recommended to use comparison test

Dec. 3. 2001 (A)

- Examples for Comparison / Limit Comparison Tests

Ex) Use Comparison Test to determine whether series converges or diverges

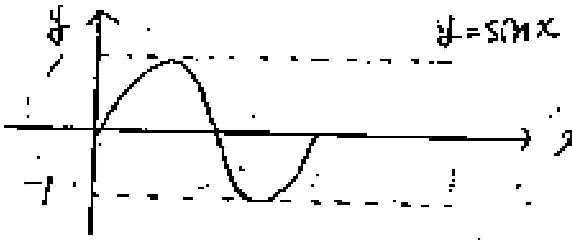
(a) $\sum_{n=1}^{\infty} \frac{\sin^2(5n)}{n^4}$ (b) $\sum_{n=3}^{\infty} \frac{1}{\ln n}$ *(c) $\sum_{n=1}^{\infty} \frac{2n-1}{\sqrt[7]{n^{12}+2n^2+19n-1}}$

*(d) $\sum_{n=1}^{\infty} \frac{n^4+3n+7}{\sqrt{n^{10}+23n^4+91}}$

Sol^n. (a) $\sum_{n=1}^{\infty} \frac{\sin^2(5n)}{n^4}$

Here $a_n = \frac{\sin^2(5n)}{n^4}$

<Note> For any real number α, $|\sin\alpha| \le 1$ hence $\sin^2\alpha \le 1$

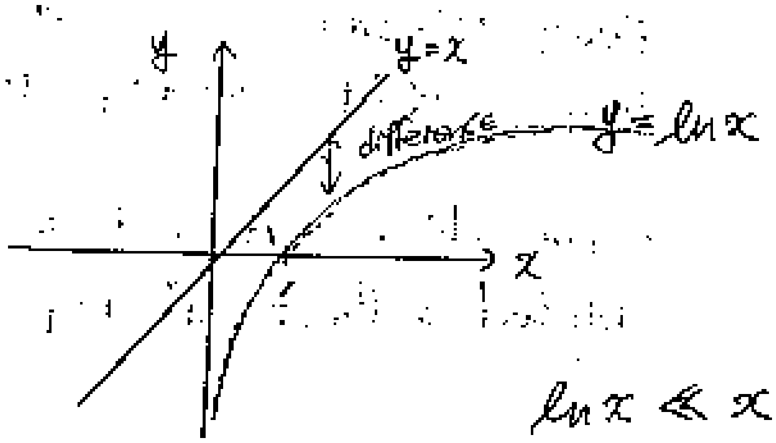


Now, $a_n = \frac{\sin^2(5n)}{n^4} \le \frac{1}{n^4} = b_n$

But $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is a p-series with $p = 4 > 1$ which converges,

hence by comparison Test $\sum a_n = \sum_{n=1}^{\infty} \frac{\sin^2(5n)}{n^4}$ converges, as well!

(b) $\sum_{n=3}^{\infty} \frac{1}{\ln n}$



Here $a_n = \frac{1}{\ln n}$

From figure $\ln n \ll n$

$\ln x \ll x$

Invert both side and do not forget to Reverse Inequality sign!

$$\frac{1}{\ln n} > \frac{1}{n}$$

that is $a_n = \frac{1}{\ln n} > \frac{1}{n} = b_n$

Sol^{n} Recall $\{a_n\}$ converges & its limit is "L" if $\lim_{n\to\infty} a_n$ exists and = "L"

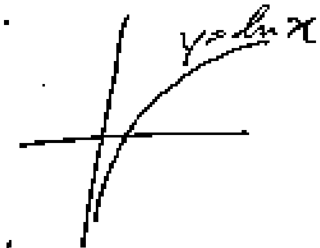
(1) $\left\{\frac{2n}{5n+1}\right\} = a_n$ $\quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{5n+1}$ (form $\frac{\infty}{\infty}$)

use leading term 技

can apply L'H rule! $\quad \lim_{n\to\infty} \frac{(2n)'}{(5n+1)'} = \lim_{n\to\infty} \frac{2}{5} = \frac{2}{5}$

$\therefore$ sequence converges to limit $\frac{2}{5}$.

(2) $\left\{\frac{\ln n}{n}\right\}$ ... here $a_n = \frac{\ln n}{n}$



$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{n} = \frac{\infty}{\infty}$ apply L'H rule

$\lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \frac{1}{\infty} = 0$ seq. converges to 0!

(5) $\left\{\left(1-\frac{2}{n}\right)^n\right\}$ Now $\lim_{n\to\infty}\left(1+\frac{-2}{n}\right)^n$ $\leftarrow$ standard limit

$= e^{\alpha}$ (here $\alpha = 2$) $= e^{-2}$

$\therefore$ sequence converges to e^{-2} or $\frac{1}{e^2}$.

** (3) $\left\{(-1)^n + \frac{1}{n}\right\}$ Here $a_n = (-1)^n + \frac{1}{n}$

Notes

1. $(-1)^n = \begin{cases} -1 & \text{if } n \text{ is odd integer } 1,3,5,\dots \\ 1 & \text{if } n \text{ is even integer } 2,4,6,\dots \end{cases}$

2. Whenever $(-1)^n$ is involved in a limit, you must rewrite it as explained in #(1) above.

In our case, $a_n = (-1)^n + \frac{1}{n} = \begin{cases} -1 + \frac{1}{n} & n \text{ odd} \\ +1 + \frac{1}{n} & n \text{ even} \end{cases}$

$\text{limit}_{n\to\infty}\, a_n = \lim_{n\to\infty}\left(-1+\frac{1}{n}\right) = -1 + \frac{1}{\infty} = -1$ (n odd)

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1 + \frac{1}{\infty} = 1$ (n even)

$\therefore \lim_{n\to\infty} a_n =$ DNE (it has more than one value) - seq. diverges!