(A)

MATH 311
ESSENTIALS

[1] Introduction to Sets

<u>Defn.(1)</u>: A set:

A Collection of distinct objects is called a set. These objects are referred to as members or elements of the set.

Sets are usually denoted by upper Case letters such as $A, B, X, \ldots$ However elements of a set are usually denoted by lower case letters such as $x, y, a, b, t, \ldots$

<u>Notations</u>: let A be a set. If "x" is an element of A, we write $x \in A$ " x belong to A ". If "x" is not a member of A, we write $x \notin A$ which reads: x does not belong to A.

<u>Description of a set</u>: There are two ways to discribe a set.

(i) <u>The Roster Method</u>

In this method we list all members of the set between a pair of braces $\{ , \}$ and separate members by Commas.

(ii) <u>The Set-builder Method</u>:

let "A" be a set and "x" be a member of A. Then Certainly "x" satisfies a certain property or membership rule say P. The set "A" may be discribed by

$$A = \{x \mid x \text{ satisfies } P\}$$

which reads: "A" is the set of all elements "x" Such that "x" satisfies P. This method of discription is called "The set-builder Method".

EX1: Let K be the set of all integers between 2, and 11 that are divisible by 3.

Indeed, elements of K are 3, 6, and 9. Hence using roster method we write $K = \{3, 6, 9\}$.

EX2: Let H be the set of all letters needed to write the name anna. Clearly $H = \{a, n\}$

EX3: Let Y be the set of all numbers between 0, and 1 excluding both. Clearly it is not at all possible to list all members of Y. Hence roster method can not be used! However using set-builder method, we describe the set Y by,

$$Y = \{t \mid 0 < t < 1\}$$

The three set Rules:

1. Order of elements in a set is unimportant. Hence the sets $\{1,2,3\}$, $\{3,1,2\}$, $\{1,3,2\}$ are the same!
2. All members of a set must be distinct.
3. The set must be well defined, that is to say: you must be able to tell whether a given object is a member or not a member of a given set (There is no maybe!!)

For Example: The collection of all "Nice" people in Calgary is not a set, since there is no criteria to define what a nice person is!!

In other words being nice or not is a matter of Taste!

EX: The set $M = \{x \mid x^2+1=0, x \in \mathbb{R}\}$ has no elements!

This is because the equation $x^2+1=0$ has no real solutions. It follows that "M" is an Empty set!

Two special sets

(1) The Empty set: A set containing no elements is called "The empty set" and may be denoted by either ϕ "phi" or $\{\ \ \}$.

(2) The Universal set:

The set containing all possible elements under certain study or discussion is called "The universal set" and may be denoted by U, or X.

EX: The set $\{x \mid x > x\} = \phi$!!

EX: The following are potential universal sets

(i) The set of all English alphbet: $\{a, b, c, \dots, x, y, z\}$.

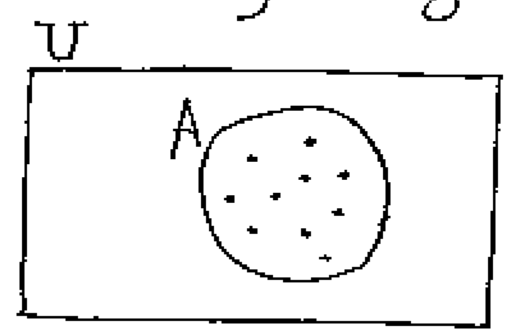
(ii) The set of all breed of dogs

(iii) The set of all students in post-secondary institutions in Alberta.

Venn-Diagram of a set:

let A be a set. The Elements of "A" may be represented by points (or dots) enclosed by an arbitrary closed figure (such as a circle). This geometric representation is known as the Venn-diagram of the set A. The universal set is usually represented by a large rectangle.

The figure shown is the Venn-diagram of the set A.



Equality of sets:

Two sets A, and B are said to be equal and we write $A = B$ if A and B have exactly same Elements.

EX: let $A = \{2, -1\}$, $B = \{t \mid t^2 - t - 2 = 0\}$.

obviously $A = B$ .. Verify!

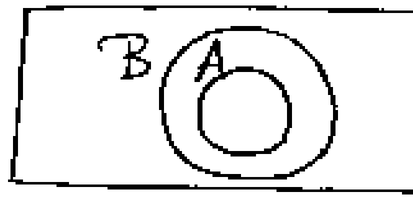
A subset:

let A, and B be sets.

The set "A" is a subset of B if Every element in "A" is also an element of B, and we write $A \subseteq B$ - reads A is contained in B.

Note that: Every set B has two obvious subsets: ϕ, and B. These subsets of B are referred to as "The Improper subsets of B". Any other subset "A" of B is called a proper subset and we write $A \subset B$ .. reads: "A" is strictly contained in B ...

The Venn-diagram of A lies entirely within the Venn-diagram of B as shown in figure.



EX: The set of all vowels $\{a, e, i, o, u\}$ is a subset of $\{a, b, c, \ldots, x, y, z\}$.

EX: $A = \{1, 3\} \subset \{x \mid 0 < x < 5\} = B$

<u>Theorem</u>: <u>Equality of sets</u>:

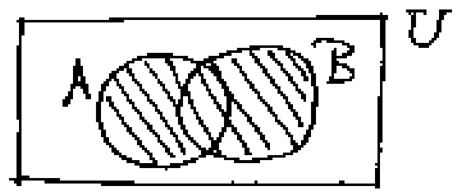
If A, and B are sets such that $A \subseteq B$, and $B \subseteq A$, then $A = B$.

<u>Operations on Sets</u>: Union, Intersection, and Complement

(i) <u>Union of sets</u>:

The Union of two sets A, and B is the set that consists of all elements that belong to either A or B. The union will be denoted by $A \cup B$: reads: "A" union "B". It follows that

$$A \cup B = \{x \mid \text{either } x \in A \text{ or } x \in B\}$$



$A \cup B$ \\\\

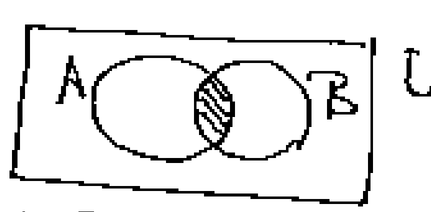
<u>Note</u>: It is obvious that $A \cup B$ is the same as $B \cup A$!

(ii) <u>Intersection of sets</u>

The intersection of two sets A, and B is the set that consists of all elements that belong to both A, and B. The intersection will be denoted by $A \cap B$: reads: "A" Intersection "B". It follows that

$$A \cap B = \{x \mid x \in A \text{ } \underline{\text{and}} \text{ } x \in B\}$$

<u>Note</u>: $A \cap B = B \cap A$!



$A \cap B$ \\\\
(Common part)

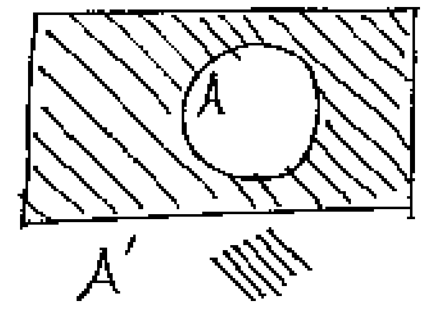
(iii) <u>Complement of a set</u>

Let "A" be a set and X be the universal set. The Complement of "A" is the set of all elements of X that do <u>not</u> belong to set "A".

The Complement of "A" may be denoted by

Either A' : reads A-prime.

or A^c : reads A-Complement.



It follows that $A' = \{x \in X \mid x \notin A\}$

<u>EX</u>: Let A be the set of all Female students in the MATH 311 Class. The universal
Then A' is the set of all male students in the Class.
set X is assumed to be the set of <u>all</u> students in MATH 311 Class.

<u>Defn.</u>: <u>Binary Operation on a set</u>

Let B be a set (non-empty). A binary operation say "$*$" defined on the set "B" is a rule that associates with each pair x, y of elements of B, a unique element z and we write

$$x * y = z$$

(You may think of the binary operation "$*$" as a chemical reaction between x, and y that produces z!).

observe that "z" may or may not belong to the set B!

<u>Definition</u>: Let B be a non-empty set and let "$*$" be a binary operation defined on B.

(i) We say, The set B is closed under the operation "$*$" if

$$x * y = z \in B \quad \text{for } \underline{\text{all}} \ x, \text{ and } y \in B$$

(ii) We say operation "$*$" is Commutative if $x * y = y * x$ for <u>all</u> $x, y \in B$.

(iii) We say operation "$*$" is associative if

$$(u * v) * w = u * (v * w) \quad \text{for all } u, v, w \in B.$$

Remark:

To establish the "Truth" of a statement, you must provide a full proof! However to establish that a statement is not always true (and hence false!), it suffices to provide one Counter Example!

Ex: let N be the set of all natural numbers and let "$\div$" be the ordinary operation of Division. Is N closed under "$\div$"?

Solution: No! N is not closed under "$\div$"

Indeed $2, 4 \in N$. However $2 \div 4 = \frac{1}{2} \notin N$!!

Ex: let $\mathbb{R}$ be the set of all real numbers and let "$-$" be the ordinary operation of subtraction. Is the operation "$-$" Commutative?

Solution: No! Since $3, 5 \in \mathbb{R}$, yet $3-5 \neq 5-3$!!

Ex: let $\mathbb{R}$ be the set of all real numbers. Define a binary operation "$\circ$" on $\mathbb{R}$ as follows:

$$x \circ y = \frac{x+y}{2} \qquad \text{for all } x, y \in \mathbb{R}$$

show that operation "$\circ$" is not associative!

Solution: we need to show $(x \circ y) \circ z$ need not equal to $x \circ (y \circ z)$!

pick $x = 2$, $y = 4$, and $z = 5$

Now $(x \circ y) \circ z = \underbrace{(2 \circ 4)}_{\text{Do 1st.}} \circ 5 = \left(\frac{2+4}{2}\right) \circ 5 = 3 \circ 5 = \frac{3+5}{2} = 4$

on the other hand,

$x \circ (y \circ z) = 2 \circ \underbrace{(4 \circ 5)}_{\text{Do 1st.}} = 2 \circ \left(\frac{4+5}{2}\right) = 2 \circ 4.5 = \frac{2+4.5}{2} = 3.25$

clearly $x \circ (y \circ z) \neq (x \circ y) \circ z$ $\therefore$ Verified!

Logical Symbols

Let p and q be statements. These statements may be connected in several ways. Let us Examine two:

(i) The If – Then Connection

If P, then q means:

The truth of "P" implies the truth of "q".

Symbolically we write

$$p \Rightarrow q \qquad \text{reads: } p \text{ implies } q.$$

or

$$q \Leftarrow p \qquad \text{reads: } q \text{ is implied by } p.$$

(ii) The If AND ONLY If Connection

p if and only if q means:

The truth of "p" implies the truth of "q" and vice-versa!

Symbolically we write

$$p \Rightarrow q \quad \underline{\text{and}} \quad q \Rightarrow p$$

(or $\quad p \Longleftrightarrow q$).

we also say: p and q are equivalent statements!

Equivalent statement: (Three or more):

Three statements p, q, and r are said to be equivalent if any statement implies the others. It follows that p, q, and r are equivalent if

$$p \Rightarrow q, \ \underline{q \Rightarrow p}, \ q \Rightarrow r, \ \underline{r \Rightarrow q}, \ r \Rightarrow p, \text{ and } \underline{p \Rightarrow r}$$

The reader may easily verify that: In order to prove that p, q, r are equivalent it suffices to only show:

$$p \Rightarrow q, \ q \Rightarrow r, \text{ and } r \Rightarrow p$$

The remaining three (underlined above) automatically follow!

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