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Support material # (1)

First Order Differential Equations : Types:

(1) Separable : equation can be put in the form $f(t)\,dt = g(x)\,dx$.

General solution : $\int f(t)\,dt = \int g(x)\,dx + C$

(2) Linear : equation can be put in the form $\frac{dx}{dt} + p(t)x = q(t)$

General solution : $x\mu = \int q\mu\,dt + C$, where $\mu(t) = I.F = e^{\int p(t)\,dt}$

(3) Bernoulli's : equation can be put in the form $\frac{dx}{dt} + p(t)x = q(t)x^n$, $n \in \mathbb{R}$, $n \neq 0, 1$

Multiply by x^{-n} to get $x^{-n}\frac{dx}{dt} + p(t)\,x^{1-n} = q(t)$. Then let

$$V = x^{1-n} \Rightarrow \frac{dV}{dt} = (1-n)\,x^{-n}\frac{dx}{dt} \Rightarrow \boxed{x^{-n}\frac{dx}{dt} = \frac{1}{1-n}\frac{dV}{dt}}$$

$$\therefore \text{Eq.} \Rightarrow \frac{1}{1-n}\frac{dV}{dt} + p(t)V = q(t) \Rightarrow \frac{dV}{dt} + P_1(t)V = q_1(t)$$

where $P_1(t) = (1-n)p(t)$, $q_1(t) = (1-n)q(t)$ .. new Eq. is Linear in V.

(4) Homogeneous : Eq. is of the form $\frac{dx}{dt} = f(t,x)$ where $f(t,x)$ remains unchanged if t, and x are replaced by λt, λx, $\lambda \in \mathbb{R}$.

Let $x = vt \Rightarrow \frac{dx}{dt} = v.1 + t\frac{dv}{dt}$. Equation becomes separable in t, and v. Finally replace v by $\frac{x}{t}$.

(5) Exact : Eq. is of the form : $M(t,x)\,dt + N(t,x)\,dx = 0$ where

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t} \quad \text{for all } t, x \in \text{some domain } D.$$

General solution is $F(t,x) = C$, where $F(t,x)$ is the sum of the two partial Integrals $\int M(t,x)\,\partial t$, $\int N(t,x)\,\partial x$ but without repeating common terms.

(6) Equations Reducible to Linear: equations of the form

$$\frac{d^n x}{dt^n} + p(t)\frac{d^{n-1}x}{dt^{n-1}} = q(t)$$

Let $v = \frac{d^{n-1}x}{dt^{n-1}}$ .. hence $\frac{dv}{dt} = \frac{d^n x}{dt^n}$. D.E becomes

$$\frac{dv}{dt} + p(t)\,v = q(t) \quad \text{which is linear in } v.$$

once v is found, integrate it $(n-1)$ times w.r.t. to t to recover "x".

Applications : Newton's Law of cooling:

The rate at which a body cools is proportional to the difference of body's temperature and the temperature of the medium in which the body is situated. Now if $x = x(t)$ is the temperature of the body at time t and "m" is the medium's temp. Then

$$\frac{dx}{dt} = K(x - m) \qquad , \; x(t_0) = x_0$$

where K is a constant and x_0 is the initial temp of body at time t_0.

This I.V.P can be treated as separable or linear, we get:

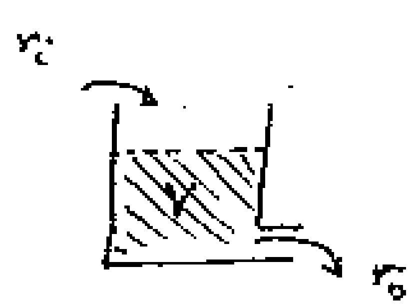
$$x(t) = m + c\,e^{Kt}$$

where c, and K are constants to be found!

Mixture Problems:

A substance (such as a salt) is allowed to flow into a certain mixture in a container at a certain rate r_i, the mixture is kept uniform by stirring and flows out of the container at another rate r_o (it is possible $r_i = r_o$ as well). Let $x = x(t)$ be the amount of substance at time t. Then

$$\frac{dx}{dt} = IN - out \qquad , \; x(t_0) = x_0$$



If c_i and c_o are the concentrations "in" and "out" respectively then $IN = c_i \cdot r_i$, $out = c_o\, r_o$

where the only quantity needed to be computed is

$$c_o = \left(\frac{x}{V}\right) = \frac{x}{V_0 + (r_i - r_o)t}$$

V = volume at time $= V_0 + (r_i - r_o)t$

Initial Volume

once again, we get a 1st. order linear or separable D.E.

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Support Material (2)

Review of Integration:

$\boxed{\text{I}}$ Integration By TABLE:

Just Match the integral with one in the table (provided earlier).
Note the importance of the fact that this table may be extended to the case where "x" is replaced by the linear function $ax+b$, $a \neq 0$. For example

$$\int \cos x \, dx = \sin x + C$$

$$\therefore \int \cos(\underbrace{ax+b}_{\text{Linear!}}) \, dx = \frac{\sin(ax+b)}{a} + C$$

Examples:

(a) $\int \sin x \, dx = -\cos x + C$

(b) $\int \sin(9x-7) \, dx = -\frac{1}{9} \cos(9x-7) + C$

(c) $\int \frac{1}{x} \, dx = \ln|x| + C$ or $\ln x + C$, $x > 0$

(d) $\int \frac{1}{3z-2} \, dz = \frac{1}{3} \ln|3z-2| + C$

(e) $\int e^{5x} \, dx = \frac{1}{5} e^{5x} + C$

(f) $\int \frac{dt}{t^2+9} = \int \frac{dt}{t^2+3^2} = \frac{1}{3} \tan^{-1}\left(\frac{t}{3}\right) + C$

(g) $\int (3t-1)^{-41} \, dt = \frac{(3t-1)^{-40}}{(-40)(3)} + C = -\frac{1}{120} (3t-1)^{-40} + C$

(h) $\int \frac{t^3+2t^2-1}{t} \, dt = \int \left(t^2 + 2t - \frac{1}{t}\right) dt = \frac{1}{3} t^3 + t^2 - \ln|t| + C$

(i) $\int \frac{1}{\cos^2(4t)} \, dt$ .. Not in Table .. rewrite as

$$\int \sec^2(4t) \, dt = \frac{1}{4} \tan(4t) + C \quad !!$$

(j) $\int \frac{1}{\sin(2x)\cos(2x)}\,dx$

1st. Recall identity: $\sin 2A = 2\sin A \cos A$

$\therefore \quad \sin 2x \cos 2x = \frac{1}{2}\sin 2(2x) = \frac{1}{2}\sin 4x$

$$\therefore \quad I = \int \frac{1}{\frac{1}{2}\sin(4x)}\,dx = 2\int \csc(4x)\,dx$$

$$= 2 \cdot \frac{1}{4}\ln|\csc(4x) - \cot(4x)| + C$$

$$= \frac{1}{2}\ln|\csc(4x) - \cot(4x)| + C$$

$\boxed{2}$ Integration by Substitution:

let $I = \int f(t)\,dt$

If Table can't be used directly to compute I, attempt a substitution say $w = w(t)$. The substitution is successful particularly if its differential dw is a multiplicative part of integrand (up to a constant multiple!)

For instance consider

$$I = \int 6\sqrt{t}\,e^{-t^{\frac{3}{2}}}\,dt = 6\int e^{-t^{\frac{3}{2}}}\,\underline{(\sqrt{t}\,dt)}$$

The suggested substitution is $w = -t^{\frac{3}{2}}$. To see why, let us compute its differential dw!

Indeed $\quad dw = -\frac{3}{2}t^{\frac{1}{2}}\,dt = -\frac{3}{2}\,\underline{(\sqrt{t}\,dt)}$

which is a multiplicative part of integrand as shown!

Now, $\quad \sqrt{t}\,dt = -\frac{2}{3}\,dw$

$$\therefore \quad I = -4\int e^{w}\,dw = -4e^{w} + C$$

$$= -4e^{-t^{\frac{3}{2}}} + C$$

Examples:

(a) $\int \frac{t^3}{\sqrt{t^4-2}}\,dt$

let $w = t^4 - 2 \Rightarrow dw = 4t^3\,dt \Rightarrow t^3\,dt = \frac{1}{4}\,dw$

$\therefore I = \frac{1}{4}\int \frac{1}{\sqrt{w}}\,dw = \frac{1}{4}\int w^{-\frac{1}{2}}\,dw = \frac{1}{4}\,\frac{w^{\frac{1}{2}}}{\frac{1}{2}} + C$

$= \frac{1}{2}\sqrt{t^4-2} + C$

(b) $\int 2te^{-t^2}\,dt$ .. let $w = -t^2 \Rightarrow dw = -2t\,dt$

or $\boxed{2t\,dt = -dw}$

$I = -\int e^w\,dw = -e^w + C = -e^{-t^2} + C.$

(c) $\int \frac{\ln^3 (x+1)^5}{x+1}\,dx$

1st. observe that $\ln^3 (x+1)^5 = \left[\ln (x+1)^5\right]^3$

$= \left[5\ln(x+1)\right]^3$

$= 125\left[\ln(x+1)\right]^3$

$\therefore I = 125\int \frac{\left[\ln(x+1)\right]^3}{x+1}\,dx$

let $w = \ln(x+1) \Rightarrow dw = \frac{1}{x+1}\,dx$

$\therefore I = 125\int w^3\,dw = \frac{125}{4}w^4 + C = \frac{125}{4}\ln^4(x+1) + C$

(d) $\int \frac{\sec(\sqrt{x})}{\sqrt{x}}\,dx$ let $w = \sqrt{x} \Rightarrow dw = \frac{1}{2\sqrt{x}}\,dx$

$\Rightarrow 2\,dw = \frac{1}{\sqrt{x}}\,dx$

$I = \int \sec(\sqrt{x})\cdot\left(\frac{1}{\sqrt{x}}\,dx\right) = \int \sec(w)\,dw = \ln\left|\sec(w) + \tan(w)\right| + C$

$= \ln\left|\sec(\sqrt{x}) + \tan(\sqrt{x})\right| + C$

$\boxed{3}$ <u>Integration by parts</u> : (a), (b), (c), (d) :

$$\int u\,dV = uV - \int V\,du \quad \text{.. see diagram}$$

$$\begin{array}{ccc} u & & dV \\ du & \xleftarrow{-\int} & V \end{array}$$

Integration by parts can be used to compute the following types of integrals:

(a) $\int t^n e^{\alpha t} dt$, $n = 1, 2, 3, \dots$, $\alpha \in \mathbb{R}$, $\alpha \neq 0$.

let $u = t^n$, $dV = e^{\alpha t}$ .. repeat n-times or use the more civilized method demonstrated in Example below!

EX $\int t^2 e^{2t} dt = \frac{1}{2} t^2 e^{2t} - \frac{1}{2} t e^{2t} + \frac{1}{4} e^{2t} + C$

$$\begin{array}{ccc} t^2 & \oplus & e^{2t} \\ 2t & \ominus & \frac{1}{2} e^{2t} \\ 2 & \oplus & \frac{1}{4} e^{2t} \\ 0 & & \frac{1}{8} e^{2t} \end{array}$$

(b) $\int t^n \cos \alpha t \, dt$, $\int t^n \sin \alpha t \, dt$, $n = 1, 2, 3, \dots$, $\alpha \in \mathbb{R}$, $\alpha \neq 0$.

let $u = t^n$, $dV = (\cos \alpha t, \text{ or } \sin \alpha t)\, dt$ .. repeat n-times or use the civilized illustration above!

EX $\int x \sin(5x)\, dx$

$= -\frac{1}{5} x \cos(5x) + \frac{1}{25} \sin(5x) + C$

$$\begin{array}{ccc} x & \oplus & \sin(5x) \\ 1 & \ominus & -\frac{1}{5}\cos(5x) \\ 0 & & -\frac{1}{25}\sin(5x) \end{array}$$

(c) $\int t^m \ln t \, dt$, $m \in \mathbb{R}$, $m \neq -1$

let $u = \ln t$, $dV = t^m dt$ .. only <u>one</u> integration by parts!

EX $\int t^{\frac{3}{4}} \ln t \, dt$

$= \frac{4}{7} t^{\frac{7}{4}} \ln t - \int \frac{1}{t} \cdot \frac{4}{7} t^{\frac{7}{4}} dt$

$= \frac{4}{7} t^{\frac{7}{4}} \ln t - \frac{4}{7} \int t^{\frac{3}{4}} dt$

$= \frac{4}{7} t^{\frac{7}{4}} \ln t - \frac{16}{49} t^{\frac{7}{4}} + C$

$$\begin{array}{ccc} \ln t & & t^{\frac{3}{4}} \\ \frac{1}{t} & \xleftarrow{-\int} & \frac{4}{7} t^{\frac{7}{4}} \end{array}$$

(d) Integral of an inverse Trigonometric or Hyperbolic function, such as $\int \tan^{-1}(t)\,dt$, $\int \cosh^{-1}(t)\,dt$, ... etc. (5)

In this case we let $u =$ The inverse function, $dv = 1.\,dt$

Ex $\int \sin^{-1}(t)\,dt$

$$= t\sin^{-1}(t) - \int \frac{t}{\sqrt{1-t^2}}\,dt$$

use substitution $w = 1-t^2$

$$= t\sin^{-1}(t) + \sqrt{1-t^2} + C$$

$$\begin{array}{ccc} \sin^{-1}(t) & & 1 \\ \frac{1}{\sqrt{1-t^2}} & \xleftarrow{-\int} & t \end{array}$$

Ex $\int \tan^{-1}(t)\,dt$

$$= t\tan^{-1}(t) - \int \frac{t}{1+t^2}\,dt$$

use $w = 1+t^2$

$$= t\tan^{-1}(t) - \frac{1}{2}\ln(1+t^2) + C$$

$$\begin{array}{ccc} \tan^{-1} t & & 1 \\ \frac{1}{1+t^2} & \xleftarrow{-\int} & t \end{array}$$

$\boxed{4}$ Integration by partial fraction Decomposition (PFD)

Consider $I = \int f(t)\,dt$ where $f(t)$ is a proper rational function. Let $f(t) = \frac{p(t)}{q(t)}$ where p, and q are polynomials.

Three Simple steps:

1. Factor denominator $q(t)$ completely into linear and/or quadratic functions
2. Decompose f into partial fractions. Three cases will be Examined.

(a) For every linear factor of q of multiplicity one say $\alpha t+\beta$, write the partial fraction $\frac{A}{\alpha t+\beta}$, A is a constant to be found!

(b) For every Linear factor of q of multiplicity r, say $(\alpha t+\beta)^r$, write the sum of r- partial fractions:

$$\frac{B_1}{(\alpha t+\beta)} + \frac{B_2}{(\alpha t+\beta)^2} + \cdots + \frac{B_r}{(\alpha t+\beta)^r}$$

(c) For every quadratic factor of q of multiplicity one, say at^2+bt+c, write the partial fraction $\dfrac{Dt+E}{at^2+bt+c}$ (6)

3. Multiply both sides of relation found in step (2) by the factored denominator $q(t)$. You get an identity. Assign any values of t to determine unknown constants.

Ex: Find $\displaystyle\int \frac{44-3x}{x^2-6x-16}\,dx$

solution $\quad \dfrac{44-3x}{x^2-6x-16} = \dfrac{44-3x}{(x-8)(x+2)} = \dfrac{A}{x-8} + \dfrac{B}{x+2}$

Multiply each side by $(x-8)\cdot(x+2)$, we get

$$44-3x = A(x+2)+B(x-8) \quad \text{-- an identity.}$$

choose any two values of x to find A, and B. Best values occur where: Denominator $=0$

$$\Rightarrow (x-8)(x+2)=0, \quad x=8,-2$$

For $x=8$: $\quad 44-3(8) = A(8+2)+0$

$20 = 10A, \quad \boxed{A=2}$

For $x=-2$: $\quad 44-3(-2) = 0 + B(-2-8)$

$50 = -10B, \quad \boxed{B=-5}$

$$\therefore \quad I = \int\left(\frac{A}{x-8}+\frac{B}{x+2}\right)dx = 2\int\frac{dx}{x-8} - 5\int\frac{dx}{x+2}$$

$$= 2\ln|x-8| - 5\ln|x+2| + C$$

EX: $\int \frac{dx}{x^2-4}$ .. Use PFD

$$\frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$$

$$\therefore \frac{1}{x^2-4} = \frac{1}{4}\frac{1}{x-2} - \frac{1}{4}\frac{1}{x+2} \quad \text{(check!)}$$

$$\therefore I = \frac{1}{4}\int \frac{1}{x-2}\,dx - \frac{1}{4}\int \frac{1}{x+2}\,dx$$

$$= \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C \overset{\text{or}}{=} \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$$

$\boxed{5}$ Integrals of powers of Trigonometric functions

EX: $\int \cos^2 x\,dx$. Use Identity $\cos^2 x = \frac{1}{2}[1+\cos 2x]$

$$\therefore I = \frac{1}{2}\int (1+\cos 2x)\,dx = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right] + C$$

EX: $\int \sin^2 x\,dx$ .. Use Identity $\sin^2 x = \frac{1}{2}[1-\cos 2x]$

$$\therefore I = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + C$$

EX: $\int \tan^2 x\,dx$ .. Use Identity $\tan^2 x = \sec^2 x - 1$

$$I = \int (\sec^2 x - 1)\,dx = \tan x - x + C$$

EX $\int \sin^3 x\,dx$ ← Isolate $\sin x$

$$= \int \sin^2 x \cdot \sin x\,dx \quad \text{.. Use identity } \sin^2 x = 1-\cos^2 x$$

$$= \int (1-\cos^2 x)\,\underline{\sin x\,dx} \quad \text{.. Now use substitution!}$$

let $w = \cos x \Rightarrow dw = -\sin x\,dx \Rightarrow -dw = \underline{\sin x\,dx}$

$$\therefore I = -\int (1-w^2)\,dw$$

$$= -w + \frac{w^3}{3} + C = -\cos x + \frac{\cos^3 x}{3} + C$$

Note: Example above illustrates how $\int \sin^n x\,dx$, $\int \cos^n x\,dx$ Can be Computed for any odd power n: $n = 3, 5, 7, \ldots$

6 Completing the Squares:

Consider the integral $I = \int G(x)\,dx$, where the integrand $G(x)$ involves a quadratic function Ax^2+Bx+C. This integral may be computed by first completing the square of Ax^2+Bx+C.

If "A" is positive, then

$$Ax^2+Bx+C = \left(\sqrt{A}\,x + \frac{B}{2\sqrt{A}}\right)^2 + \left(C - \frac{B^2}{4A}\right)$$

EX: Find $\int \frac{dx}{x^2+4x+40}$

solution: 1st Complete squares: Here $A=1>0$, $B=4$, and $C=40$

$$x^2+4x+40 = \left(\sqrt{1}\,x + \frac{4}{2\sqrt{1}}\right)^2 + \left(40 - \frac{(4)^2}{4(1)}\right)$$

$$= (x+2)^2+36$$

$\therefore\ I = \int \frac{dx}{(x+2)^2+36}$ .. Now let $t = x+2 \Rightarrow dt = dx$

$$= \int \frac{dt}{t^2+36} = \frac{1}{6}\tan^{-1}\left(\frac{t}{6}\right)+C = \frac{1}{6}\tan^{-1}\left(\frac{x+2}{6}\right)+C$$

EX: $\int \frac{1}{\sqrt{8x-x^2}}\,dx$ .. complete squares!

Note: For $8x-x^2$, $A=-1<0$!! can't proceed!

Trick: write $8x-x^2 = -[x^2-8x]$ now: Complete squares!

$$= -[(x-4)^2-16] \leftarrow \text{check!}$$

$$= 16-(x-4)^2$$

$I = \int \frac{1}{\sqrt{16-(x-4)^2}}\,dx$ .. let $t = x-4 \Rightarrow dt = dx$

$$= \int \frac{1}{\sqrt{16-t^2}}\,dt = \sin^{-1}\left(\frac{t}{4}\right)+C = \sin^{-1}\left(\frac{x-4}{4}\right)+C$$

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Support Material #(3)

Linear Differential equations of order n

(1) The equation is of the form $L(x) = E(t)$, where
$L = a_0(t)D^n + a_1(t)D^{n-1} + \cdots + a_{n-1}(t)D + a_n(t)$, $D = \frac{d}{dt}$ is the so called differential operator. Note that in general $D^k = \frac{d^k}{dt^k}$. The k^{th} order Derivative.

(2) It will be assumed that DE is normal on an interval I, meaning:
$a_0(t), a_1(t), \ldots, a_n(t)$, and $E(t)$ are Continuous on I and $a_0(t) \neq 0$ on I.

(3) If $E(t) = 0$, equation becomes $L(x) = 0$ and is referred to as a homogeneous D.E. However if $E(t)$ is non-zero, the D.E is called non-homogeneous.

4) The Homogeneous Equation: $L(x) = 0$.
The general solution of the homogeneous eq. may be denoted by $x_h(t)$ (and sometimes called "The Complementary function". The solution is given by,

$$x_h(t) = c_1 h_1(t) + c_2 h_2(t) + \cdots + c_n h_n(t)$$

where $c_1, c_2, \ldots, c_n \in \mathbb{R}$ are arbitrary constants and the set $\{h_1, h_2, \ldots, h_n\}$ is a fundamental set, meaning: (1) $h_1(t), h_2(t), \ldots h_n(t)$ are solutions to $L(x) = 0$
(2) $h_1(t), h_2(t), \ldots h_n(t)$ are linearly independent on I.

Test for independence: If
$$W(h_1, h_2, \ldots h_n)(t_0) = \begin{vmatrix} h_1 & h_2 & \cdots & h_n \\ h_1' & h_2' & \cdots & h_n' \\ h_1^{(n-1)} & h_2^{(n-1)} & \cdots & h_n^{(n-1)} \end{vmatrix}$$
is not zero at some selected value $t_0 \in I$, then $h_1, h_2, \ldots, h_n$ are independent on I!

(5) The Non-homogeneous equation: $L(x) = E(t)$
Let $x_h(t)$ be the general solution of the associated homogeneous eq. $L(x) = 0$ and $x_p(t)$ be the solution of the non-homogeneous eq. $L(x) = E(t)$ Containing No arbitrary Constants (such a solution is called a particular solution $p(t)$,
Then general solution to non-homogeneous eq. is $x(t) = x_h(t) + x_p(t)$

An Important Special Case:

Linear Differential Equation with CONSTANT Coefficients

The eq. is of the form $a_0 \frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_{n-1} \frac{dx}{dt} + a_n x = E(t)$

where $a_0, a_1, \ldots,$ and a_n are CONSTANT real numbers. Using operator notation, we write equation in the form

$$(a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n)\, x = E(t)$$

or

$$\boxed{P(D)\, x = E(t)}$$

where $P(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$ = polynomial in D of degree n.

The Associated homogeneous Equation: $P(D)\, x = 0$

If D is replaced by say r, we get the polynomial $P(r)$ which is referred to as the characteristic polynomial. The equation $P(r) = 0$ has exactly n-roots counting multiplicities.

Case 1: If $P(r)$ has m real distinct roots $r_1, r_2, \ldots,$ and r_m; then the part of x_h associated with these m roots is

$$c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_m e^{r_m t} \qquad c_1, c_2, \ldots c_m \in \mathbb{R}$$

Case 2: If $P(r)$ has a real root r of multiplicity K, then the part of x_h associated with this root is

$$(b_1 + b_2 t + \cdots + b_k t^{K-1})\, e^{rt} \qquad b_1, b_2, \ldots b_K \in \mathbb{R}$$

Case 3: If $P(r)$ has a pair of Complex Conjugate roots $\alpha \pm i\beta$, of multiplicity ..., then the part of x_h associated with it is

$$e^{\alpha t}(a_1 \cos\beta t + a_2 \sin\beta t) \qquad a_1, a_2 \in \mathbb{R}$$

However if $\alpha \pm i\beta$ has multiplicity K, then the part of x_h associated with it is

$$e^{\alpha t}\Big((a_1 + a_2 t + \cdots + a_k t^{K-1}) \cos\beta t + (b_1 + b_2 t + \cdots + b_K t^{K-1}) \sin\beta t \Big),$$

$$a_1, a_2, \ldots a_K;\ b_1, b_2, \ldots b_K \in \mathbb{R}$$

— — — — — To be continued!

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Support Material # (4)

The non-homogeneous linear D.E with constant coefficients.

Consider the D.E $P(D)x = E(t)$.

The General solution of this equation is given by

$$x(t) = x_h(t) + x_p(t) \quad \text{or}$$
$$= H(t) + P(t)$$

where $x_h(t)$ or $H(t)$ is the general solution to the associated homogeneous equation which we have already discussed in support material # (3), and $x_p(t)$ or $P(t)$ is the so called "Particular Solution".

The method that will be used to find $P(t)$ is called "The method of undetermined coefficients" or for short "The UC method". The method can be applied only if $E(t)$ is one of the following functions listed below. These functions are called UC functions:

(1) e^{at}, $a \neq 0$ (2) $\sin bt$ or $\cos bt$, $b \neq 0$

(3) A polynomial in t of degree n, say $a_0 + a_1 t + \cdots + a_n t^n$

(4) A product of the functions (1) — (3) above

Note: $E(t)$ could be also a linear combination of functions (1) — (4).

Ex: Method can be applied to

(1) $E(t) = e^{7t}$ (2) $E(t) = t^4 + 7e^{-2t}$

(3) $E(t) = t\sin 2t - 9 + e^t$, ... etc.

Defn.: The UC set: let f be a UC function type (1) — (3). The UC set of "f" is the set S that contains f together with all of its different Derivatives (up to a constant Multiple!

The following table summarizes the UC sets for various UC functions

The UC function	The Corresponding UC set S
e^{at}	$\{e^{at}\}$
$\cos bt$	$\{\sin bt, \cos bt\}$
$\sin bt$	$\{\sin bt, \cos bt\}$
polynomial of degree n: $a_0 + a_1 t + \cdots + a_n t^n$	$\{1, t, t^2, \ldots, t^n\}$

Note : UC set of a product

Consider the UC function $t^2 \sin 3t$. This function is a product of t^2 " with UC set $S_1 = \{1, t, t^2\}$. and $\sin 3t$ " with a UC set $\{\sin 3t, \cos 3t\}$.

The UC set for $t^2 \sin 3t$ is obtained by listing all possible products of a member of S_1 and a member of S_2.

$\therefore$ The UC set for $t^2 \sin 3t$ is

$$S = \{\sin 3t, \cos 3t, t\sin 3t, t\cos 3t, t^2 \sin 3t, t^2 \cos 3t\}$$

$S_1 = \{1, t, t^2\}$

$S_2 = \{\sin 3t, \cos 3t\}$

(form all possible products!).

Defn. : A Revised UC set

Let H be the set of all solutions to the homogeneous D.E $P(D)x = 0$, and let S be a UC set. If H and S have a member in common, we must multiply each member of S by t^K, where K is the smallest positive integer so that resulting set S', and H have no members in common. The new set S' is then so called Revised Set.

We now outline the UC method for finding a particular solution $x_p(t)$ for $P(D)x = E(t)$ where $E(t)$ is a l.c. of the "m" UC functions $f_1, f_2, \ldots,$ and f_m.

The steps:

(1) Find the General solution $x_h(t)$ to associated homog. eq. and hence write down the set H of all linearly independent solutions

(2) Determine the UC sets $S_1, S_2, \ldots, S_m$ for $f_1, f_2, \ldots, f_m$ respectively.

(3) Revise all sets that need to be revised and leave the others unchanged.. Now you have a collection of "m" revised / unrevised sets.

(4) write x_p as l.c of sets of step (3). Use unknown coefficients $K_1, K_2, \ldots$ etc.

(5) Determine these unknown coefficients by substituting $x_p, x_p', \ldots$ etc. into D.E.

An illustration: Find general solution to

$$x'' - 2x' - 3x = 2e^t - 10\sin t$$

First: $x_h(t)$: consider $(D^2 - 2D - 3)x = 0$

$$r^2 - 2r - 3 = 0 \Rightarrow r = -1, 3 \text{ (Simple)}$$

$$x_h(t) = c_1 e^{-t} + c_2 e^{3t}$$

$\therefore$ e^{-t}, e^{3t} are the linearly indep. solutions $\Rightarrow H = \{e^{-t}, e^{3t}\} \ldots (*)$

Next: $x_p(t)$: R.H.S $= E(t) = 2e^t - 10\sin t$

$\therefore$ E is l.c. of e^t, $\sin t$

The UC sets are $S_1 = \{e^t\}$, $S_2 = \{\sin t, \cos t\}$

Note that neither S_1 nor S_2 have a member in common with H.

It follows that neither sets need to be revised!

Now, x_p = l.c of members of S_1, S_2

$$\therefore x_p = K_1 e^t + K_2 \sin t + K_3 \cos t \quad \ldots (1)$$

To Find K_1, K_2, K_3, we need

$$x_p' = K_1 e^t + K_2 \cos t - K_3 \sin t \quad \ldots (2)$$

and $$x_p'' = K_1 e^t - K_2 \sin t - K_3 \cos t \quad \ldots (3)$$

Sub. x_p, x_p', and x_p'' into D.E $x'' - 2x' - 3x = 2e^t - 10 \sin t$ to get,

$$(K_1 e^t - K_2 \sin t - K_3 \cos t) - 2(K_1 e^t + K_2 \cos t - K_3 \sin t)$$
$$- 3(K_1 e^t + K_2 \sin t + K_3 \cos t) = 2e^t - 10 \sin t$$

Group like terms:

$$-4K_1 e^t + (2K_3 - 4K_2) \sin t + (-4K_3 - 2K_2) \cos t = 2e^t - 10 \sin t + 0 . \cos$$

<u>Compare</u>

$$-4K_1 = 2 \Rightarrow K_1 = -\tfrac{1}{2}$$

$$\left.\begin{aligned} 2K_3 - 4K_2 &= -10 \\ -4K_3 - 2K_2 &= 0 \end{aligned}\right\} \text{solve: } K_2 = 2, \quad K_3 = -1$$

$$\therefore x_p = K_1 e^t + K_2 \sin t + K_3 \cos t$$
$$= -\tfrac{1}{2} e^t + 2 \sin t - \cos t$$

General solution $x(t) = x_h + x_p$

$$= c_1 e^{-t} + c_2 e^{3t} - \tfrac{1}{2} e^t + 2 \sin t - \cos t, \quad c_1, c_2 \in \mathbb{R}$$

Ex : Let $P(D)x = E$ be a linear equation with constant coefficients. (5

The characteristic polynomial of the associated homogeneous eq. $P(D)x = 0$ has the four roots $\{0, 0, -4, 5\}$.

(a) what is the general solution to associated homogeneous eq.?

(b) what form a particular solution x_p have if

$$E(t) = 9 + 3e^{2t}\sin 5t + 4t^2 e^{-4t} \text{ (Do not solve for coefficients!)}$$

Solution : (a) $r = 0, 0 \Rightarrow e^{0t}, te^{0t}$ are solutions

$r = -4 \Rightarrow e^{-4t}$ is a solution

$r = 5 \Rightarrow e^{5t}$ is a solution

$$\therefore X_h(t) = c_1 e^{0t} + c_2 t e^{0t} + c_3 e^{-4t} + c_4 e^{5t}$$

$$\overset{\text{or}}{=} c_1 \cdot 1 + c_2 t + c_3 e^{-4t} + c_4 e^{5t} \quad c_1, c_2, c_3, c_4 \in \mathbb{R}$$

Note : The set H of all independent solutions is

$$H = \{1, t, e^{-4t}, e^{5t}\} \text{ - - - } (*)$$

(b) $E = 9 + 3e^{2t}\sin 5t + 4t^2 e^{-4t}$

E is l.c of $9 = 9t^0$ (poly. of degree 0)

$e^{2t}\sin 5t$, and $t^2 e^{-4t}$

The three UC sets are

$$S_1 = \{1\}, \quad S_2 = \{e^{2t}\sin 5t, e^{2t}\cos 5t\},$$

and $S_3 = \{e^{-4t}, te^{-4t}, t^2 e^{-4t}\}$

which of these sets need to be revised?

Indeed. the sets S_2, H have no members in common $\Rightarrow S_2$ need no revision!

However S_1 and H have "1" in common $\Rightarrow S_1$ need to be revised and that revised set is $S' = \{t^2\}$

Similarly S_3 need to be revised and that $S_3' = \{te^{-4t}, t^2e^{-4t}, t^3e^{-4t}\}$

$\therefore x_p = $ l.c of members of $S_1', S_2, S_3' = K_1 t^2 + K_2 e^{2t}\sin 5t + K_3 e^{2t}\sin 5t$

The method of variation of parameters

Consider the linear 2nd. order D.E

$$\left(a_0(t)D^2 + a_1(t)D + a_2(t)\right)x = E(t)$$

Assume that the general solution to the associated homogeneous equation is known, say

$$H(t) = c_1 h_1(t) + c_2 h_2(t)$$

where c_1, and c_2 are arbitrary constant. let us seek a particular solution of the form

$$p(t) = c_1(t)\, h_1(t) + c_2(t)\, h_2(t)$$

where $c_1(t)$, and $c_2(t)$ are now functions of t. These functions are determined by the equations

$$c_1'(t) = \frac{\begin{vmatrix} 0 & h_2(t) \\ \frac{E(t)}{a_0(t)} & h_2'(t) \end{vmatrix}}{\begin{vmatrix} h_1(t) & h_2(t) \\ h_1'(t) & h_2'(t) \end{vmatrix}} \quad , \quad c_2'(t) = \frac{\begin{vmatrix} h_1(t) & 0 \\ h_1'(t) & \frac{E(t)}{a_0(t)} \end{vmatrix}}{\begin{vmatrix} h_1(t) & h_2(t) \\ h_1'(t) & h_2'(t) \end{vmatrix}}$$

<u>Note</u>: The denominator determinant is $W(h_1, h_2)(t)$ "The wronskian!"

Now, c_1, and c_2 can be found and adding zero as a constant of integration. OR if constants K_1, and K_2 are added instead, you would get general solution directly!

<u>EX</u>: solve, $t^2x'' - 6tx' + 10x = 3t^4 + 6t^3$ given that $h_1(t) = t^2$, and $h_2(t) = t^5$ are linearly independent solutions of associated homog. eq.

Solution: Here $H(t) = c_1h_1 + c_2h_2 = c_1t^2 + c_2t^5$

Assume $p(t) = c_1(t)\,t^2 + c_2(t)\,t^5$

(Verify!)

$$\therefore\ c_1'(t) = -(t+2) \quad , \quad c_2'(t) = \frac{t+2}{t^3}$$

$$\therefore\ c_1(t) = -\tfrac{1}{2}t^2 - 2t \qquad c_2(t) = -\frac{1}{t} - \frac{1}{t^2}$$

$$\therefore\ p(t) = \left(-\tfrac{1}{2}t^2 - 2t\right)t^2 + \left(-\frac{1}{t} - \frac{1}{t^2}\right)t^5 = -\frac{3}{2}t^4 - 3t^2$$

$\therefore$ General soln. $x(t) = c_1t^2 + c_2t^5 - \frac{3}{2}t^4 - 3t^2$.

AMAT 307

Support Material sheet (5)

Linear systems of Differential Equations:

A system of "n" linear first order differential equations in the unknowns $x_1, x_2, \ldots,$ and x_n is of the form

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + f_1(t)$$
$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + f_2(t)$$
$$\vdots$$
$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + f_n(t)$$

we shall assume that the coefficients a_{ij} $i,j = 1,2,\ldots n$ are constant real numbers, and the functions $f_1(t), f_2(t), \ldots, f_n(t)$ are continuous for all t in some interval I.

Matrix form of linear system:

If we let $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\bar{E}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$, and $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \overset{\text{or}}{=} (a_{ij})_{i,j}$

the system above takes the simple form

$$D\bar{x} = A\bar{x} + \bar{E}(t)\text{, where } A \text{ is an } n\times n \text{ matrix, } D \equiv \frac{d}{dt}$$

If $\bar{E}(t) \equiv 0$, the system is said to be homogeneous. Otherwise it is a non-homogeneous system.

Note that $D\bar{x} = A\bar{x}$ is the homogeneous system associated with (or related to) the non-homogeneous system $D\bar{x} = A\bar{x} + \bar{E}(t)$

General solution: Consider D.E $D\bar{x} = A\bar{x} + \bar{E}(t)$ ---(*)

If $\bar{x}_h(t)$ is a formula that describes all solutions to the associated homogeneous equation $D\bar{x} = A\bar{x}$, and $\bar{x}_p(t)$ is a particular solution to non-homogeneous system $D\bar{x} = A\bar{x} + \bar{E}(t)$, then $\bar{x}(t) = \bar{x}_h(t) + \bar{x}_p(t)$ is the so called General solution to (*)

Note: $\bar{x}_h(t)$ usually contains "n" arbitrary constant. However $\bar{x}_p(t)$ contains no arbitrary constants whatsoever!

(I) The Homogeneous Equation

Consider the homogeneous equation $D\bar{x} = A\bar{x}$ --- (*)

Def. : Fundamental set

The collection of "n" vectors $\bar{h}_1(t), \bar{h}_2(t), \dots, \bar{h}_n(t)$ is said to generate the General solution of (*) (or Simply a fundamental set) if :

(1) $\bar{h}_i(t)$, $i=1,2,\dots n$ are Solutions to D.E (*)

(2) $\bar{h}_1(t), \bar{h}_2(t), \dots,$ and $\bar{h}_n(t)$ are linearly independent on some interval I

Note: Test for Independence: let $[\bar{h}_1(t)\ \bar{h}_2(t) \cdots \bar{h}_n(t)]$ be the $n \times n$ matrix whose Column vectors are $\bar{h}_i(t)$ $i=1,2,\dots n$.

Define the Wronskian of $h_1, h_2, \dots,$ and h_n by,

$$W(\bar{h}_1, \bar{h}_2, \dots \bar{h}_n)(t) = \det[\bar{h}_1\ \ \bar{h}_2 \cdots \bar{h}_n]$$

The set of vectors $\{\bar{h}_1, \bar{h}_2, \dots, \bar{h}_n\}$ is independent on I provided,

$$W(\bar{h}_1, \bar{h}_2, \dots \bar{h}_n)(t_0) \not\equiv 0 \text{ for any } t_0 \in I.$$

Ex : Given the system

$$\frac{dx}{dt} = 5x - 2y,$$
$$\frac{dy}{dt} = x + 2y$$

(a) write system in matrix form

(b) verify that $\bar{h}_1(t) = e^{3t}\begin{bmatrix}1\\1\end{bmatrix}$, $\bar{h}_2(t) = e^{4t}\begin{bmatrix}2\\1\end{bmatrix}$ generate the general solution to system.

Solution : (a) let $\bar{x} = \begin{bmatrix}x\\y\end{bmatrix}$, $A = \begin{bmatrix}5&-2\\1&2\end{bmatrix}$ .. hence system becomes

$$D\bar{x} = A\bar{x} \quad \text{where } A = \begin{bmatrix}5&-2\\1&2\end{bmatrix}, \ D \equiv \frac{d}{dt}$$

(b) let $h_1(t) = e^{3t}\begin{bmatrix}1\\1\end{bmatrix} = \bar{x}$

L.H.S of D.E $= D\bar{x} = D\, e^{3t}\begin{bmatrix}1\\1\end{bmatrix} = 3e^{3t}\begin{bmatrix}1\\1\end{bmatrix}$

R.H.S of D.E $= A\bar{x} = \begin{bmatrix}5&-2\\1&2\end{bmatrix}e^{3t}\begin{bmatrix}1\\1\end{bmatrix} = e^{3t}\begin{bmatrix}5&-2\\1&2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = e^{3t}\begin{bmatrix}3\\3\end{bmatrix} = 3e^{3t}\begin{bmatrix}1\\1\end{bmatrix}$

Clearly L.H.S = R.H.S $\Rightarrow \bar{h}_1$ is a Solution!

Similarly you can verify $\bar{h}_2$ is a solution.

Are they Independent on $(-\infty, \infty)$? Indeed $W(h_1, h_2)(0) = \det[\bar{h}_1\ \ \bar{h}_2]\Big|_{t=0}$

$= \det\begin{bmatrix}e^{3t} & 2e^{4t}\\ e^{3t} & e^{4t}\end{bmatrix}\Big| = \begin{vmatrix}1&2\\1&1\end{vmatrix} \neq 0$ .. yes! $\Rightarrow \{\bar{h}_1, \bar{h}_2\}$ is Fundamental set

Note: If $\bar{h}_1(t), \bar{h}_2(t), \ldots,$ and $\bar{h}_n(t)$ is a collection which generates the general solution to $D\bar{x} = A\bar{x}$, then that general solution $\bar{x}$ is a linear combination of these "n" vectors, namely

$$\bar{x}(t) = c_1\bar{h}_1(t) + c_2\bar{h}_2(t) + \cdots + c_n\bar{h}_n(t).$$

In the preceding example, the general solution is given by

$$\bar{x}(t) = c_1\bar{h}_1(t) + c_2\bar{h}_2(t)$$
$$= c_1 e^{3t}\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{4t}\begin{bmatrix}2\\1\end{bmatrix} \quad \cdots\cdots (*)$$

Defn.: An I.V.P for system:

It consists of (1) D.E: $D\bar{x} = A\bar{x} + \bar{E}(t)$

(2) An initial condition $\bar{x}(t_0) = \bar{x}_0$

EX: Find solution to I.V.P $D\bar{x} = A\bar{x}$, $A = \begin{bmatrix}5 & -2\\1 & 2\end{bmatrix}$, $\bar{x}(0) = \begin{bmatrix}1\\3\end{bmatrix}$

solution: This is the preceding Example where we found ((*) above)

that $\bar{x}(t) = c_1 e^{3t}\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{4t}\begin{bmatrix}2\\1\end{bmatrix}$

need to find c_1, c_2 !! by applying $\bar{x}(0) = \begin{bmatrix}1\\3\end{bmatrix}$ to get

$$\begin{bmatrix}1\\3\end{bmatrix} = c_1 e^0\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^0\begin{bmatrix}2\\1\end{bmatrix}$$
$$\begin{bmatrix}1\\3\end{bmatrix} = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}c_1 + 2c_2\\c_1 + c_2\end{bmatrix}$$

$\Rightarrow \begin{matrix}1 = c_1 + 2c_2\\3 = c_1 + c_2\end{matrix}$ solve: $c_1 = 5$, $c_2 = -2$

$\therefore$ solution to I.V.P is $\bar{x}(t) = 5e^{3t}\begin{bmatrix}1\\1\end{bmatrix} - 2e^{4t}\begin{bmatrix}2\\1\end{bmatrix}$

$$\begin{bmatrix}x\\y\end{bmatrix} \overset{\text{or}}{=} \begin{bmatrix}5e^{3t} - 4e^{4t}\\5e^{3t} - 2e^{4t}\end{bmatrix}$$

Eigenvalues and Eigenvectors of a square matrix

Def: Let A be a square matrix of order n. The scalar λ (real or Complex) is called an Eigenvalue of A if there exists a non-zero vector $\bar{v}$ in $\mathbb{R}^n$ such that $A\bar{v} = \lambda\bar{v}$ or equivalently

$$(A-\lambda I)\bar{v} = \bar{0}$$

The non-zero vector $\bar{v}$ is an Eigenvector Corresponding to λ.

The system $(A-\lambda I)\bar{v} = \bar{0}$ has a non-Trivial Solution iff:

$$\boxed{\det(A-\lambda I) = 0}$$

Note that $\det(A-\lambda I)$ is a polynomial in λ of degree n which we call "The characteristic polynomial of A" and denote it by $C_A(\lambda)$.

The results above may be summarized in the following Powerful Theorem!

Theorem: Let A be a square matrix of order n

(1) The Eigenvalues of A are the solutions of the equation

$$\det(A-\lambda I) = 0$$

This equation has Exactly $\underline{\underline{n}}$ roots (Counting multiplicities!)

(2) Corresponding Eigenvectors are solutions to homogeneous system

$$(A-\lambda I)\bar{v} = \bar{0}$$

Note: About the matrix $A-\lambda I$:

This matrix Can be obtained from A by simply subtracting λ from each main-diagonal entry!

$$\therefore A-\lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$$

EX: Find Eigenvalues and Eigenvectors of each of the following matrices

(1) $A = \begin{bmatrix} 4 & 9 \\ -1 & -6 \end{bmatrix}$ (2) $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}$ (3) $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

(4) $A = \begin{bmatrix} 3 & -10 \\ 1 & 1 \end{bmatrix}$ (5) $A = \begin{bmatrix} -2 & 1 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

solutions: Refer to class notes!!

<u>Answers:</u> (1) $\lambda_1 = -5$, $\bar{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; $\lambda_2 = 3$, $\bar{v}_2 = \begin{bmatrix} 9 \\ -1 \end{bmatrix}$

(2) $\lambda_1 = -1$, $\bar{v}_1 = \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix}$, $\lambda_2 = 1$, $\bar{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, $\lambda_3 = 4$, $\bar{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(3) $\lambda_{1,2} = 0, 0$, $\bar{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\lambda_3 = 6$, $\bar{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

(4) $\lambda_{1,2} = 2 \pm 3i$, $\bar{v}_{1,2} = \begin{bmatrix} 10 \\ 1 \mp 3i \end{bmatrix}$

(5) $\lambda_1 = -2$, $\bar{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\lambda_{2,3} = 1, 1$, $\bar{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ "short!!"

<u>Solution to a homogeneous system:</u>

Consider the homogeneous system $D\bar{x} = A\bar{x}$ ---- $(*)$

<u>Theorem:</u> let $\bar{h}(t) = e^{\lambda t}\bar{v}$, $\bar{v}$ is a constant vector. Then $\bar{h}(t)$ is a solution to D.E $(*)$ iff λ is an Eigenvalue of A and $\bar{v}$ is a corresponding Eigenvector!

Therefore the problem of solving $(*)$ reduces to the problem of finding Eigenvalues / Eigenvectors of A!!

<u>The following three Cases will be Examined:</u>

<u>Case (1):</u> If the Eigenvalues of A, say $\lambda_1, \lambda_2, \ldots$ and λ_n are real and distinct and if $\bar{v}_1, \bar{v}_2, \ldots,$ and $\bar{v}_n$ are Corresponding Eigenvectors, the general solution is given by L.C of $\bar{h}_i(t) = e^{\lambda_i t}\bar{v}_i$, $i = 1, 2, \ldots n$

$$\therefore \quad \bar{x}(t) = c_1 e^{\lambda_1 t}\bar{v}_1 + c_2 e^{\lambda_2 t}\bar{v}_2 + \cdots + c_n e^{\lambda_n t}\bar{v}_n .$$

<u>Case (2):</u> If A has a pair of Complex Conjugate Eigenvalues $\lambda = \alpha \pm i\beta$, $\alpha, \beta \in \mathbb{R}$ of multiplicity one (non-repeated), then choose one of them say $\lambda = \alpha + i\beta$ and determine a Corresponding Eigenvector $\bar{v}$. The functions $\bar{h}_1(t) = Re(e^{\lambda t}\bar{v})$, $\bar{h}_2(t) = Im(e^{\lambda t}\bar{v})$ are linearly independent solutions to D.E.

<u>Case (3):</u> <u>If A has an Eigenvalue λ of multiplicity K</u>

Case (3): If A has an Eigenvalue λ of multiplicity K.

Assume that λ has "p" linearly independent Eigenvectors. Now consider two subcases: (1) $p = K$; and (2) $p < K$.

In subcase (1): $p = K$ (that is the number of independent Eigenvectors is the same as the multiplicity), let these Eigenvectors be $\bar{v}_1, \bar{v}_2, \ldots,$ and $\bar{v}_K$. Hence the functions $\bar{h}_1(t) = e^{\lambda t}\bar{v}_1$, $\bar{h}_2(t) = e^{\lambda t}\bar{v}_2$, ..., and $h_K(t) = e^{\lambda t}\bar{v}_K$ are independent solution to DE $D\bar{x} = A\bar{x}$. Therefore we are "done!"

EX: $D\bar{x} = A\bar{x}$, $A = \begin{bmatrix} 3 & +1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix}$

The Eigenvalues of A are $\lambda = 1, 2,$ and 2. Find general solution to D.E.

Solution: $A - \lambda I = \begin{bmatrix} 3-\lambda & 1 & -1 \\ 1 & 3-\lambda & -1 \\ 3 & 3 & -1-\lambda \end{bmatrix}$

For $\lambda = 1$ (Simple): solve $(A - \lambda I)\bar{v} = \bar{0}$, $\lambda = 1$

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ 3 & 3 & -2 \end{bmatrix} \bar{v} = \bar{0}$$

$$\therefore \bar{v} = \begin{bmatrix} +\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} \\ +\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

Note: (use cross product method only for Simple Eigenvalues!)

is an Eigenvector

$\therefore$ 1st solution $\bar{h}_1(t) = e^{1t}\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

For $\lambda = 2$ (Multiplicity Two): we may get one or Two independent Eigenvectors -- let us find out!

solve $\quad (A-\lambda I)\bar{v} = \bar{0}, \quad \lambda = 2$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 3 & 3 & -3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \bar{0}$$

clearly system reduces to the single equation

$$a + b - c = 0$$

(one equation in three unknowns $\Rightarrow$ $3-1 = 2$- parameter family)

let $b = t$, $c = s$ -- hence $a = -b + c = -t + s$

$$\therefore \bar{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -t+s \\ t \\ s \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ There are Two independent Eigenvectors: $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(subcase (1)!)

$\therefore$ The other two solutions of D.E are

$$\bar{h}_2(t) = e^{2t}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \qquad \bar{h}_3(t) = e^{2t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ general solution

$$\bar{x}(t) = \text{l.c of } h_i\text{'s}$$
$$= c_1 e^{t}\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

<u>sub case (2)</u> : Very Very Important

Here $p < K$, that is the number of independent Eigenvectors is Less than the multiplicity.

let us illustrate the case where λ has multiplicity Two. The generalization is straight forward.

let λ be an Eigenvalue of multiplicity $K = 2$. Assume that λ has <u>only</u> one independent Eigenvector $\bar{\alpha}$. Therefore a first solution is given by $\bar{h}_1(t) = e^{\lambda t}\bar{\alpha}$.

A second solution $\bar{h}_2(t)$ is given by

$$\bar{h}_2(t) = (t\bar{\alpha} + \bar{\beta})e^{\lambda t}$$

where $\bar{\beta}$ is a solution to the non-homogeneous system

$$(A-\lambda I)\bar{\beta} = \bar{\alpha}$$

<u>EX</u>: Consider $D\bar{x} = A\bar{x}$, $A = \begin{bmatrix} -2 & 1 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

A has Eigenvalue $\lambda = -2$ (Simple) with a corresponding Eigen vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and a Eigenvalue $\lambda = 1$ (multiplicity Two) but only one corresponding Eigenvector $\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$. Find general solution.

Also find solution with $\bar{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

<u>Solution</u>: For $\lambda = -2$, a solution is $e^{-2t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

For $\lambda = 1$ (Multiplicity Two), we need <u>Two</u> independent solutions.

Let $\bar{\alpha}$ = The only Eigenvector for $\lambda = 1$ = $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$

$\therefore$ one solution is $e^{1t}\bar{\alpha} = e^{1t}\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$

The other is given by $e^{1t}[t\bar{\alpha} + \bar{\beta}]$ where β is determined by $(A-\lambda I)\bar{\beta} = \bar{\alpha}$, $\lambda = 1$

$$\begin{bmatrix} -2-1 & 1 & 5 \\ 0 & 1-1 & 2 \\ 0 & 0 & 1-1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & 5 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

$\Rightarrow \left.\begin{array}{l} -3a + b + 5c = 1 \\ 2c = 3 \end{array}\right\}$ Two eqs. in three unknowns $\Rightarrow$ one parameter family

$\Rightarrow$ one choice (can't be c .. it is either a or b)

Know $2c = 3 \Rightarrow c = \frac{3}{2}$

Eq. (1) $\Rightarrow -3a + b + \frac{15}{2} = 1$. let a = say $\frac{5}{2}$

$\therefore b = 1$

$$\therefore\ \bar{\beta} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ 1 \\ \frac{3}{2} \end{bmatrix}$$

There for the last solution is $e^{t}\left[t\bar{\alpha} + \bar{\beta}\right]$

$$= e^{t}\left[t\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{5}{2} \\ 1 \\ \frac{3}{2} \end{bmatrix}\right]$$

$$\overset{\text{or}}{=} e^{t}\begin{bmatrix} t + \frac{5}{2} \\ 3t + 1 \\ 0t + \frac{3}{2} \end{bmatrix}$$

General solution

$$\bar{X}(t) = c_1 e^{-2t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + c_3 e^{t}\begin{bmatrix} t + \frac{5}{2} \\ 3t + 1 \\ 0t + \frac{3}{2} \end{bmatrix}$$

$c_1, c_2, c_3 \in \mathbb{R}$

Next : apply condition $\bar{X}(0) = \bar{0}$

obviously $c_1 = c_2 = c_3 = 0$

$\therefore\ \bar{X}(t) = \vec{0}$

END

TESTS FOR CONVERGENCE OF A SERIES OF POSITIVE TERMS

Defn : The series $\sum_{n=1}^{\infty} a_n$ is said to be a Series of positive terms if $a_n > 0$ for all $n \geq n_0$ for some positive integer n_0. The Convergence or divergence of the series may be determined by applying an approperiate test from the following list: "FIVE in list"

$\boxed{1}$ The Divergence Test (OR: The nth term Test)

If $\lim_{n\to\infty} a_n = \ell$ where $\ell \neq 0$, the series diverges!

However if $\lim_{n\to\infty} a_n = 0$, the Test Fails, that is the series may converge or may diverge.

EX $\sum_{n=1}^{\infty} \cos\left(\frac{2}{n}\right)$. Here $a_n = \cos\left(\frac{2}{n}\right)$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos\left(\frac{2}{n}\right) = \cos\left(\frac{2}{\infty}\right) = \cos 0 = 1 \neq 0.$$

$\Rightarrow$ series Diverges!

EX $\sum_{n=1}^{\infty} \frac{n+5}{n^3+4}$. Here $a_n = \frac{n+5}{n^3+4}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+5}{n^3+4} = \lim_{n\to\infty} \frac{n}{n^3} = \lim_{n\to\infty} \frac{1}{n^2} = 0$$

$\therefore$ Test Fails!

In fact: we shall show later that the series converges!

EX $\sum_{n=1}^{\infty} n \sin\left(\frac{3}{n}\right)$. Here $a_n = n \sin \frac{3}{n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n \sin\left(\frac{3}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{3}{n}\right)}{\frac{1}{n}} \leftarrow \text{apply L'H rule}$$

$$= \lim_{n\to\infty} \frac{-\frac{3}{n^2}\cos\left(\frac{3}{n}\right)}{-\frac{1}{n^2}} = 3 \lim_{n\to\infty} \cos\left(\frac{3}{n}\right) = 3\cdot 1 = 3 \neq 0$$

$\Rightarrow$ series diverges!

$\boxed{2}$ **The Ratio Test** : consider series of positive terms $\sum a_n$.

(2)

let $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$. Then

(1) Series Converges if $0 \le L < 1$,

(2) series diverges if $L > 1$,

(3) Test Fails if $L = 1$.

EX $\sum \frac{3^n}{n!}$. Here $a_n = \frac{3^n}{n!}$, $a_{n+1} = \frac{3^{n+1}}{(n+1)!} = \frac{3^n \cdot 3}{(n+1)\, n!}$

$$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^n \cdot 3}{(n+1)\, n!} \cdot \frac{n!}{3^n} = \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1$$

$\Rightarrow$ series Converges.

EX $\sum \frac{n^n}{n!}$. Here $a_n = \frac{n^n}{n!}$, $a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!} = \frac{\cancel{(n+1)}\,(n+1)^n}{\cancel{(n+1)}\; n!}$

$$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^n}{\cancel{n!}} \, \frac{\cancel{n!}}{n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$$

$$= e > 1$$

$\Rightarrow$ series diverges

EX $\sum \frac{n+1}{n^2+2}$. Here $a_n = \frac{n+1}{n^2+2}$, $a_{n+1} = \frac{(n+1)+1}{(n+1)^2+2}$

$$= \frac{n+2}{n^2+2n+3}$$

$$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+2)}{(n^2+2n+3)} \cdot \frac{(n^2+2)}{(n+1)}$$

$$= \lim_{n\to\infty} \frac{L.T}{L.T} = \lim_{n\to\infty} \frac{n^3}{n^3} = 1$$

$\Rightarrow$ Test Fails!

$\boxed{3}$ The nth Root Test: consider $\sum a_n$, $a_n > 0$

let $\lim_{n\to\infty} \sqrt[n]{a_n} = L$, then

series converges if $0 \le L < 1$, diverges if $L > 1$. However if $L = 1$, the Test Fails.

EX $\sum \left(1+\frac{4}{n}\right)^{n^2}$. Here $a_n = \left(1+\frac{4}{n}\right)^{n^2} \Rightarrow \sqrt[n]{a_n} = \left(1+\frac{4}{n}\right)^{\frac{n^2}{n}}$
$= \left(1+\frac{4}{n}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(1+\frac{4}{n}\right)^n = e^4 > 1$$
$\Rightarrow$ Series diverges.

Note: standard limit: $\lim_{n\to\infty} \left(1+\frac{\alpha}{n}\right)^n = e^\alpha$.

EX $\sum \frac{n^4}{2^n}$. Here $a_n = \frac{n^4}{2^n} \Rightarrow \sqrt[n]{a_n} = \frac{\sqrt[n]{n^4}}{2}$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{\sqrt[n]{n^4}}{2} = \frac{1}{2} < 1 \Rightarrow \text{series converges.}$$

Note: standard limit:
$$\lim_{n\to\infty} \sqrt[n]{n^p} = 1 \quad \text{for all } p \in \mathbb{R}.$$

EX $\sum \left(1+\frac{1}{3n}\right)^n$. Here $a_n = \left(1+\frac{1}{3n}\right)^n$
$\Rightarrow \sqrt[n]{a_n} = \left[\left(1+\frac{1}{3n}\right)^n\right]^{\frac{1}{n}} = 1+\frac{1}{3n}$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(1+\frac{1}{3n}\right) = 1+0 = 1$$
$\Rightarrow$ Test Fails

$\boxed{4}$ <u>The Integral Test</u> : Consider $\sum a_n$, $a_n > 0$

Let $f(n) = a_n$ and consider the function $f(x)$ obtained by replacing "n" by "x".

If (1) $f(x)$ is <u>positive</u> and is <u>continuous</u> for $x \geq x_0$

(2) $f(x)$ is strictly decreasing for $x \geq x_0$

Then the series and the improper integral $J = \int_{x_0}^{\infty} f(x)\,dx$

either both converge or both diverge.

<u>Remark</u> : The Improper Integral J converges if $J < \infty$ and diverges if $J = +\infty$.

Ex $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$. Here $a_n = \frac{1}{n(\ln n)^3} = f(n)$

$\therefore f(x) = \frac{1}{x(\ln x)^3}$, $x \geq 2$

clearly : (1) $f(x) = \frac{1}{x(\ln x)^3}$ is positive and is continuous for $x \geq 2$

(2) $f'(x) = -\left[\frac{x \cdot 3(\ln x)^2 \cdot \frac{1}{x} + (\ln x)^3}{x^2(\ln x)^6}\right] < 0$ for $x \geq 2$

$\Rightarrow f$ is strictly decreasing for $x \geq 2$

$\therefore$ Can apply Integral Test.

Now, consider $J = \int_{x_0}^{\infty} f(x)\,dx = \int_{2}^{\infty} \frac{1}{x(\ln x)^3}\,dx = \int_{2}^{\infty} \frac{1}{(\ln x)^3} \cdot \left(\frac{1}{x}\,dx\right)$

let $w = \ln x \Rightarrow dw = \frac{1}{x}\,dx$

$\therefore J = \int \frac{1}{w^3}\,dw = -\frac{1}{2}\,\frac{1}{w^2} = -\frac{1}{2} \cdot \frac{1}{(\ln x)^2}\Big|_2^{\infty}$

$= -\frac{1}{2}\left[\cancelto{0}{\frac{1}{(\ln \infty)^2}} - \frac{1}{(\ln 2)^2}\right] < \infty$

$\Rightarrow J$ converges, and hence so does the Series!

5 The Limit Comparison Test

Let $\sum a_n$, $\sum b_n$ be series of positive terms and let

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

(1) If L is not zero or infinity, then $\sum a_n$, $\sum b_n$ have Same behaviour: Either both converge or both diverge.

☆ (2) If $L = 0$ and $\sum b_n$ converge, so does $\sum a_n$.

☆ (3) If $L = +\infty$ and $\sum b_n$ diverge, so does $\sum a_n$.

Remarks (1) The Limit Comparison Test requires that $\sum b_n$ to be of Known behaviour (Convergent / Divergent).

(2) The most popular Comparison series $\sum b_n$ are

(a) The p-series $\sum \frac{1}{n^p}$, $p>0$. The p-series converges if $p>1$ and diverges if $0<p\leq 1$.

(b) The Geometric Series $\sum ar^{n-1}$. The series converges if $-1<r<1$, and diverges if either $r\geq 1$ or $r\leq -1$.

(3) We recommend that the two cases ☆ be avoided. This can be done by choosing $\sum b_n$ properly! (see (4)!)

(4) In order to choose $\sum b_n$ properly, the following principle may be applied:

The Informal principle:

The numerator / Denominator of a_n may be replaced by its "Dominant term". For instance if numerator / denominator involves a polynomial expression, this polynomial expression is replaced by its "leading term". Also for expression such as $3^n + n^2$, the n^2 may be dropped and hence $3^n + n^2$ may be replaced by 3^n, ... etc.

EX: $\sum \frac{n^3+2n^2+10}{\sqrt{n^8+2n^7+5}}$

Here $a_n = \frac{n^3+2n^2+10}{\sqrt{n^8+2n^7+5}}$. Choose $b_n = \frac{L.T}{L.T} = \frac{n^3}{\sqrt{n^8}} = \frac{1}{n}$

Now, $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3+2n^2+10}{\sqrt{n^8+2n^7+5}} \cdot \frac{n}{1}$

$= \lim_{n\to\infty} \frac{n^3}{\sqrt{n^8}} \cdot \frac{n}{1} = 1 \neq 0 \text{ or } \infty$

$\therefore$ $\sum a_n$, $\sum b_n = \sum \frac{1}{n}$ have <u>Same</u> behaviour!

But $\sum \frac{1}{n}$ is a p-series with $p=1$ hence it diverges, and hence so does our series $\sum a_n$!

EX $\sum \frac{4^n+1}{3^n+n}$. Here $a_n = \frac{4^n+1}{3^n+n}$.

Note that 3^n is the dominant term of 3^n+n since 3^n is very large compared with n. Hence we may choose $b_n = \frac{4^n}{3^n}$

Now, $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{4^n+1}{3^n+n} \cdot \frac{3^n}{4^n}$

$= \lim_{n\to\infty} \frac{4^n+1}{4^n} \cdot \lim_{n\to\infty} \frac{3^n}{3^n+n}$

$= \lim (1+\frac{1}{4^n}) \cdot \lim_{n\to\infty} \frac{1}{1+\frac{n}{3^n}} = (1+0) \cdot \frac{1}{(1+0)}$

$= 1 \neq 0 \text{ or } \infty$

$\therefore$ $\sum a_n$, $\sum b_n = \sum \frac{4^n}{3^n}$ have same behaviour.

But $\sum b_n = \sum \frac{4^n}{3^n} = \sum (\frac{4}{3})^n$ is a geometric series with common ratio $r = \frac{4}{3} > 1$ hence it diverges. Therefore $\sum a_n$ diverges as well.

[6] The Comparison Test

Let $\sum a_n, \sum b_n$ be series of positive terms.

(1) If $a_n \leq b_n$ for $n \geq n_0$ and if $\sum b_n$ converges, then $\sum a_n$ converges as well.

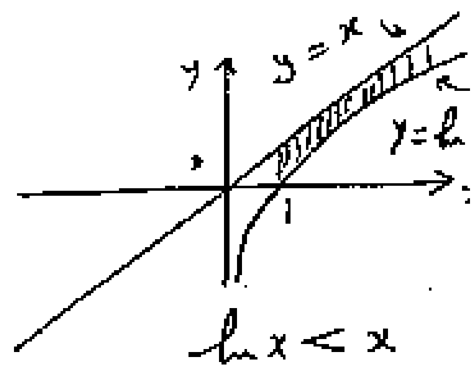
In words: If the bigger series $\sum b_n$ converges, so does the smaller one $\sum a_n$!

(2) If $a_n \geq b_n$ for $n \geq n_0$ and if $\sum b_n$ diverges, then $\sum a_n$ diverges as well

In words: If the smaller series $\sum b_n$ diverges, so does the bigger one $\sum a_n$!

Ex $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$. Here $a_n = \frac{1}{\ln n}$

clearly $\ln n < n$ (Refer to graph)



$\ln x < x$ for all $x \geq 1$

$$\therefore \underbrace{\frac{1}{\ln n}}_{a_n} > \underbrace{\frac{1}{n}}_{b_n}$$

Now, $\sum b_n = \sum \frac{1}{n}$ (the harmonic series) diverges, and hence $\sum a_n = \sum \frac{1}{\ln n}$ diverges as well.

Ex $\displaystyle\sum \frac{\sin^2(n^3)}{n^4}$.

Here $a_n = \frac{\sin^2(n^3)}{n^4}$. Note that $|\sin\theta| \leq 1, \theta \in \mathbb{R}$

$$\therefore a_n = \frac{\sin^2(n^3)}{n^4} \leq \frac{1}{n^4} = b_n$$

But $\sum b_n = \sum \frac{1}{n^4}$ (p-series with $p = 4 > 1$) and hence converges.

$$\therefore \sum \frac{\sin^2(n^3)}{n^4} \text{ converges as well.}$$

Alternating Series:

Let $a_n > 0$, $n = 1, 2, 3, \ldots$ Then a series of the form,

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

is called an alternating series.

Note: The series $-a_1 + a_2 - a_3 + \cdots = \sum (-1)^n a_n$ is also an alternating series!

EX $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ is an alternating series.

Alternating Series Test: Libentz Test

Given an alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$. The series converges if the following two conditions are met:

(1) $\lim_{n \to \infty} a_n = 0$ (2) $a_n = f(n)$ is decreasing.

Note: If condition (1) is not satisfied, the series certainly diverges. However if condition (2) is not met, the series may or may not converge. This case is beyond the scope of this course!

Ex: ① $\sum_{n=1}^{\infty} (-1)^{n-1} \sin\left(\frac{1}{n^3}\right)$, ② $\sum_{n=1}^{\infty} (-1)^{n-1} \cos\left(\frac{4}{n}\right)$

#① Here $a_n = \sin\left(\frac{1}{n^3}\right)$.

Condition 1: $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \sin\left(\frac{1}{n^3}\right) = \sin 0 = 0$ ✓

Condition 2: Let $a_n = f(n) = \sin\left(\frac{1}{n^3}\right) \Rightarrow f(x) = \sin\left(\frac{1}{x^3}\right)$

$f'(x) = -\frac{3}{x^4} \cos\left(\frac{1}{x^3}\right) < 0$, $x \geq 1 \Rightarrow f(x)$ is decreasing ✓

$\therefore$ series converges!

#② Here $a_n = \cos\left(\frac{4}{n}\right)$.

Condition 1: $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \cos\left(\frac{4}{n}\right) = \cos 0 = 1 \neq 0 \Rightarrow$ series diverges!

Summary : Tests applied to series of positive terms expect last one (Alternating Series)

TEST NAME	USE TEST IF	TEST STATEMENT(S)	TEST FAILS IF	ADDITIONAL COMMENTS
Divergence OR nth. Term Test	$\lim_{n\to\infty} a_n$ is Fairly easy to Compute	If $\lim_{n\to\infty} a_n \neq 0$ Series Diverges	$\lim_{n\to\infty} a_n = $ zero	
Ratio Test	a_n involves powers, factorial or both	If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ series Converges if $L<1$ series diverges if $L>1$	$L = 1$	1) Never use ratio Test if a_n involves only poly. Expressions. 2 $n! = n(n-1)!$ $= n(n-1)(n-2)!, ...$ etc.
nth. Root Test	a_n involves powers that depends on "n"	If $\lim \sqrt[n]{a_n} = L$ series Converges if $L<1$ series diverges if $L>1$	$L = 1$	special limits $\lim_{n\to\infty} (1+\frac{\alpha}{n})^n = e^\alpha$ $\lim_{n\to\infty} \sqrt[n]{n^p} = 1, \ p\in\mathbb{R}$
Integral Test	$\int f(x)\,dx$ is easy to Compute Note: $f(x)$ is the function obtained from a_n by replacing n by x	If ① $f(x) > 0$, $f(x)$ is Continuous for $x \geq x_0$ ② $f(x)$ is decreasing for $x \geq x_0$ series Converges if $J < \infty$ series diverges if $J = \infty$ where $J = \int_{x_0}^{\infty} f(x)\,dx$	Conditions (1), (2) are not met.	

Limit Comparison TEST	Mainly if "a_n" involve polynomial expressions or expressions of the form r^n for some real number r.	If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, $L \neq 0$ or ∞ Then either both $\sum a_n$, $\sum b_n$ Converge or both diverge	Failure Can be avoided by proper choice of Comparison series $\sum b_n$	Most popular forms of $\sum b_n$ are (1) p-series $\sum \frac{1}{n^p}$ (Converge if $p>1$), (diverge if $0 \le p < 1$) (2) Geometric series $\sum a r^{n-1}$ Converge if $-1<r<1$ diverge otherwise.
Comparison Test	No particular form! You must have a "feel" for! Require extensive experience!!	If $a_n \le b_n$, $n \ge n_0$ and if $\sum b_n$ Converges, so does $\sum a_n$ If $a_n \ge b_n$, $n \ge n_0$ and if $\sum b_n$ diverges, so does $\sum a_n$.	Your guess about series behaviour is Incorrect	Not easy to apply. Use this Test as your last option!!
Alternating series Test	series is alternating $\sum (-1)^{n-1} a_n$ or $\sum (-1)^n a_n$	If (1) $\lim_{n\to\infty} a_n = 0$ (2) $\{a_n\}$ is decreasing series Converges. If condition (1) is not met series diverges.	If Condition # (2) is not met	Failure of alternating series Test is Beyond scope of this Course!

The second-order D.E of Cauchy-Euler's Type.

A D.E of the form $a_0t^2x'' + a_1tx' + a_2x = 0$, $t>0$ or $(a_0t^2D^2 + a_1tD + a_2)x = 0$, where a_0, a_1, and a_2 are constant real number is called "a Cauchy-Euler's D.E".

The general solution can be determine by finding two linearly independent solutions of the form $x = t^r$, where r is a real or complex number. If $x = t^r$ is substituted into D.E we get a quadratic eq. in r. Three cases arise:

1. If roots r_1 and r_2 are real and distinct, the general solution is given by $x(t) = c_1 t^{r_1} + c_2 t^{r_2}$
2. If roots r_1 and r_2 are real and equal, say $r_1 = r_2 = r$, then the general solution is given by $x(t) = c_1 t^r + c_2 t^r \ln t$, $t>0$
3. If roots r_1 and r_2 are complex conjugate, say $r_{1,2} = \alpha \pm i\beta$, $\alpha, \beta \in \mathbb{R}$,

the general solution is given by

$$x(t) = c_1 t^\alpha \cos(\beta \ln t) + c_2 t^\alpha \sin(\beta \ln t)$$

Examples: In each case, find general solution

(a) $t^2x'' + 2tx' - 6x = 0$ (c) $t^2x'' - tx' + 2x = 0$

(b) $4t^2x'' + x = 0$ (d) $t^2x'' + tx' + 9x = 0$

Solution (a) $t^2x'' + 2tx' - 6x = 0$ ← Cauchy-Euler's type.

let $x = t^r \Rightarrow x' = rt^{r-1},\ x'' = r(r-1)t^{r-2}$

$$\text{D.E} \Rightarrow \quad t^2 \cdot r(r-1)t^{r-2} + 2t \cdot rt^{r-1} - 6t^r = 0 \qquad \div t^r$$

$$r(r-1) + 2r - 6 = 0$$

$r^2 + r - 6 = 0$, $(r+3)(r-2) = 0$, $r = -3, +2$

(Real, distinct)

$\therefore\ x(t) = c_1 t^{-3} + c_2 t^2$. $c_1, c_2 \in \mathbb{R}$

(b) $4t^2 x'' + x = 0$

let $x = t^r \Rightarrow x' = rt^{r-1},\ x'' = r(r-1)t^{r-2}$

D.E $\Rightarrow$ $4r(r-1) + 1 = 0$ (check!)

$4r^2 - 4r + 1 = 0,\ (2r-1)^2 = 0$

$r = \frac{1}{2}, \frac{1}{2}$ (Real & equal!)

$\therefore\ x(t) = c_1 t^{\frac{1}{2}} + c_2 t^{\frac{1}{2}} \ln t,\ t > 0.$

(c) $t^2 x'' - tx' + 2x = 0$

let $x = t^r$ .. we get,

$r(r-1) - r + 2 = 0,\ r^2 - 2r + 2 = 0$

$r = \frac{+2 \pm \sqrt{4 - 8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$

(Complex Conjugate with $\alpha = 1,\ \beta = 1$)

$\therefore\ x(t) = c_1 t^1 \cos(1.\ln t) + c_2 t^1 \sin(1.\ln t)$

$= c_1 t \cos(\ln t) + c_2 t \sin(\ln t),\ t > 0.$

(d) $t^2 x'' + tx' + 9x = 0$

let $x = t^r \Rightarrow x' = rt^{r-1},\ x'' = r(r-1)t^{r-2}$

$\therefore$ D.E $\Rightarrow$ $t^2 r(r-1)t^{r-2} + trt^{r-1} + 9t^r = 0$

$[r(r-1) + r + 9]t^r = 0 \qquad \div t^r \neq 0$

$\therefore\ r^2 - r + r + 9 = 0$

$r^2 = -9,\ r = \pm 3i = 0 \pm 3i\ (\alpha = 0, \beta = 3)$

$\therefore\ x(t) = c_1 t^0 \cos(3\ln t) + c_2 t^0 \sin(3\ln t)$

or $x(t) = c_1 \cos(3\ln t) + c_2 \sin(3\ln t),\ c_1, c_2 \in \mathbb{R}$

Cauchy - Euler's D.E : change of Independent variable:

An equation of the form

$$\left(a_0 t^n D^n + a_1 t^{n-1} D^{n-1} + \cdots + a_{n-1} t D + a_n\right) x = E(t)$$

where $a_0, a_1, \ldots, a_n$ are constant real numbers with $a_0 \neq 0$, and D is the differential operator " $\frac{d}{dt}$ " is called a Cauchy-Euler D.E. We shall assume either $t \in (0, \infty)$ or $t \in (-\infty, 0)$

The change of the independent variable

$$t = e^u \quad , \quad t > 0$$

(or $t = -e^u$ if $t < 0$), transforms D.E into a linear D.E with constant coefficients.

Indeed $t = e^u \Rightarrow \frac{dt}{du} = e^u \Rightarrow \frac{dt}{du} = t$ or $\boxed{\frac{du}{dt} = \frac{1}{t}}$

Now, $Dx = \frac{dx}{dt} = \frac{dx}{du} \cdot \frac{du}{dt}$ "By chain rule!"

$$\therefore Dx = \frac{dx}{du} \cdot \frac{1}{t}$$

$$\Rightarrow \boxed{t Dx = \frac{dx}{du}} \quad \text{-------- (1)}$$

Differentiating each side w.r.to t we get

$$t D^2 x + 1 . Dx = \frac{d}{dt}\left(\frac{dx}{du}\right) \xrightarrow[\text{chain rule}]{\text{By}} \frac{d}{du}\left(\frac{dx}{du}\right) \frac{du}{dt}$$

$$\therefore t D^2 x + Dx = \frac{d^2 x}{du^2} \cdot \frac{1}{t}$$

Multiply each side by t

$$\therefore t^2 D^2 x + t Dx = \frac{d^2 x}{du^2}$$

Applying (1) we get

$$\boxed{t^2 D^2 x = \frac{d^2x}{du^2} - \frac{dx}{du}} \quad \text{------ (2)}$$

Introduce the Differential operator $\theta = \frac{d}{du}$. Then we can rewrite (1), (2) respectively as

$$tDx = \theta x \quad \text{or} \quad \boxed{tD = \theta}$$

$$t^2D^2x = \theta^2 x - \theta x \quad \text{or} \quad \boxed{t^2D^2 = \theta(\theta-1)}$$

In general we can prove that

$$t^m D^m = \theta(\theta-1) \cdots (\theta-m+1),$$

$m = 1, 2, 3, \cdots$

Therefore D.E is transformed into the D.E

$$\left[a_0 \theta(\theta-1) \cdots (\theta-n+1) + a_1 \theta(\theta-1) \cdots (\theta-n+2) + \cdots + a_{n-1}\theta + a_n\right] x = E(e^u) \stackrel{\text{or}}{=} F(u)$$

"Since $t = e^u$" That is D.E become

$$P(\theta) x = F(u) \quad , \quad \theta = \frac{d}{du}$$

where P is a polynomial in θ of degree n.

Ex: Find general solution to

$$(t^3D^3+5t^2D^2+4tD)x = \frac{6}{t^2} + 2\ln t, \quad t>0.$$

Solution: The D.E is of Cauchy-Euler's Type. let $t=e^u$, and let $\theta = \frac{d}{du}$

$$\therefore \quad tD = \theta, \quad t^2D^2 = \theta(\theta-1), \quad t^3D^3 = \theta(\theta-1)(\theta-2)$$

D.E $\Rightarrow$

$$[\theta(\theta-1)(\theta-2) + 5\theta(\theta-1) + 4\theta]\, x = \frac{6}{(e^u)^2} + 2\ln e^u$$

$$[\theta^3 - 3\theta^2 + 2\theta + 5\theta^2 - 5\theta + 4\theta]\, x = 6e^{-2u} + 2u$$

$$(\theta^3 + 2\theta^2 + \theta)x = 6e^{-2u} + 2u$$

char. eq. $P(r) = r^3 + 2r^2 + r = 0$

$r(r^2+2r+1) = 0$

$r = 0$ (multiplicity 1),

$r = -1$ (~ 2)

$$\therefore \quad x_h = c_1 e^{0.u} + c_2 e^{-1.u} + c_3 u e^{-1.u}$$

$$= c_1 \cdot 1 + c_2 e^{-u} + c_3 u e^{-u}$$

Next, $E(u) = 6e^{-2u} + 2u$

$= $ l.c of Two UC functions: e^{-2u}, u

Now $H = \{1, e^{-u}, ue^{-u}\}$

For e^{-2u}, UC set is $S_1 = \{e^{-2u}\}$ ← Needs no revision!

For u, UC set is $S_2 = \{1, u\} \Rightarrow S_2' = \{u, u^2\}$

$\therefore\ X_p = A e^{-2u} + Bu + Cu^2$

$\therefore\ X_p' = -2A e^{-2u} + B + 2Cu,$

$x_p'' = 4A e^{-2u} + 2C\ ,\quad x_p''' = -8A e^{-2u}$

D.E $\quad (\theta^3 + 2\theta^2 + \theta) X = 6 e^{-2u} + 2u$

or $\quad X''' + 2X'' + X' = 6 e^{-2u} + 2u$

substitute x', x'', and x''' from above we get

$$-8A e^{-2u} + 2(4A e^{-2u} + 2C) + (-2A e^{-2u} + B + 2Cu) = 6 e^{-2u} + 2u$$

rarranging we get,

$$-2A e^{-2u} + 2Cu + (4C + B) = 6 e^{-2u} + 2u + 0$$

Comparing, we get,

$-2A = 6\ ,\quad 2C = 2,\quad 4C + B = 0$

$\Rightarrow \boxed{A = -3}\ ,\quad \boxed{C = 1}\ ,\quad \boxed{B = -4}$

$\therefore\ x_p = A e^{-2u} + Bu + Cu^2 = -3 e^{-2u} - 4u + u^2$

General sol $\quad X = x_h + x_p = c_1 + c_2 e^{-u} + c_3 u e^{-u} - 3 e^{-2u} - 4u + u^2$

But $t = e^u$ (and hence $u = \ln t,\ t > 0$)

$\therefore\ X(t) = c_1 + c_2 t^{-1} + c_3 t^{-1} \ln t - 3t^{-2} - 4 \ln t + (\ln t)^2,\ t > 0$

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POWER Series Solution to a D.E : A FAST METHOD!

Given D.E $\quad a_0(t)\,x'' + a_1(t)\,x' + a_2(t)\,x = 0$, where a_0, a_1, a_2 are Continuous on some interval I. Assume $t=0$ is an ordinary point of D.E; that is $a_0(0) \neq 0$. Therefore the D.E have two linearly independent solutions of the form

$$x(t) = \sum_{n=0}^{\infty} a_n t^n \quad , \quad a_0 \neq 0$$

To find the coefficients a_n, we simply substitute x, x', and x'' into D.E to get an identity. Finally we equate all coefficients of various powers of t to zero and solve resulting equations.

An Illustration: Find a power series solution to D.E

$$(t^2+1)\,x'' + t\,x' + 2tx = 0 \quad , \quad x(0) = 2 \quad x'(0) = 3$$

in powers of t upto an including the fourth power of t.

solution: let $\quad x(t) = \sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \cdots$

that is

$$x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \cdots \quad (1)$$

$$x'(t) = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \cdots \quad (2)$$

$$x''(t) = 2a_2 + 6a_3 t + 12a_4 t^2 + \cdots \quad (4)$$

From (1): $\quad x(0) = 2 \quad \Rightarrow \quad \boxed{2 = a_0}$

From (2) $\quad x'(0) = 3 \quad \Rightarrow \quad \boxed{3 = a_1}$

sub. (1), (2), (3) into D.E: $\; t^2 x'' + x'' + t x' + 2t x = 0 \;$ to get:

$$t^2(2a_2 + 6a_3 t + 12a_4 t^2 + \cdots) + (2a_2 + 6a_3 t + 12a_4 t^2 + \cdots)$$
$$+\, t(a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \cdots) + 2t(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \cdots)$$
$$= 0$$

rearrange (grouping various powers of t):

$$2a_2 + (6a_3 + a_1 + 2a_0)\,t + (2a_2 + 12\underline{\underline{a_4}} + 2a_2 + 2a_1)\,t^2 + \cdots$$
$$= 0$$

stop! upto here since we want upto a_4

$\therefore \; 2a_2 = 0 \quad \Rightarrow \quad \boxed{a_2 = 0}$

$6a_3 + a_1 + 2a_0 = 0$ .. but $a_1 = 3, a_0 = 2$

$\therefore \; 6a_3 + 3 + 4 = 0 \quad \Rightarrow \quad \boxed{a_3 = -\frac{7}{6}}$

Finally $\quad 2a_2 + 12a_4 + 2a_2 + 2a_1 = 0$ .. but $a_2 = 0, a_1 = 3$

$\therefore \; 0 + 12a_4 + 0 + 6 = 0$

$\Rightarrow \boxed{a_4 = -\frac{1}{2}}$

$\therefore$ Solution to I.V.P $\quad x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \dots$

$$= 2 + 3t + 0\,t^2 - \frac{7}{6}t^3 - \frac{1}{2}t^4 + \dots$$

<u>About Convergence of power series:</u>

The minimum radius of Convergence of power series is determined as follows:

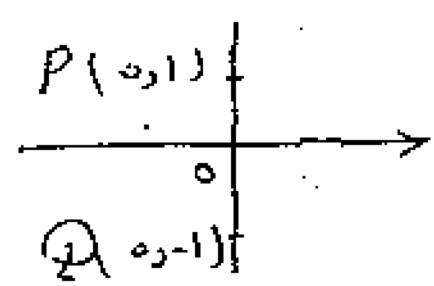
(1) Find all zeros of leading Coefficient $a_0(t) = t^2 + 1$ in Complex plane:

$$t^2 + 1 = 0, \quad t^2 = -1 = i^2$$
$$t = \pm i$$

The Root $t = +i$ is represented by the point $P(0,1)$

$t = -i$ ~ ~ ~ ~ ~ $Q(0,-1)$

(2) Find distance from origin to P, and Q, and choose the smaller



Indeed, distances are both 1 unit

$\therefore$ The least radius of Convergence is $R = 1$

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