec{w}$ in $V$ and all scalars $K$, and $l$, then we call "$V$" a vector space and we called the objects in $V$ vectors.

**Addition Axioms: A1 → A5:**

**A1** $\vec{u}+\vec{v}$ is in $V$ "That is $V$ is closed under addition."

**A2** $\vec{u}+\vec{v} = \vec{v}+\vec{u}$ "That is addition operation is Commutative."

**A3** $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$

"That is addition operation is associative."

**A4** There is an object in $V$ called "The zero vector", denoted by $\vec{0}$ such that

$$\vec{u}+\vec{0} = \vec{0}+\vec{u} = \vec{u} \quad \text{for all } \vec{u} \text{ in } V$$

**A5** For each $\vec{u}$ in $V$, there is an object $-\vec{u}$ in $V$, called the negative of $\vec{u}$ with the property

$$\vec{u}+(-\vec{u}) = (-\vec{u})+\vec{u} = \vec{0}$$

**Scalar Multiplication Axioms: S1 → S5:**

**S1** $K\vec{u}$ is in $V$ for any scalar $K$ and any object $\vec{u}$ in $V$

"That is $V$ is closed under scalar Multiplication."

S2 $\quad K(\vec{u}+\vec{v}) = K\vec{u} + K\vec{v}$
S3 $\quad (K+\ell)\vec{u} = K\vec{u} + \ell\vec{u}$ } Distributive Properties.

S4 $\quad K(\ell\vec{u}) = (K\ell)\vec{u}$

S5 $\quad 1\,\vec{u} = \vec{u}$

Defn. (2) : Real and Complex Vector spaces

The scalars we have mentioned in definition (1) above are in general members of some Field $\mathbb{F}$. In particular the scalars may be "Real Numbers" or "Complex Numbers".

Vector spaces in which the scalars are "Real Numbers" are called "Real Vector Spaces" whereas vector spaces in which the scalars are "Complex Numbers" are called "Complex Vector Spaces".

In this course we shall only discuss "Real Vector Spaces".

Defn. (3) : The set $\mathbb{R}^n$ - The n-space :

let $n$ be a positive integer. An ordered n-tuple is a sequence of $n$ real numbers denoted by $(a_1, a_2, \dots, a_n)$. The set of all ordered n-tuple is denoted by $\mathbb{R}^n$ : reads $\mathbb{R}$-$n$ or n-space.
It follows that
$$\mathbb{R}^n = \{(a_1, a_2, \dots, a_n) \mid a_i, \ i=1,2,\dots n \text{ is a real number}\}.$$

Special Cases

1. If $n=1$, each ordered n-tuple consists of a single real number. Hence the set $\mathbb{R}^1$ may be viewed as the well known set of real numbers $\mathbb{R}$. We shall write $\mathbb{R}^1 = \mathbb{R}$.

2. If $n = 2$ or $3$, it is common to use the terms : ordered pair and ordered triple rather than ordered 2-tuple and ordered 3-tuple respectively.

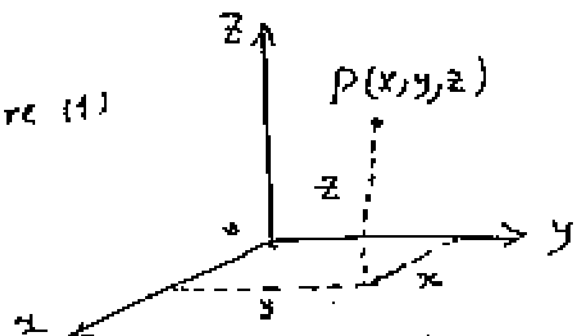
## Geometric Interpretation

Consider the set $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$.

An ordered triple $(x, y, z)$ in $\mathbb{R}^3$ may be viewed as either a point $P$ in 3-space, in which case $x$, $y$, and $z$ are the coordinates of $P$ (figure (1)), or may be viewed as a position vector $\vec{u}$ in 3-space, in which case $x$, $y$ and $z$ are the components of $\vec{u}$ (figure (2)).

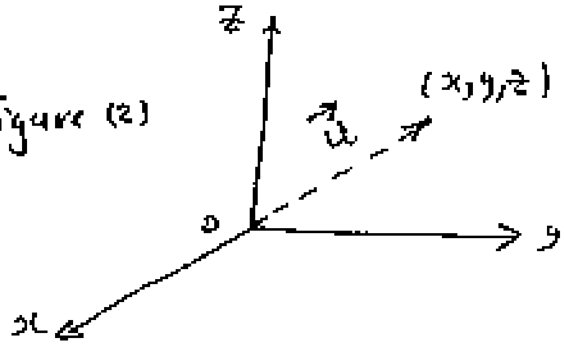
It follows that an ordered $n$-tuple $(a_1, a_2, \dots, a_n)$ may be viewed as either a generalized point or as a generalized position vector in $n$-space.

Figure (1)




$(x, y, z)$ is viewed as a point $P$ in $\mathbb{R}^3$

Figure (2)




$(x, y, z)$ is viewed as a position vector $\vec{u}$ in $\mathbb{R}^3$

**Def. (4) : Equal vectors in $\mathbb{R}^n$**

Let $\bar{u} = (x_1, x_2, \dots, x_n)$, $\bar{v} = (y_1, y_2, \dots, y_n)$ be vectors in $\mathbb{R}^n$. We shall say $\bar{u}$, and $\bar{v}$ are equal and write $\bar{u} = \bar{v}$ if : corresponding components of $\bar{u}, \bar{v}$ are equal that is : $\vec{u} = \vec{v}$ if $x_i = y_i$ $i = 1, 2, \dots,$ and $n$.

**Def. (5) : Standard operations in $\mathbb{R}^n$**

1. The sum of vectors $\bar{u} = (x_1, x_2, \dots, x_n)$, and $\bar{v} = (y_1, y_2, \dots, y_n)$ in $\mathbb{R}^n$ is denoted and defined by

$$\bar{u} + \bar{v} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

Note : $\bar{u} + \bar{v}$ is also a vector in $\mathbb{R}^n$ !

2. For any scalar $K$ and a vector $\bar{u} = (x_1, x_2, \dots, x_n)$ in $\mathbb{R}^n$, the scalar multiple of $\vec{u}$ by $K$ is denoted and defined by

$$K\bar{u} = (Kx_1, Kx_2, \dots, Kx_n)$$

Note : $K\vec{u}$ is also a vector in $\mathbb{R}^n$ !

The operations of addition and scalar multiplication defined above are called

" The Standard operations in $\mathbb{R}^n$ ".

<u>Defn. (6)</u> : <u>A matrix - The Set $M_{mn}$</u>

A matrix is a rectangular array of numbers. These numbers are referred to as the entries of the matrix. If the rectangular array consists of $m$ rows ad $n$ columns, one says the matrix is of order $m$ by $n$ ad we write $m \times n$.

Matrices are usually denoted by upper case letters such as $A, B, X, \ldots$. However entries of a matrix are usually denoted by the corresponding lower case letters. In particular if "$A$" is a matrix, the entry of $A$ which lies in the $i$th row ad the $j$th column is denoted by $a_{ij}$. Therefore the matrix "$A$" may be described by

$$A = (a_{ij}) \qquad i = 1, 2, \ldots m \; ; \; j = 1, 2, \ldots n$$

$$\text{or} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The set of all $m \times n$ matrices with real entries will be denoted by $M_{mn}$. For instance $M_{22}$ is the set of all square matrices of order "2" with real entries.

<u>Defn. (7)</u> : <u>Equal matrices</u>

Two matrices $A$, ad $B$ are said to be equal ad we write $A = B$ if the following two conditions are met

(i) $A$, ad $B$ have the <u>Same</u> order say $m \times n$

(ii) Corresponding entries of $A$, ad $B$ are equal, that is

$$a_{ij} = b_{ij} \qquad i = 1, 2, \ldots, m \; ; \; j = 1, 2, \ldots n.$$

**Def. (7)***: Standard operations in $M_{mn}$

let $A = (a_{ij})$, $B = (b_{ij})$ $i=1,2,\dots m$; $j=1,2,\dots n$ be matrices in $M_{mn}$.

(1) The sum of $A$, and $B$ is the $m \times n$ matrix denoted and defined by

$$A + B = (a_{ij} + b_{ij}) \quad i=1,2,\dots m;\ j=1,2,\dots n$$

(2) For any scalar $K$, the product $KA$ is the $m \times n$ matrix defined by

$$KA = (K a_{ij}) \quad i=1,2,\dots m;\ j=1,2,\dots n$$

The addition and scalar multiplication operations defined above are the usual addition and scalar multiplication of matrices and are referred to as the "Standard operation in $M_{mn}$".

**Def. (8)**: A polynomial function: The Set $P_n$:

A real-valued polynomial of the real variable "$x$" is a function $P$ of the form

$$p(x) = a_0 + a_1 x + a_2 x + \dots + a_n x^n$$

where $n \geq 0$ is an integer and $a_0, a_1, \dots,$ and $a_n$ are real numbers with $a_n \neq 0$. The real numbers $a_0, a_1, \dots, a_n$ are the Coefficients of the polynomial where as "$n$" is its degree, and we write $\deg(p) = n$.

EX $p(x) = 1 + 2x + x^2$ is a polynomial of degree 2.

EX $p(x) = c$, $c \neq 0$ is a polynomial of degree zero.

Note: By our definition above $f(x) = 0$ is not a polynomial function!

This function is called "The zero function" and is denoted by $0$.

The set of all polynomial functions of degree n or less together with the zero function will be denoted by $P_n$. There fore if $q \in P_n$, then $\deg(q) \leq n$ or $q \equiv 0$.

## Def. (9): Equal polynomials

Let $p$, and $q$ be polynomials of degree $n$, namely

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n, \quad q(x) = b_0 + b_1 x + \cdots + b_n x^n$$

The two polynomials are said to be equal and we write $p = q$ if:

corresponding coefficients of $p$, and $q$ are equal, that is if

$$a_i = b_i \qquad i = 0, 1, 2, \cdots n.$$

In particular $a_0 + a_1 x + \cdots + a_n x^n = 0$ for all $x$ will mean:

$$a_0 = a_1 = a_2 = \cdots = a_n.$$

## Def. (10): Standard operations in $\mathcal{P}_n$

Let $p$, $q$ be elements in $\mathcal{P}_n$ and $K$ be any scalar.

1. The sum of $p$, and $q$ is denoted and defined by

$$(p+q)(x) = p(x) + q(x)$$

2. The scalar multiple of $p$ by $K$ is denoted and defined by

$$(Kp)(x) = K\, p(x)$$

The addition and scalar multiplication operations above are the usual addition and scalar multiplication of function and are referred to as "The Standard Operations in $\mathcal{P}_n$".

## Def. (11): The Zero Vector space

Let $V$ be a set that consists of a single object which we denote by $\vec{0}$. Define addition and scalar multiplication on $V$ by

$$(i) \quad \vec{0} + \vec{0} = \vec{0} \qquad\qquad (ii) \quad K\vec{0} = \vec{0} \quad \text{for any } K \in \mathbb{R}$$

It is obvious that all the ten axioms of vector space are satisfied. Hence the set $V = \{\vec{0}\}$ together with operations (i), (ii) is a vector space which we call "The Zero Vector Space".

Theorem (1) : Three special Vector spaces

(i) The set $\mathbb{R}^n$ together with the standard operations in $\mathbb{R}^n$ is a vector space called " The Euclidean n-space "

(ii) The set $M_{mn}$ together with the standard operations in $M_{mn}$ is a vector space.

(iii) The set $P_n$ together with the standard operations in $P_n$ is a vector space.

proof. : Obvious.. In each case verify that the ten axioms of a vectorspace are satisfied.

Note that $\mathbb{R}$, the set of all real number satisfy all ten axioms!

Theorem (2) : The additive Cancellation Law

Let $\vec{u}, \vec{v}$, and $\vec{w}$ be vectors in a vector space V. If

$$\vec{u} + \vec{v} = \vec{u} + \vec{w}$$

then

$$\vec{v} = \vec{w}.$$

proof. : Given $\bar{u} + \bar{v} = \bar{u} + \bar{w}$

Adding $-\vec{u}$ to each side we get

$$-\vec{u} + (\vec{u} + \vec{v}) = -\vec{u} + (\vec{u} + \vec{w}) \quad \text{" by } \underline{A5}, -\vec{u} \text{ exists! "}$$

$$\Rightarrow (-\vec{u} + \vec{u}) + \vec{v} = (-\vec{u} + \vec{u}) + \vec{w} \quad \text{" by } \underline{A3} \text{ "}$$

$$\vec{0} + \vec{v} = \vec{0} + \vec{w} \quad \text{" by } \underline{A5} \text{ "}$$

$$\vec{v} = \vec{w} \quad \text{" by } \underline{A4} \text{ "}$$

Remark : Difference of vectors :

If $\vec{x}, \vec{y}$ are vectors in a vector space V, we define the difference $\vec{x} - \vec{y}$ by $\vec{x} - \vec{y} = \vec{x} + (-\vec{y})$

Theorem (3) : Vector Equation

Let $\vec{u}, \vec{v}$ be vectors in a vector space $V$. The Equation

$$\vec{x} + \vec{u} = \vec{v}$$

has a unique solution $\vec{x}$ in $V$ given by $\vec{x} = \vec{v} - \vec{u}$.

proof. $\vec{x} + \vec{u} = \vec{v} \quad \ldots (*)$

If $\vec{x} = \vec{v} - \vec{u}$ is a solution to equation $(*)$, the equation must become a true statement upon replacing $\vec{x}$ by $\vec{v} - \vec{u}$!

Indeed L.H.S of $(*)$ $= \vec{x} + \vec{u}$

$$= (\vec{v} - \vec{u}) + \vec{u} = \left(\vec{v} + (-\vec{u})\right) + \vec{u}$$

$$= \vec{v} + (-\vec{u} + \vec{u}) = \vec{v} + \vec{0} = \vec{v} = \text{R.H.S of } (*).$$

$\therefore$ $\vec{x} = \vec{v} - \vec{u}$ is a solution to Equation $(*)$. To prove uniqueness, assume that eq. $(*)$ has two solutions $\vec{x}_1, \vec{x}_2$. Hence,

$$\vec{x}_1 + \vec{u} = \vec{v} \quad \ldots (1)$$

$$\vec{x}_2 + \vec{u} = \vec{v} \quad \ldots (2)$$

From (1), (2), we get: $\vec{x}_1 + \vec{u} = x_2 + \vec{u}$. But then by Cancellation Law, $\vec{x}_1 = \vec{x}_2$. This proves uniqueness!

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Theorem (4) : Properties of vectors in a vector space

Let $\vec{u}$ be a vector in a vector space $V$ and $K$ be a scalar.

(i) $0\vec{u} = \vec{0}$

(ii) $K\vec{0} = \vec{0}$

(iii) $(-1)\vec{u} = -\vec{u}$

(iv) If $K\vec{u} = \vec{0}$, then either $K = 0$ or $\vec{u} = \vec{0}$.

proof.: (i) $0\vec{u} = 0\vec{u} + \vec{0} \quad \ldots (1)$

on the other hand

$$0\vec{u} = (0+0)\vec{u} \overset{S3}{=\!=} 0\vec{u} + 0\vec{u} \quad \ldots (2)$$

From (1), (2): equating Right hand sides we get,

$$\cancel{0\vec{u}} + 0\vec{u} = \cancel{0\vec{u}} + \vec{0}$$

By cancellation Law: $0\vec{u} = \vec{0}$ .. proved!

(ii) Easy!

(iii) By $\underline{A5}$: $\vec{0} = -\vec{u} + \vec{u}$ --- (3)

on the other hand:

$$\vec{0} = 0\vec{u} = [(-1)+1]\vec{u}$$
$$\underline{\underline{S3}}\ (-1)\vec{u} + 1\vec{u}$$
$$\therefore \vec{0}\ \underline{\underline{S5}}\ (-1)\vec{u} + \vec{u} \quad --- (4)$$

From (3), (4): equating we get,

$$-\vec{u} + \cancel{\vec{u}} = (-1)\vec{u} + \cancel{\vec{u}}$$
$$\Rightarrow -\vec{u} = (-1)\vec{u} \quad \text{.. proved!}$$

(iv) If $K=0$, there is nothing to prove!

Assume $K \neq 0$ - hence $\frac{1}{K}$ exists

Now, $K\vec{u} = \vec{0}$

$$\therefore \frac{1}{K}(K\vec{u}) = \frac{1}{K}(\vec{0}) \quad \text{.. Apply } \underline{\underline{S4}}:$$
$$(\tfrac{1}{K}K)\vec{u} = \vec{0}$$
$$1\vec{u} = \vec{0} \quad \text{.. Apply } \underline{\underline{S5}}$$
$$\vec{u} = \vec{0} \quad \text{.. proved!}$$

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<u>Remark</u>: The properties we have proved simply tell us:
Addition and scalar multiplication of vectors obey same rules of real Numbers !!!

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