MATH 311

ASSigment (I)

SOLUTIONS

(1) we need to verify all ten axioms $A1 \to A5$; $S1 \to S5$

A1: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2 + 1)$ obviously $\in V$

A2: $L.H.S = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2 + 1)$

$R.H.S = (x_2, y_2) + (x_1, y_1) = (x_2 + x_1, y_2 + y_1 + 1) = (x_1 + x_2, y_1 + y_2 + 1)$

$\therefore L.H.S = R.H.S$

A3 $\{(x_1, y_1) + (x_2, y_2)\} + (x_3, y_3) = (x_1 + x_2, y_1 + y_2 + 1) + (x_3, y_3)$

$= ((x_1 + x_2) + x_3, (y_1 + y_2 + 1) + y_3 + 1)$

$= (x_1 + x_2 + x_3, y_1 + y_2 + y_3 + 2) = L.H.S$

$(x_1, y_1) + \{(x_2, y_2) + (x_3, y_3)\} = (x_1, y_1) + (x_2 + x_3, y_2 + y_3 + 1)$

$= (x_1 + (x_2 + x_3), y_1 + (y_2 + y_3 + 1) + 1)$

$= (x_1 + x_2 + x_3, y_1 + y_2 + y_3 + 2) = R.H.S$

$L.H.S = R.H.S$

A4: By inspection the zero object is $\vec{o} = (0, -1)$. This is because

$(x_1, y_1) + (0, -1) = (x_1 + 0, y_1 - 1 + 1) = (x_1, y_1)$

A5: By inspection $(-x_1, -y_1 - 2)$ is the negative of (x_1, y_1). This is because

$(x_1, y_1) + (-x_1, -y_1 - 2) = (0, -1) = \vec{o}$

S1 $K(x, y) = (Kx, Ky + K - 1)$ obviously $\in V$ for all $K \in \mathbb{R}$

S2 $L.H.S. = K\{(x_1, y_1) + (x_2, y_2)\} = K(x_1 + x_2, y_1 + y_2 + 1)$

$= (K(x_1 + x_2), K(y_1 + y_2 + 1) + K - 1)$

$R.H.S.$ $K(x_1, y_1) + K(x_2, y_2) = (Kx_1, Ky_1 + K - 1) + (Kx_2, Ky_2 + K - 1)$

$= (Kx_1 + Kx_2, Ky_1 + K - 1 + Ky_2 + K - 1 + 1)$

$= (K(x_1 + x_2), K(y_1 + y_2 + 1) + K - 1)$

$L.H.S = R.H.S$

S3 $L.H.S = (K + \ell)(x_1, y_1) = ((K + \ell)x_1, (K + \ell)y_1 + (K + \ell) - 1) = (Kx_1 + \ell x_1, Ky_1 + \ell y_1 + K + \ell - 1)$

$R.H.S = K(x_1, y_1) + \ell(x_1, y_1) = (Kx_1, Ky_1 + K - 1) + (\ell x_1, \ell y_1 + \ell - 1)$

$= (Kx_1 + \ell x_1, Ky_1 + \ell y_1 + K + \ell - 1)$ $\therefore L.H.S = R.H.S$

$\underline{S4}$: L.H.S $(K\ell)(x_1,y_1) = (K\ell x_1, K\ell y_1 + K\ell - 1)$

$$\begin{aligned} R.H.S &= K(\ell(x_1,y_1)) = K((\ell x_1, \ell y_1 + \ell - 1) \\ &= (K(\ell x_1), K(\ell y_1 + \ell - 1) + K - 1) \\ &= (K\ell x_1, K\ell y_1 + K\ell - 1) \end{aligned}$$

$\therefore$ L.H.S $=$ R.H.S

$\underline{S5}$ L.H.S $1(x_1,y_1) = (1x_1, 1y_1 + 1 - 1) = (x_1, y_1)$

R.H.S $= (x_1, y_1)$

$\therefore$ L.H.S $=$ R.H.S

It follows that V is a vector space under the stated operations.

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(2) Since addition on V is the <u>standard</u> matrix addition, we know that axioms $A1 \rightarrow A5$ are all satisfied.

however let us find out which of the remaining axioms $S1 \rightarrow S5$ fail to hold!

For $\underline{S1}$: By defn. $KA = KA^t$, $K \in \mathbb{R}$

clearly $KA^t \in V$ .. hence $KA \in V \Rightarrow S1$ is true!

For $\underline{S2}$: let $A, B \in V$, $K \in \mathbb{R}$

L.H.S $= K(A+B) = K(A+B)^t = K(A^t + B^t) = KA^t + KB^t$

R.H.S. $= KA + KB = KA^t + KB^t$

L.H.S $=$ R.H.S $\Rightarrow S2$ is true!

For $\underline{S3}$: Let $K, \ell \in \mathbb{R}$, $A \in V$

L.H.S $= (K+\ell)A = (K+\ell)A^t$

R.H.S $= KA + \ell A = KA^t + \ell A^t = (K+\ell)A^t$

$\therefore$ L.H.S $=$ R.H.S $\Rightarrow S3$ is true!

For $\underline{S4}$ L.H.S $= (K\ell)A = K\ell A^t$

R.H.S $= K(\ell A) = K(\ell A^t) = (K\ell)(A^t)^t = K\ell A$

$\therefore$ L.H.S $\neq$ R.H.S for all $A \in V$

For $\underline{S5}$: $L.H.S = 1A = 1A^t = A^t$

$R.H.S = A$

$\therefore L.H.S \neq R.H.S$

$\therefore$ V is not a vector space under stated operations. The only axioms that failed are S4, and S5.

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(3) (a) $K\vec{u} = l\vec{u} \Rightarrow (K-l)\vec{u} = \vec{0}$

$\Rightarrow$ Either $K - l = 0$ or $\vec{u} = \vec{0}$

Since $\vec{u} \neq \vec{0}$, we must have $K - l = 0$ or $K = l$.

(b) $K\bar{u} = K\vec{v} \Rightarrow K(\bar{u} - \bar{v}) = \vec{0}$

But $K \neq 0$, hence $\bar{u} - \bar{v} = 0$ or $\bar{u} = \vec{v}$.

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(4) Indeed $U = \{p(x) \mid p(2) = 0\}$. Clearly U is non-empty.

Since $p = x - 2 \in U$.

Now, let $p(x), q(x) \in U$ .. hence $p(2) = 0$, $q(2) = 0$

Now, $(p+q)(2) = p(2) + q(2) = 0 + 0 = 0 \Rightarrow p+q \in U$

$\Rightarrow$ A1 is true

Next, let $K \in \mathbb{R}$, $p \in U$ .. hence $p(2) = 0$

Now $(Kp)(2) = Kp(2) = K \cdot 0 = 0 \Rightarrow Kp \in U$

$\Rightarrow$ S1 is true

$\therefore U$ is a subspace of P_2

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(5) $W = \{A \mid \det A \neq 0\}$ is not a subspace of M_{22}.

Indeed $\det 0 = 0 \Rightarrow$ The zero matrix $\notin W$.

$\underline{OR}$: we shall show by an example that W is not closed under addition! let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ .. clearly $A, B \in W$ since $\det A \neq 0$, $\det B \neq 0$. Now $A + B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, $\det(A+B) = 0$

$\Rightarrow A + B \notin W$.

(6) (a) Let $(x,y,z) \in \mathbb{R}^3$. If S spans $\mathbb{R}^3$, then

$$(x,y,z) = \text{l.c of } (2,1,-3), (4,1,2), (8,-1,8)$$
$$= c_1(2,1,-3) + c_2(4,1,2) + c_3(8,-1,8)$$

$$\Rightarrow \quad \begin{aligned} 2c_1 + 4c_2 + 8c_3 &= x \\ c_1 + c_2 - c_3 &= y \\ -3c_1 + 2c_2 + 8c_3 &= z \end{aligned} \quad \Rightarrow \quad A\vec{x} = \vec{b}$$

where $A = \begin{bmatrix} 2 & 4 & 8 \\ 1 & 1 & -1 \\ -3 & 2 & 8 \end{bmatrix}$, $\det A = \begin{vmatrix} 2 & 4 & 8 \\ 1 & 1 & -1 \\ -3 & 2 & 8 \end{vmatrix} \xrightarrow[C_2 \to C_2 + C_3]{C_1 \to C_1 + C_3}$

$$= \begin{vmatrix} 10 & 12 & 8 \\ 0 & 0 & -1 \\ 5 & 10 & 8 \end{vmatrix} = -(-1)\begin{vmatrix} 10 & 12 \\ 5 & 10 \end{vmatrix} \neq 0$$

$\Rightarrow$ system $A\bar{x} = \vec{b}$ has a solution for every column vector $\vec{b}$. Hence S spans $\mathbb{R}^3$ that is $\mathbb{R}^3 = \text{span}(S)$.

(b) Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22}$ be an arbitrary matrix. If S spans M_{22}

then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = c_1\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} + c_3\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} + c_4\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

$$\Rightarrow \quad \begin{aligned} c_1 - c_2 + c_3 + c_4 &= a \\ c_1 - c_2 \qquad\qquad &= b \\ c_3 + c_4 &= c \\ c_2 - c_3 \qquad &= d \end{aligned} \quad \Rightarrow \quad A\vec{x} = \vec{b}$$

where $A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$, $\det A = \begin{vmatrix} 1 & -1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \end{vmatrix} \xleftrightarrow{R_2 \to R_2 - R_1}$

$$= \begin{vmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & -1 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix} = 0$$

$\Rightarrow A\vec{x} = \vec{b}$ may not be consistent for some column vectors $\vec{b}$.

Hence S is <u>not</u> a spanning set for M_{22}

7. $x_1 + x_2 + Kx_3 = 0$

$x_1 + Kx_2 + x_3 = 0$

$Kx_1 + x_2 + x_3 = 0$

The idea is to bring Augmented matrix to row-Echelon form or to a form as close as possible avoiding division by quantities involving the unknown K.

Now, $[A \mid B] = \left[\begin{array}{ccc|c} 1 & 1 & K & 0 \\ 1 & K & 1 & 0 \\ K & 1 & 1 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - KR_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & K & 0 \\ 0 & K-1 & 1-K & 0 \\ 0 & 1-K & 1-K^2 & 0 \end{array}\right]$

$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & K & 0 \\ 0 & \boxed{K-1} & 1-K & 0 \\ 0 & 0 & \boxed{2-K-K^2} & 0 \end{array}\right]$ - - - - (*)

Note: Significant values of K are determined by

$K - 1 = 0 \Rightarrow K = 1$

$2 - K - K^2 = 0 \Rightarrow K^2 + K - 2 = 0 \Rightarrow K = 1$ or -2

Therefore, three Cases will be Examined: $K = 1, -2$, $K \neq 1, -2$

Case (1): If $K = 1$, (*) $\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$, $n - r = 3 - 1 = 2$-parameter family

$\therefore$ subspace is a plane through origin in $\mathbb{R}^3$ Since its dim. is 2

Case (2): If $K = -2$, (*) $\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \to -\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

So $n - r = 3 - 2 = 1$-parameter family.

$\therefore$ subspace is a st. line through origin in $\mathbb{R}^3$ Since its dim. is 1

Case (3): If $K \neq 1, -2$ say $K = 2$, (*) $\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right]$

$\xrightarrow{R_3 \to -\frac{1}{4}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow$ system has only the Trivial solution $(x, y, z) = (0, 0, 0)$. Hence Subspace is the origin $\mathbb{R}^3$.

Note: No value of K in which subspace is all of $\mathbb{R}^3$

(8) Let c_1, c_2, and c_3 be scalars such that

$$p = c_1 P_1 + c_2 P_2 + c_3 P_3$$

$$2 - x + x^2 = c_1(1+x) + c_2(1+x^2) + c_3(x+x^2)$$

$$\Rightarrow \begin{aligned} c_1 + c_2 &= 2 \\ c_1 + c_3 &= -1 \\ c_2 + c_3 &= 1 \end{aligned}$$

solving we get, $c_1 = 0$, $c_2 = 2$, $c_3 = -1$

$\therefore$ The coordinate vector of P relative to B is

$$(P)_B = (c_1, c_2, c_3) = (0, 2, -1)$$

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(9) $W = \{(a,b,c,d) \mid d = a+b,\ c = a-b\}$

$= \{(a, b, a-b, a+b) \mid a, b \in \mathbb{R}\}$

Now let $\vec{w} \in W$. Therefore

$\vec{w} = (a, b, a-b, a+b) = (a, 0, a, a) + (0, b, -b, b)$

$= a(1,0,1,1) + b(0,1,-1,1)$

$\therefore \vec{w} = $ l.c. of $(1,0,1,1), (0,1,-1,1)$

$\therefore W = \text{span}\{(1,0,1,1), (0,1,-1,1)\}$ --- (1)

clearly the set $\{(1,0,1,1), (0,1,-1,1)\}$ is independent --- (2)

Since neither vector is a scalar multiple of the other

(1), (2) $\Rightarrow$ The set $B = \{(1,0,1,1), (0,1,-1,1)\}$

is a basis for W and hence $\dim(W) = 2$.

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(10) Let c_1, c_2 be scalars such that

$$c_1(\bar{u} - \bar{v}) + c_2(\bar{u} + \bar{v}) = \vec{0} \quad --- (*)$$

$$\Rightarrow (c_1 + c_2)\bar{u} + (-c_1 + c_2)\bar{v} = \vec{0}$$

But $S = \{\bar{u}, \bar{v}\}$ is indep. $\Rightarrow$ $\begin{aligned} c_1 + c_2 &= 0, \\ -c_1 + c_2 &= 0 \end{aligned}$

solve: $c_1 = c_2 = 0$ $\Rightarrow$ $(*)$ has only the Trivial Solution $\therefore$ hence $S' = \{\bar{u} - \bar{v}, \bar{u} + \bar{v}\}$ is indep. #

MATH 311

Support Material

SOLUTIONS TO

Assignment (II)

(1) (a) $$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 + R_1}]{R_2 \to R_2 - 2R_1}$$

$$\begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_2]{R_1 \to R_1 - 4R_2} \begin{bmatrix} 1 & -3 & 0 & -14 & 13 & 28 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[R_2 \to R_2 + 2R_3]{R_1 \to R_1 - 13R_3} \begin{bmatrix} 1 & -3 & 0 & -14 & 0 & -37 \\ 0 & 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

- The non-zero rows of R form a basis for row(A).

$\therefore$ A basis for row$(A) = \{(1,-3,0,-14,0,-37), (0,0,1,3,0,4), (0,0,0,0,1,5)\}$

Since 1st., 3rd. and 5th. Columns of R contain leading ones, the Corresponding Columns of A form a basis for Col(A)

$\therefore$ A basis for Col$(A) = \{(1,2,2,-1), (4,9,9,-4), (5,8,9,-5)\}$

Finally: Consider the system $A\vec{x} = \vec{0}$. The system reduces to

$$\begin{aligned} x_1 - 3x_2 - 14x_4 - 37x_6 &= 0 \Rightarrow x_1 = 3x_2 + 14x_4 + 37x_6 \\ x_3 + 3x_4 + 4x_6 &= 0 \Rightarrow x_3 = -3x_4 - 4x_6 \\ x_5 + 5x_6 &= 0 \Rightarrow x_5 = -5x_6 \end{aligned}$$

here $n - r = 6 - 3 = 3$ - parameter family where the parameters are $x_2 = t$, $x_4 = p$, and $x_6 = q$

$$\therefore (x_1, x_2, x_3, x_4, x_5, x_6) = (3t + 14p + 37q, t, -3p - 4q, p, -5q, q)$$

$$= t(3,1,0,0,0,0) + p(14,0,-3,1,0,0) + q(37,0,-4,0,-5,1)$$

$\therefore$ A basis for null(A) is $\{(3,1,0,0,0,0), (14,0,-3,1,0,0), (37,0,-4,0,-5,1)\}$

(b) $\operatorname{rank}(A) = \#$ of leading ones of $R = 3$

$\operatorname{nullity}(A) = \#$ of columns of $A - \operatorname{rank}(A)$

$= 6 - 3 = 3$

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(2) The problem is equivalent to finding a basis for the

(a) $\operatorname{col}(A)$ where A is the matrix whose column vectors are the vectors in S: $\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4$, and $\bar{v}_5$.

$$\therefore A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix} \xrightarrow[R_4 \to R_4 - 3R_1]{R_2 \to R_2 + 2R_1} \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & -3 & 3 & 4 & 1 \\ 0 & 0 & 0 & -13 & -13 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 - 3R_1]{R_1 \to R_1 + 2R_2} \begin{bmatrix} 1 & 0 & 2 & 8 & 9 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & 0 & 0 & -5 & -5 \\ 0 & 0 & 0 & -13 & -13 \end{bmatrix} \xrightarrow[R_3 \to \frac{R_3}{-5}]{R_2 \to -R_2}$$

$$\begin{bmatrix} 1 & 0 & 2 & 8 & 9 \\ 0 & 1 & -1 & -3 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -13 & -13 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 + 3R_3 \\ R_4 \to R_4 + 13R_3}]{R_1 \to R_1 - 8R_3} \begin{array}{c} \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} \bar{w}_1 & \bar{w}_2 & \bar{w}_3 & \bar{w}_4 & \bar{w}_5 \end{array} \end{array} = R$$

$\therefore$ Basis for W = basis for $\operatorname{col}(A)$

$= \{(1,-2,0,3), (2,-5,-3,6), (2,-1,4,-7)\}$

(b) $\dim W = 3$

(c) By inspection $\bar{w}_3 = 2\bar{w}_1 - 1 \cdot \bar{w}_2$, These are dependency

$\bar{w}_5 = \bar{w}_1 + \bar{w}_2 + \bar{w}_5$ equations.

$\therefore$ The v_i's satisfy the same equations and we get,

$\bar{v}_3 = 2\bar{v}_1 - \bar{v}_2$ or $(0,1,3,0) = 2(1,-2,0,3) - (2,-5,-3,6)$

$\bar{v}_5 = \bar{v}_1 + \bar{v}_2 + \bar{v}_3$ or $(5,-8,1,2) = (1,-2,0,3) + (2,-5,-3,6) + (2,-1,4,-$

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(3) Know $\text{rank}(A) = 4$

$$\dim[\text{row}(A)] = \dim[\text{Col}(A)] = \text{rank}(A) = 4$$

$$\dim[\text{null}(A)] = \#\text{ of columns of } A - \text{rank}(A)$$
$$= 10 - 4 = 6$$

Likewise, $\dim[\text{null}(A^t)] = \#\text{ of columns of } A^t - \text{rank}(A^t)$
$$= 7 - 4 \text{ (as well)}$$
$$= 3$$

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(4) $A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 3 & 2 & 1 & 1 \\ 1 & 0 & 1 & 3 \end{bmatrix}$

First find U from $[A \mid I_3] \longrightarrow [R \mid U]$

$$[A \mid I_3] = \left[\begin{array}{cccc|ccc} 1 & 1 & 0 & -1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 3R_1}$$

$$\left[\begin{array}{cccc|ccc} 1 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 4 & -3 & 1 & 0 \\ 0 & -1 & 1 & 4 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to -R_2} \left[\begin{array}{cccc|ccc} 1 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 3 & -1 & 0 \\ 0 & -1 & 1 & 4 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 + R_2]{R_1 \to R_1 - R_2} \left[\begin{array}{cccc|ccc} 1 & 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 1 & -1 & -4 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & 1 \end{array}\right] = [R \mid U]$$

$$\therefore U = \begin{bmatrix} -2 & 1 & 0 \\ 3 & -1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

Next, Find V from $[R^t \mid I_4] = \left[\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \mid V^t \right]$

$$[R^t \mid I_4] = \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 3 & -4 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_4 \to R_4 - 3R_1]{R_3 \to R_3 - R_1}$$

$$\left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -4 & 0 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_4 \to R_4 + 4R_2]{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 4 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} \mid V^t \right] \Rightarrow V^t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -3 & 4 & 0 & 1 \end{bmatrix} \Rightarrow V = \begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and $\text{rank}(A) = 2$

(5) Let $\bar{u} = (u_1, u_2, u_3)$, $\bar{v} = (v_1, v_2, v_3)$, and $\bar{w} = (w_1, w_2, w_3)$ be vectors in $\mathbb{R}^3$ and let $K \in \mathbb{R}$ be a scalar.

<u>axiom P1</u>: Is $< \bar{u}, \bar{v} > = < \bar{v}, \bar{u} >$?

$$L.H.S = <\bar{u}, \bar{v}> = u_1 v_1 + u_3 v_3 \quad \Rightarrow \quad L.H.S = R.H.S$$
$$R.H.S = <\bar{v}, \bar{u}> = v_1 u_1 + v_3 u_3$$

$\therefore$ Axiom P1 is true!

<u>axiom P2</u>: Is $<\bar{u}+\bar{v}, \vec{w}> = <\bar{u}, \bar{w}> + <\bar{v}, \bar{w}>$?

$$L.H.S \; <\bar{u}+\bar{v}, \bar{w}> = < (u_1+v_1, u_2+v_2, u_3+v_3), (w_1, w_2, w_3)>$$
$$= (u_1+v_1) w_1 + (u_3+v_3) w_3$$
$$= (u_1 w_1 + u_3 w_3) + (v_1 w_1 + v_3 w_3)$$
$$= <\bar{u}, \bar{w}> + <\bar{v}, \bar{w}> = R.H.S$$

$\therefore$ axiom P2 is true!

<u>axiom P3</u> Is $<K\bar{u}, \bar{v}> = K < \bar{u}, \bar{v}>$?

$$L.H.S \; <K\bar{u}, \bar{v}> = < (Ku_1, Ku_2, Ku_3), (v_1, v_2, v_3)>$$
$$= K u_1 v_1 + K u_3 v_3$$
$$= K(u_1 v_1 + u_3 v_3) = K <\bar{u}, \bar{v}>$$
$$= R.H.S$$

$\therefore$ axiom P3 is true!

<u>axiom P4</u>: Is $<\bar{u}, \bar{u}> \geq 0$ and $<\bar{u}, \bar{u}> = 0 \Rightarrow \bar{u} = \vec{0}$?

No!, an example

let $\bar{u} = (0,1,0)$ .. clearly $\vec{u} \neq \vec{0}$

However,
$$<\bar{u}, \bar{u}> = <(0,1,0), (0,1,0)>$$
$$= (0)(0) + (0)(0) = 0$$

$\therefore <\bar{u}, \bar{v}> = u_1 v_1 + u_3 v_3$ is not an i.p on $\mathbb{R}^3$

(6) $A - B = \begin{bmatrix} 2 & 6 \\ 9 & 4 \end{bmatrix} - \begin{bmatrix} -4 & 7 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ 8 & -2 \end{bmatrix}$

$\therefore\ d(A,B) = \| A - B \| = \sqrt{\langle A-B, A-B \rangle}$

$= \sqrt{(6)^2 + (-1)^2 + (8)^2 + (-2)^2} = \sqrt{105}$

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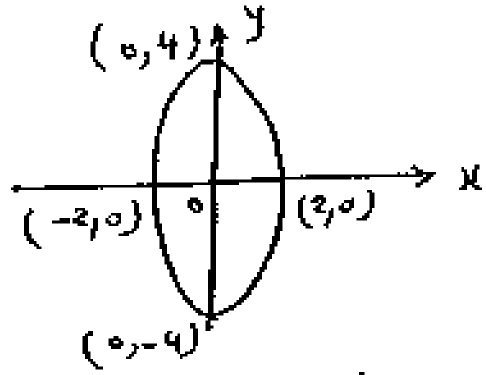
(7) let $\bar{u} = (x,y)$

$\|\bar{u}\|^2 = \langle \bar{u}, \bar{u} \rangle = \langle (x,y), (x,y) \rangle = \frac{1}{4}x^2 + \frac{1}{16}y^2$

The unit circle in $\mathbb{R}^2$ is given by $\|\bar{u}\|^2 = 1$

$\Rightarrow\ \frac{1}{4}x^2 + \frac{1}{16}y^2 = 1$

which is an equation of an ellipse centred at the origin and of semi-axis 2, 4 units



(8) $\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\,dx$ .. In particular if $q = p$,

$\langle p, p \rangle = \int_{-1}^{1} p^2(x)\,dx$

Now, $\langle 1, 1 \rangle = \int_{-1}^{1} 1^2\,dx = x\Big|_{-1}^{1} = 2 \Rightarrow \|1\|^2 = 2 \Rightarrow \|1\| = \sqrt{2}$

Next, $\langle x^2, x^2 \rangle = \int_{-1}^{1} (x^2)^2\,dx = \int_{-1}^{1} x^4\,dx = \frac{2}{5}$

$\Rightarrow\ \|x^2\|^2 = \frac{2}{5} \Rightarrow \|x^2\| = \sqrt{\frac{2}{5}}$.

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(9) $W = \text{span}\{(1,0,-2,3), (1,-1,2,0)\}$.

First observe that if A is the matrix whose row vectors are the vectors in the spanning set, then $W^{\perp}$ is null(A)!

$\therefore$ To Find a basis for $W^{\perp}$ is the same as finding a basis for null(A)!

Now, let $A = \begin{bmatrix} 1 & 0 & -2 & 3 \\ 1 & -1 & 2 & 0 \end{bmatrix}$

solve system: $A\vec{x} = \vec{0}$:

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & 0 \\ 1 & -1 & 2 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & 0 \\ 0 & -1 & 4 & -3 & 0 \end{array}\right] \xrightarrow{R_2 \to -R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & -4 & 3 & 0 \end{array}\right] \qquad n - r = 4 - 2 = 2 \text{ - parameter family}$$

let $z = t$, $w = s$

$\therefore$ $x - 2z + 3w = 0$, $\quad x = 2t - 3s$

$y - 4z + 3w = 0$, $\quad y = 4t - 3s$

$\therefore$ $(x, y, z, w) = (2t - 3s, 4t - 3s, t, s)$

$= t(2, 4, 1, 0) + s(-3, -3, 0, 1)$

$\therefore$ A basis for $W^{\perp}$ is $\{(2,4,1,0), (-3,-3,0,1)\}$ and

$\dim W^{\perp} = 2$.

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(10) $B = \{\bar{u}_1 = (1,1,1), (1,1,0), \bar{u}_3 = (1,0,0)\}$

1st. Construct an orthogonal basis $B' = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ as follows

1 Take $\bar{v}_1 = \bar{u}_1 = (1,1,1)$

2 $\bar{v}_2 = \bar{u}_2 - \dfrac{\langle \bar{u}_2, \bar{v}_1 \rangle}{\langle \bar{v}_1, \bar{v}_1 \rangle} \bar{v}_1 = (1,1,0) - \dfrac{1+2+0}{1+2+3}(1,1,1)$

$= (1,1,0) - \frac{1}{2}(1,1,1)$

$= \frac{2}{2}(1,1,0) - \frac{1}{2}(1,1,1)$

$= \frac{1}{2}[2(1,1,0) - (1,1,1)] = \cancel{\frac{1}{2}}(1,1,-1)$

dropped!

$\therefore$ Take $\bar{v}_2 = (1,1,-1)$

3 $\bar{v}_3 = \bar{u}_3 - \dfrac{\langle \bar{u}_3, \bar{v}_1 \rangle}{\langle \bar{v}_1, \bar{v}_1 \rangle} \bar{v}_1 - \dfrac{\langle \bar{u}_3, \bar{v}_2 \rangle}{\langle \bar{v}_2, \bar{v}_2 \rangle} \bar{v}_2$

$= (1,0,0) - \dfrac{1+0+0}{1+2+3}(1,1,1) - \dfrac{1+0+0}{1+2+3}(1,1,-1)$

$= (1,0,0) - \frac{1}{6}(1,1,1) - \frac{1}{6}(1,1,-1)$

$$\therefore \quad \bar{v}_3 = \frac{6}{6}(1,0,0) - \frac{1}{6}(1,1,1) - \frac{1}{6}(1,1,-1)$$

$$= \frac{1}{6}\left[6(1,0,0) - (1,1,1) - (1,1,-1)\right] = \frac{1}{6}(4,-2,0) = \frac{2}{6}(2,-1,0)$$

Take $\bar{v}_3 = (2,-1,0)$.

$\therefore$ An orthogonal basis is $B' = \{(1,1,1), (1,1,-1), (2,-1,0)\}$

<u>Next</u>: Construct an orthonormal basis $B'' = \{\bar{w}_1, \bar{w}_2, \bar{w}_3\}$ as follows:

$\bar{v}_1 = (1,1,1) \Rightarrow \|\bar{v}_1\|^2 = 1+2+3 = 6 \Rightarrow \|\bar{v}_1\| = \sqrt{6}$

$\bar{v}_2 = (1,1,-1) \Rightarrow \|\bar{v}_2\|^2 = 1+2+3 = 6 \Rightarrow \|\bar{v}_2\| = \sqrt{6}$

$\bar{v}_3 = (2,-1,0) \Rightarrow \|\bar{v}_3\|^2 = 4+2+0 = 6 \Rightarrow \|\bar{v}_3\| = \sqrt{6}$

$$\therefore \quad \bar{w}_i = \frac{\bar{v}_i}{\|v_i\|} \qquad i = 1,2,3$$

$$\therefore \quad B'' = \left\{ \frac{1}{\sqrt{6}}(1,1,1),\ \frac{1}{\sqrt{6}}(1,1,-1),\ \frac{1}{\sqrt{6}}(2,-1,0) \right\}$$

$$\text{or} \quad \left\{ \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right), \left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0\right) \right\}$$

END