MATH 211/221

Support Material (1)

Matrices and Vectors

A matrix is a rectangular array of numbers. These numbers are called "Entries" of the matrix. If a matrix has m rows and n columns, one says that the matrix is of order $m \times n$ (m by n).

In particular: (1) If a matrix has n rows as well as n columns the matrix is said to be a square matrix of order n.

(2) If a matrix has one row and n columns, one says it is a row-matrix of order n.

(3) If a matrix has one column and m-rows, the matrix is said to be a column matrix of order m.

(4) A matrix of order 1×1, say $[b]$ may be identified by "b" itself and we write $[b] = b$.

Notations: Matrices are usually denoted by upper case letters such as $A, B, C, \ldots$ etc. However entries of a matrix are usually denoted by lower case letters. In particular if B is a matrix of order $m \times n$, then the entry of B which lies in the ith row and jth column may be denoted by b_{ij} $i = 1, 2, \ldots m$, $j = 1, 2, \ldots n$. Hence

$$B = \begin{bmatrix} b_{11} & b_{12} \cdots & b_{1n} \\ b_{21} & b_{22} \cdots & b_{2n} \\ \vdots & & \\ b_{m1} & b_{m2} \cdots & b_{mn} \end{bmatrix} \quad \text{or} \quad B = (b_{ij}) \quad \begin{matrix} i = 1, 2, \ldots m \\ j = 1, 2, \ldots n \end{matrix}$$

An Exception: Row and Column matrices are usually denoted by lower case letters with an arrow on top!! such as $\vec{x}, \vec{y}, \vec{r}, \vec{a}, \ldots$ etc. and may be called Vectors!

Examples: $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 7 & -1 \end{bmatrix}$ is a matrix of order 2×3 (2 by 3)

$C = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 7 & 3 \\ 1 & 1 & -7 \end{bmatrix}$ is a square matrix of order 3

$\vec{x} = \begin{bmatrix} 7 \\ 0 \\ 9 \\ 2 \end{bmatrix}$ is a column matrix (or column vector) of order 4.

Special matrices:

(1) The Zero matrix: This is a matrix all of its entries are zeros and may be denoted by O (or $\vec{0}$ for vectors!)

Ex $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is the zero matrix of order 4×1

$O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is the zero matrix of order 2×3

(2) The unit or Identity matrix

1st. Let A be a square matrix of order n, say $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

Main-diagonal entries.

The entries $a_{11}, a_{22}, \ldots,$ and a_{nn} are the so called Main-diagonal entries of A. The other entries of A are called "The off-diagonal entries".

Now, we are ready to define the unit matrix:

The unit matrix is a square matrix (order say n) whose diagonal entries are 1's and off-diagonal entries are all 0's.

This matrix is denoted by I or I_n.

EX $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$, ---- $I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$.

Arithematics of matrices:

(1) Equal Matrices: Matrices A, and B are said to be equal if and only if the following two conditions are met:

(1) A, B have same order, say $m\times n$

(2) Corresponding entries of A, and B are equal! $\boxed{a_{ij} = b_{ij}}$

EX: If $\begin{bmatrix} 2a-b & -3c \\ 1+d & a+b \end{bmatrix} = \begin{bmatrix} -2 & 15 \\ -7 & 8 \end{bmatrix}$, Find a, b, c, and d.

Solution: Equating corresponding entries, we get: $2a-b=-2$, $-3c=15$, $1+d=-7$, $a+b=8$

solve to get: $a=2$, $b=6$, $c=-5$, and $d=-8$

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(2) Addition of matrices: Let A, B be matrices. The sum of A, and B is defined only if A and B are of the same order $m\times n$. The sum is denoted by $A+B$ and is the $m\times n$ matrix obtained from A, B by adding corresponding entries! $A+B=(a_{ij})+(b_{ij})=(a_{ij}+b_{ij})$.

EX $A=\begin{bmatrix} 1 & 9 \\ 3 & -1 \end{bmatrix}$, $B=\begin{bmatrix} 2 & 5 \\ -1 & 7 \end{bmatrix}$. Then $A+B=\begin{bmatrix} 1+2 & 9+5 \\ 3+(-1) & -1+7 \end{bmatrix}=\begin{bmatrix} 3 & 14 \\ 2 & 6 \end{bmatrix}$.

(3) Scalar multiplication: Let K be a scalar (real or complex number). Then the product of K, and matrix A is denoted by KA and is obtained by simply multiplying every entry of A by K!

EX If $A=\begin{bmatrix} 4 & -1 & 2 \\ 1 & 7 & 3 \end{bmatrix}$, then $3A=\begin{bmatrix} (3)(4) & (3)(-1) & (3)(2) \\ (3)(1) & (3)(7) & (3)(3) \end{bmatrix}=\begin{bmatrix} 12 & -3 & 6 \\ 3 & 21 & 9 \end{bmatrix}$

Likewise $(-1)A$ or simply $-A=\begin{bmatrix} -4 & +1 & -2 \\ -1 & -7 & -3 \end{bmatrix}$. The matrix "$-A$" may be called "The negative of A.

(4) Difference of matrices: Let A, B be $m\times n$ matrices we define $A-B$ by $A+(-B)$. Therefore: To subtract two matrices, just subtract corresponding entries!!

EX $A=\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $B=\begin{bmatrix} t \\ e^t \end{bmatrix}$. Find $B-A$, $B-t^2A$, $B-B$.

soln. $B-A=\begin{bmatrix} t \\ e^t \end{bmatrix}-\begin{bmatrix} 2 \\ -1 \end{bmatrix}=\begin{bmatrix} t-2 \\ e^t-(-1) \end{bmatrix}=\begin{bmatrix} t-2 \\ e^t+1 \end{bmatrix}$

$B-t^2A=\begin{bmatrix} t \\ e^t \end{bmatrix}-t^2\begin{bmatrix} 2 \\ -1 \end{bmatrix}=\begin{bmatrix} t \\ e^t \end{bmatrix}-\begin{bmatrix} 2t^2 \\ -t^2 \end{bmatrix}=\begin{bmatrix} t-2t^2 \\ e^t+t^2 \end{bmatrix}$

$B-B=\begin{bmatrix} 0 \\ 0 \end{bmatrix}=\vec{0}$!!

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(5) Product of Matrices:

Let A be an $m\times p$ matrix and B be an $q\times n$ matrix. Then the product of A, and B in that order is defined only if $p=q$. Further if $AB=C$, then C is of order $m\times n$ (Refer to display below).

To find entries of C, say c_{ij} we simply multiply corresponding entries of ith row of A, and jth Column of B, and add up!

$$\underset{m\times p}{A}\cdot\underset{q\times n}{B}=\underset{m\times n}{C}$$

Ex: Let $A=\begin{bmatrix}2&1&10\\-1&4&3\end{bmatrix}$, $B=\begin{bmatrix}7&-1&0\\2&1&5\\9&-8&3\end{bmatrix}$

Then AB is defined and the product is of order 2×3!

$$\underset{2\times3}{A}\cdot\underset{3\times3}{B}=\underset{2\times3}{C}$$

Now $AB=\begin{bmatrix}2&1&10\\-1&4&3\end{bmatrix}\begin{bmatrix}7&-1&0\\2&1&5\\9&-8&3\end{bmatrix}=\begin{bmatrix}c_{11}&c_{12}&c_{13}\\c_{21}&c_{22}&c_{23}\end{bmatrix}$

For example to Find c_{13}: circle 1st. row of A, 3rd. Col. of B as shown. Now multiply/add up: $(2)(0)+(1)(5)+(10)(3)$
$=0+5+30=35$ .. etc.

$$\therefore\begin{bmatrix}2&1&10\\-1&4&3\end{bmatrix}\begin{bmatrix}7&-1&0\\2&1&5\\9&-8&3\end{bmatrix}=\begin{bmatrix}106&-81&35\\28&-19&29\end{bmatrix}$$

Remarks: Given matrices A,B. Any of the following may occur:

(a) Neither AB nor BA are defined
(b) Both AB and BA are defined
(c) Either AB or BA is defined
(d) AB, and BA are defined, but may or may not be equal!

Ex $A=\begin{bmatrix}1\\3\end{bmatrix}$, $B=\begin{bmatrix}7&3\end{bmatrix}$

$\therefore AB=\begin{bmatrix}1\\3\end{bmatrix}\begin{bmatrix}7&3\end{bmatrix}=\begin{bmatrix}7&3\\21&9\end{bmatrix}$, $BA=\begin{bmatrix}7&3\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}=[16]$

clearly $BA\neq AB$!!

Power of a square matrix

Let A be a square matrix of order n and let r be a positive integer ($r = 1, 2, 3, \dots$). Then by A^r we shall mean: $\underbrace{A \cdot A \cdots A}_{r\text{- times}}$

In particular $A^0 = I_n$ = Identity matrix.

Ex: If $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$. Find A^2.

solution $A^2 = A \cdot A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

Warning: $A^2 \neq \begin{bmatrix} 1^2 & 3^2 \\ 2^2 & (-1)^2 \end{bmatrix}$!!

Transpose of a matrix: let $A = (a_{ij})$ be an $m \times n$ matrix. The transpose of A is the matrix obtained by interchange its rows and columns and will be denoted by A^t. It follows that

$$A^t = (a_{ji}) \text{ and is of order } \underline{n \times m}.$$

Ex: If $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 7 \end{bmatrix}$, then $A^t = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 1 & 7 \end{bmatrix}$

Properties of Transpose:

(1) $(A \pm B)^t = A^t \pm B^t$

(2) $(AB)^t = B^t A^t$, $(ABC)^t = C^t B^t A^t$

reverse

(3) $I_n^t = I_n$, $O_{m\times n}^t = O_{n\times m}$

(4) $(KA)^t = K A^t$, $K \in \mathbb{R}$.

(5) $(A^t)^t = A$

Symmetric / skew symmetric matrices: Let A be a square matrix.

(1) The matrix A is said to be symmetric if and only if $A^t = A$

(2) The matrix A is said to be skew symmetric if and only if $A^t = -A$.

Ex: $A = \begin{bmatrix} 2 & 1 & 5 \\ 1 & 7 & -3 \\ 5 & -3 & 20 \end{bmatrix}$. Clearly $A^t = A \Rightarrow A$ is symmetric.

Ex: $B = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 5 \\ -2 & -5 & 0 \end{bmatrix}$. clearly $B^t = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 5 \\ -2 & -5 & 0 \end{bmatrix} = -B$

$\Rightarrow B$ is skew symmetric

observe that: Main-diagonal entries of a skew-symmetric matrix are all zeros

<u>Elementary row operations</u>: These are the permissible operations that may performed on the rows of a matrix A. Let R_i denote the $i^{\underline{th}}$ row of A. Here is a list of the three row-operations:

(1) Interchanging the $i^{\underline{th}}$ and $j^{\underline{th}}$ rows of A. This operation may be symbolized by $\quad R_i \leftrightarrow R_j$

(2) Multiplying the $i^{\underline{th}}$ row by a non-zero scalar c. This operation may be symbolized by $\quad R_i \rightarrow c R_i \quad , c \neq 0$

(3) Adding K-times of $j^{\underline{th}}$ row to the $i^{\underline{th}}$ row. This operation may be symbolized by $\quad R_i \rightarrow R_i + K R_j$. <u>Only</u> the $i^{\underline{th}}$ row is changed.

<u>EX</u>: $A = \begin{bmatrix} 5 & 3 & 2 \\ 1 & 7 & -1 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 0 & 1 & 3 \\ 1 & 7 & -1 \\ 5 & 3 & 2 \end{bmatrix} = B$

EX: $A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 4 & 3 & 1 & 1 \\ 2 & 1 & -1 & 5 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 - 3R_1} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 1 & 6 & -5 & -2 \\ 2 & 1 & -1 & 5 \end{bmatrix} = B$

EX $\quad A = \begin{bmatrix} 1 & 5 \\ 2 & -1 \end{bmatrix} \xrightarrow{R_2 \rightarrow \frac{R_2}{3}} \begin{bmatrix} 1 & 5 \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} = B$

<u>Def.</u>: If A and B are matrix and if B is obtained from A by performing a sequence of elementary row operations, one says A and B are equivalent.

<u>Def.</u>: <u>Row-echelon matrices</u>

A matrix is said to be in Row-echelon form if the following conditions are met:

(1) All rows consisting entirely of zeros are grouped at the bottom of matrix.

(2) For every non-zero row, the first non-zero entry is a "1" called the leading one for that row.

(3) The Leading one of a row lies to the <u>right</u> of all the leading ones of the rows above it.

In other words leading ones form a stair-case starting high at the upper left Corner of matrix $\begin{bmatrix} \boxed{1} & * & * & * \\ 0 & \boxed{1} & * & * \\ 0 & 0 & \boxed{1} & * \end{bmatrix}$

If in addition to the three conditions above, the following condition (4) is satisfied, one says the matrix is in Reduced row-echelon form.

A row-echelon matrix or a reduced row-echelon matrix may be denoted by R.

Condition (4): The leading one of a row is the only non-zero entry in its column.

EX $\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is a row-echelon matrix

EX $\begin{bmatrix} 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is a reduced row-echelon matrix

EX $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is not a row-echelon matrix since leading one of R_2 lies to the left of leading one of R_1.

Theorem: Every $m \times n$ matrix A is equivalent to a row-echelon matrix (or reduced row-echelon matrix) R. That is to say R can be obtained from A by performing a sequence of elementary row operations.

EX: Find a reduced row-echelon matrix R equivalent to

$$A = \begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 0 & 1 & 5 \\ 5 & -1 & 6 & 17 \end{bmatrix}$$

solution $A = \begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 0 & 1 & 5 \\ 5 & -1 & 6 & 17 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 1 & 5 \\ 2 & -1 & 3 & 2 \\ 5 & -1 & 6 & 17 \end{bmatrix}$

$$\xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & -1 & 1 & -8 \\ 0 & -1 & 1 & -8 \end{bmatrix} \xrightarrow{R_2 \to -R_2} \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & -1 & 8 \\ 0 & -1 & 1 & -8 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & -1 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R \quad \#$$

Rank of matrix: let A be matrix and let R be a row-echelon matrix equivalent to A. Then # of leading ones of R

say "r" is called "The Rank of A" and we write rank(A) = r.

In Example above: rank(A) = 2.

<u>System of Linear equations</u>: A system of "m" equations in the "n" unknowns $x_1, x_2, \ldots,$ and x_n is of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\;\; \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

which can be written in the simple matrix form $A\vec{x} = \vec{b}$ where $A = (a_{ij})\ \begin{smallmatrix} i=1,2,\ldots,m \\ j=1,2,\ldots,n \end{smallmatrix}$ is the coefficient matrix, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is the unknown column vector and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ is constant term column.

The system is non-homogeneous if $\vec{b} \neq \vec{0}$. If $\vec{b} = \vec{0}$, we get $A\vec{x} = \vec{0}$ which we call a homogeneous system.

<u>Nature of solution of a linear system</u>: consider $A\vec{x} = \vec{b}$.

A linear system may have either

(1) A unique solution (2) An infinitely-many solutions, or

(3) No solution at all.

A system having a solution is called a consistent linear system. However system having <u>no</u> solutions are Inconsistent!

<u>Augmented matrix</u>: This is the matrix obtained from "A" by inserting "b" as a last column. Augmented matrix is thus denoted by $[A \,\vdots\, \vec{b}]$.

EX: For the system
$$\begin{aligned} 3x - 2y + 3z &= 9 \\ x - y \qquad &= 4 \end{aligned}$$

The augmented matrix is $\left[\begin{array}{ccc|c} 3 & -2 & 3 & 9 \\ 1 & -1 & 0 & 4 \end{array}\right]$.

<u>Ex</u>: write down a system having the augmented matrix $\left[\begin{array}{cc|c} 2 & 1 & -1 \\ 2 & 4 & 0 \\ 1 & -1 & 7 \end{array}\right]$.

<u>solution</u>: we know this system has <u>3</u> equations in two unknowns, say u, and v. Hence system is given by $\begin{cases} 2u + v = -1 \\ 2u + 4v = 0 \\ u - v = 7 \end{cases}$

To solve a system of linear equations just bring its augmented matrix to row-echelon form (or reduced row-echelon form). (9)

Theorem: Very Important.

Consider the system $A\vec{x} = \vec{b}$, A is an $m\times n$ matrix. let $\text{rank}(A) = r_1$, $\text{rank}[A|\vec{b}] = r_2$. Then

(1) If $r_1 \neq r_2$, system has no solution (Inconsistent)

(2) If $r_1 = r_2 = \#$ of unknowns n, the system has exactly one solution

(3) If $r_1 = r_2 = $ say $r < n$, the system has an infinitely-many solution. In fact the system has $n-r$-parameter family of solutions. These parameters are the unknowns whose leading ones are missing!

EX: solve
$$\begin{cases} 2x+3y-4z = 9 \\ x+y-z = 1 \\ 3x+4y-5z = 0 \end{cases}$$

solution: Augmented matrix $[A|\vec{b}] = \left[\begin{array}{ccc|c} 2 & 3 & -4 & 9 \\ 1 & 1 & -1 & 1 \\ 3 & 4 & -5 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2}$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 3 & -4 & 9 \\ 3 & 4 & -5 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -2 & 7 \\ 0 & 1 & -2 & -3 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -2 & 7 \\ 0 & 0 & 0 & -10 \end{array}\right] \xrightarrow{R_3 \to \frac{R_3}{-10}} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -2 & 7 \\ 0 & 0 & 0 & 1 \end{array}\right] \text{ stop!}$$

$\text{rank}(A) = 2$, $\text{rank}[A|\vec{b}] = 3 \Rightarrow$ No solution!

EX solve
$$x_1 - 2x_2 + 5x_3 - 2x_4 = 3$$
$$2x_1 - 4x_2 + 9x_3 + x_4 = -1$$

solution
$$\left[\begin{array}{cccc|c} 1 & -2 & 5 & -2 & 3 \\ 2 & -4 & 9 & 1 & -1 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & -2 & 5 & -2 & 3 \\ 0 & 0 & -1 & 5 & -7 \end{array}\right]$$

$$\xrightarrow{R_2 \to -R_2} \left[\begin{array}{cccc|c} 1 & -2 & 5 & -2 & 3 \\ 0 & 0 & 1 & -5 & 7 \end{array}\right] \xrightarrow{R_1 \to R_1 - 5R_2} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 23 & -32 \\ 0 & 0 & 1 & -5 & 7 \end{array}\right] \quad (*)$$

$\text{rank}(A) = \text{rank}(A|\vec{b}) = 2 < n = 3$

$\Rightarrow$ Infinitely-many solutions. Since $n - r = 3 - 2 = 1$ ... we have a two-parameter family. Since leading ones for x_2, x_4 are missing, we let $x_2 = t, \quad x_4 = s \qquad t, s \in \mathbb{R}$

It follows from (*) that $\quad x_1 - 2x_2 + 23x_4 = -32$

$\qquad\qquad\qquad\qquad\qquad x_3 - 5x_4 = 7$

Hence $x_1 = -32 + 2x_2 - 23x_4 = -32 + 2t - 23s$

$x_3 = 7 + 5x_4 \qquad = 7 \qquad - 5s$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -32 + 2t - 23s \\ t \\ 7 \quad - 5s \\ s \end{pmatrix} = \begin{pmatrix} -32 \\ 0 \\ 7 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -23 \\ 0 \\ -5 \\ 1 \end{pmatrix} \quad t, s \in \mathbb{R}$$

Inverse of a Square matrix:

Let A be a square matrix of order n. We say that "A" is invertible if there exists a matrix X such that $AX = I_n$ and $XA = I_n$.
The matrix X is then called "The Inverse of A" and we write $X = A^{-1}$! It follows that A^{-1} has the properties (1) $AA^{-1} = I_n$.
(2) $A^{-1}A = I_n$.

properties of Inverse: let A, B, and C be invertible $n \times n$ matrices.

(1) $(A^{-1})^{-1} = A$ $\qquad$ (2) $(A^{-1})^t = (A^t)^{-1}$

(3) $(KA)^{-1} = K^{-1}A^{-1}$, $\quad K \in \mathbb{R}, K \neq 0$

(4) $I_n^{-1} = I_n$ $\qquad$ (5) $(AB)^{-1} = B^{-1}A^{-1}$, $\quad (ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

reverse

Determinant of a square matrix

Let $A = (a_{ij})$ be a square matrix of order n. The determinant of "A" is a real number denoted by $\det A$ or $\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$.

Computing Determinants:

Case (1): Determinant of a matrix of order 2×2:

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. we define $\det A$ by

$$\det A \overset{\text{or}}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Case (2): Determinant of a matrix of order 3x3.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix of order 3.

Def.: Cofactor of an entry:

The Cofactor of an entry a_{ij} of A is denoted by c_{ij} and is defined by

$$c_{ij} = (-1)^{i+j} \cdot \text{The } 2\times 2 \text{ determinant obtained from } A \text{ by deleting } i\text{th. row and } j\text{th. column}$$

For instance $C_{23} = (-1)^{2+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$ - (*)

$$\begin{bmatrix} a_{11} & a_{12} & \cancel{a_{13}} \\ \cancel{a_{21}} & \cancel{a_{22}} & \cancel{a_{23}} \\ a_{31} & a_{32} & \cancel{a_{33}} \end{bmatrix}$$

"Delete 2nd. row and 3rd. Column"

The sign pattern for determinant of order 3x3

Since $(-1)^{i+j} = \pm 1$, the signs $(-1)^{i+j}$ are shown

$$\begin{bmatrix} (-1)^{1+1} = + & (-1)^{1+2} = - & (-1)^{1+3} = + \\ (-1)^{2+1} = - & (-1)^{2+2} = + & (-1)^{2+3} = - \\ (-1)^{3+1} = + & (-1)^{3+2} = - & (-1)^{3+3} = + \end{bmatrix} \rightarrow \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Laplace Expansion of det A:

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

To find det A: choose any row (or a column). Then find the Cofactors of the entries of the chosen row (or Column).

For instance if 1st. row is chosen, the cofactors of its entries are c_{11}, c_{12}, and c_{13}

$$\therefore \det A = a_{11} c_{11} + a_{12} c_{12} + a_{13} c_{13}$$

... etc. Note that according to (*) above:

$$c_{11} = + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad c_{12} = - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad c_{13} = + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

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Remark: Determinants of order $n\times n$, $n\geq 4$

This is a straight forward generalization to determinants of order 3×3 which we have discussed earlier.

EX: Find det A if $A=\begin{bmatrix}1&2&-1\\3&0&0\\4&-1&3\end{bmatrix}$.

solution $\det A=\begin{bmatrix}1&2&-1\\3&0&0\\4&-1&3\end{bmatrix}$ $\longrightarrow$ Sign pattern $[\ominus\oplus\ominus]$ for R_2

let us use 2nd row to find det A. The easiest!

$$\det A=\ominus 3\begin{vmatrix}2&-1\\-1&3\end{vmatrix}+\cancel{0|\ \ |}-\cancel{0|\ \ |}$$

$$=-3[6-1]=-15.$$

properties of determinants:

Action of elementary row-operations on a determinant

(1) If $A\xrightarrow{R_i\to R_j}B$, then $\det B=-\det A$

In words: Interchange two rows of a determinant changes its value by a negative sign.

(2) If $A\xrightarrow[c\neq 0]{R_i\to cR_i}B$, then $\det B=c\det A$

In words: Multiplying a row of a determinant by $c\neq 0$, change its value by c-time!

Ex $\det A=\begin{vmatrix}1&2&-1\\3&1&1\\0&0&2\end{vmatrix}$. clearly $\det A=-10$

Now $\det\begin{bmatrix}7&14&-7\\3&1&1\\0&0&2\end{bmatrix}=7(-10)=-70$ .. because 1st row is seven times that of det A!

(3) If $A\xrightarrow{R_i\to R_i+KR_j}B$, then

$$\det A=\det B$$

This is a very powerful property which says: Applying an elementary row operation of type (III) on "A" Does not change its determinant.

Other properties: let A be a square matrix of order n

(1) If A has two Identical rows, then $\det A = 0$

(2) If A has a zero row, then $\det A = 0$

(3) $\det A = \det A^t$. This powerful property tell us:

(a) we can use column to expand determinants

(b) All properties that hold for rows of a determinant, remain true for columns.

(c) we are permitted column operation on a determinant.

usually we use row or column operations to get as many zeros as possible in a row (or column) then use it to expand $\det A$.

Ex: let $B = \begin{bmatrix} 2 & 2 & 9 \\ 1 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}$.. Indeed $\det B = 0$ since its 1st. and 2nd. columns are identical.

Ex: let $C = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -2 \\ 1 & 1 & -4 \end{bmatrix}$. Find $\det C$

let us 1st. apply column operations to get as many zeros in a row!

$$C = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -2 \\ 1 & 1 & -4 \end{bmatrix} \xrightarrow[C_3 \to C_3 + 4C_1]{C_2 \to C_2 - C_1} \begin{bmatrix} 4 & -3 & 15 \\ 2 & 1 & 6 \\ 1 & 0 & 0 \end{bmatrix}$$

Get zeros

$$\therefore \det C = +1 \begin{vmatrix} -3 & 15 \\ 1 & 6 \end{vmatrix} = -33$$

More properties of determinants: let A, B be square matrices of order n and $K \in \mathbb{R}$ be scalar

(1) $\det(AB) = (\det A) \det B$

(2) $\det I_n = 1$, $\det O_{n\times n} = 0$

(3) $\det A^{-1} = \dfrac{1}{\det A}$ provided A^{-1} exists

(4) $\det A^t = \det A$

(5) $\det(KA) = K^n \det A$

(6) If $r = 1, 2, 3, \ldots$, then $\det(A^r) = (\det A)^r$.

Ex: If $A = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$, Find $\det A^{100}$.

solution $\det A = \begin{vmatrix} 3 & -1 \\ -5 & 2 \end{vmatrix} = 6 - 5 = 1$

$\therefore \det A^{100} = (\det A)^{100} = (1)^{100} = 1$

Ex: If $A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ find $\det A^{-1}$.

1st. $\det A = +1 \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = 1 \Rightarrow \det A^{-1} = \frac{1}{1} = 1$ as well.

Ex: If $A = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix}$. Find $\det(A - \lambda I_2)$ and solve $\det(A - \lambda I_2) = 0$.

Solution $A - \lambda I_2 = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 4 \\ 5 & 3-\lambda \end{bmatrix}$

$\therefore \det(A - \lambda I_2) = (2-\lambda)(3-\lambda) - 20 = \lambda^2 - 5\lambda - 14$

Now $\det(A - \lambda I_2) = 0 \Rightarrow \lambda^2 - 5\lambda - 14 = 0 \Rightarrow \lambda = 7, -2$

Computing Inverse of a square matrix:

Case (1): let A be a square matrix of order 2. Then "A" is invertible iff $\det A \neq 0$ and that A^{-1} is given by

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{where } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Ex: Find B^{-1} if $B = \begin{bmatrix} 7 & 3 \\ 1 & 9 \end{bmatrix}$

Solution: $\det B = (7)(9) - (1)(3) = 60 \neq 0 \Rightarrow B^{-1}$ Exists.

Now $B^{-1} = \frac{1}{60} \begin{bmatrix} 9 & -3 \\ -1 & 7 \end{bmatrix}$

Case (2): let A be a square matrix of order n, $n \geq 2$.

The matrix "A" is invertible iff $\det A = 0$.

Defn.: Adjoint of a square matrix:

Let $A = (a_{ij})$ $i,j = 1, 2, \ldots n$ and let c_{ij} be cofactor of a_{ij}

The adjoint of "A" is denoted and defined by

$$\operatorname{adj} A = (c_{ij})^t \quad i,j = 1,2,\dots,n$$

<u>In words</u>: adj A = The Transpose of the matrix of all Cofactors of A

<u>Theorem</u>: $A(\operatorname{adj} A) = (\det A) I_n$, $(\operatorname{adj} A) A = (\det A) I_n$

In particular if A is invertible, $\boxed{A^{-1} = \frac{1}{\det A} \operatorname{adj} A}$

This last powerful relation is used to Compute A^{-1}.

<u>EX</u>: If $A = \begin{bmatrix} 3 & 1 & -1 \\ 4 & 1 & 0 \\ 0 & -1 & 3 \end{bmatrix}$. Find A^{-1}.

<u>solution</u>: $\operatorname{adj} A = (c_{ij})^t = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}^t = \begin{bmatrix} +\begin{vmatrix} 1 & 0 \\ -1 & 3 \end{vmatrix} & -\begin{vmatrix} 4 & 0 \\ 0 & 3 \end{vmatrix} & +\begin{vmatrix} 4 & 1 \\ 0 & -1 \end{vmatrix} \\ -\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix} & +\begin{vmatrix} 3 & -1 \\ 0 & 3 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 0 & -1 \end{vmatrix} \\ +\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} & +\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} \end{bmatrix}^t$

$$= \begin{bmatrix} 3 & -12 & -4 \\ -2 & 9 & 3 \\ 1 & -4 & -1 \end{bmatrix}^t = \begin{bmatrix} 3 & -2 & 1 \\ -12 & 9 & -4 \\ -4 & 3 & -1 \end{bmatrix}$$

and $\det A = 1$. Hence $A^{-1} = \frac{1}{1} \begin{bmatrix} 3 & -2 & 1 \\ -12 & 9 & -4 \\ -4 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 1 \\ -12 & 9 & -4 \\ -4 & 3 & -1 \end{bmatrix}$

<u>Definition</u>: <u>upper</u> / <u>lower Triangular matrices</u>: let A be square matrix. A matrix "A_1" is an upper Triangular matrix if all entries below the main-diagonal are zeros .. Likewise a matrix "A_2" is a lower triangular if all entries above the main-diagonal are zeros.

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \text{, upper Triangular.}$$

$$A_2 = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix} \text{, Lower Triangular}$$

<u>Note</u>: $\det A_1$ = product of main-diagonal entries $= a_{11} a_{22} \cdots a_{nn}$.
$\det A_2 = a_{11} a_{22} \cdots a_{nn}$ as well.

FACTS ABOUT LINEAR SYSTEMS

[a] Homogeneous systems of linear Equations

Consider the homogeneous system $A\vec{x} = \vec{0}$, where A is an $m \times n$ matrix consisting of m equations in the n-unknowns $x_1 = x_2 = \cdots = x_n$.

Fact 1: Every homogeneous system is consistent; meaning: It has at least one solution.

Fact 2: Every homogeneous system has a trivial solution $x_1 = x_2 = \cdots = x_n$. Any other solution is called a non-trivial solution.

Fact 3: If $m < n$, the homogeneous system has a non-Trivial solution (in fact an-infinitely-many).

Fact 4: If $m = n$ and $\det A = 0$, the system has a non-Trivial solution. However if $\det A \neq 0$, the homogeneous system has only the Trivial solution $\vec{x} = \vec{0}$.

Fact 5: If $m > n$, and if B is a sub-matrix of order $n \times n$ such that $\det B \neq 0$, the system has only the Trivial solution. Otherwise it has a non-Trivial solution.

Ex: The homogeneous system

$$\begin{aligned} x + 2y + 3z &= 0 \\ 3x - 7y + z &= 0 \\ 4x - 5y + 4z &= 0 \end{aligned}$$

has a non Trivial solution since $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -7 & 1 \\ 4 & -5 & 4 \end{bmatrix}$, $\det A = 0$ « verify! »

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[b] Non-homogeneous systems of linear equations

Consider the non-homogeneous system $A\vec{x} = \vec{b}$ where A is an $\underline{\underline{n \times n}}$ matrix.

(1) If $\det A \neq 0$, the system has the unique solution $\vec{x} = A^{-1}\vec{b}$.

(2) If $\det A = 0$, the system may have either no solution or an-infinitely-many solution depending on the column $\vec{b}$.

EX: The system

$$\begin{aligned} 3x - 7y &= 4 \\ 2x + y &= -1 \end{aligned}$$

has a unique solution since $A = \begin{bmatrix} 3 & -7 \\ 2 & 1 \end{bmatrix}$ is invertible ($\det A = 3 + 14 = 17 \neq 0$). Further $A^{-1} = \frac{1}{17}\begin{bmatrix} 1 & 7 \\ -2 & 3 \end{bmatrix}$. Hence the unique solution is

$$\vec{x} = A^{-1}\vec{b} = \frac{1}{17}\begin{bmatrix} 1 & 7 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{17} \\ -\frac{11}{17} \end{bmatrix}.$$

END