Applied Mathematics 307 – Differential Equations for Engineers

$1.\ Separable: f(x)dx = g(t)dt \ \leftarrow form$

$2.\ Linear\ First\ Order\ ODE: \frac{dx}{dt} + p(t)x = q(t) \ \leftarrow form$

$$Integrating\ Factor\ (IF) = \mu(t) = e^{\int p(t)dt};\ x\mu = \int q\mu dt + c$$

$3.\ Bernoulli's\ DE: \frac{dx}{dt} + p(t)x = q(t)x^n,\ n \neq 0, 1 \ \leftarrow form$

$$Use\ transformation: V = x^{1-n},\ \frac{dV}{dt} = \cdots \frac{dx}{dt}$$

$$\mapsto reduces\ DE\ to\ linear,\ first\ order\ in\ dep.\ var\ V$$

$\mapsto NOTE: linear\ and\ Bernoulli\ has\ 3\ terms!$

$4.\ Homogeneous\ DE: \frac{dx}{dt} = f(t,x) \ \leftarrow form$

$NOTE: homogeneous\ function\ of\ degree\ n\ (of\ 2\ variables): f(\lambda t, \lambda x) = \lambda^n f(t,x),\ \lambda \in \mathbb{R}$

$\mapsto here,\ n = 0 \Leftrightarrow f(t,x)\ is\ homog.\ function\ of\ degree$

$Use\ transformation: x = vt,\ \frac{dx}{dt} = v + \cdots \frac{dv}{dt}\ (chain\ rule: v\ is\ a\ function\ not\ const!)$

$5.\ Exact\ DE:$

$NOTE: Exact\ differential: M(t,x)dt + N(t,x)dx\ is\ exact\ if\ it\ is\ total\ differential\ of\ f(t,x)$

$(ie,\ df = M(t,x)dt + N(t,x)dx)$

$NOTE: Exact\ DE: DE\ M(t,x)dt + N(t,x)dx = \underline{0}\ is\ exact\ if\ M(t,x)dt + N(t,x)dx\ is\ exact$

$\mapsto Test\ for\ exactness: DE\ is\ exact\ if\ \frac{\partial M}{\partial x} = \frac{\partial N}{\partial t} \leftarrow opposite\ variables\ (M\ with\ x\ etc.)$

$M(t,x)dt + N(t,x)dx = 0\ (exact\ DE) \ \leftarrow formdd$

$F(t,x) = c,\ F(t,x)\ is\ sum\ of\ 2\ partial\ integrals\ \int M(t,x)dt,\ \&\ \int N(t,x)dx\ w/out\ repeating$

$common\ terms$

$5.1\ Integrating\ factor\ for\ non-exact\ DE: non\ exact\ DE: M(t,x)dt + N(t,x)dx = 0$

$\mu = \mu(t,x)\ is\ IF\ of\ DE\ if\ DE\ becomes\ exact\ upon\ multiplying\ both\ sides\ of\ DE\ by\ \mu(t,x)$

$6.\ DE\ of\ n^{th}\ order\ reducible\ to\ first\ order: \frac{d^n x}{dt^n} + p(t)\frac{d^{n-1}x}{dx^{n-1}} = q(t) \leftarrow form\ (note\ consecutive\ deg)$

$$Use\ transformation: V = \frac{d^{n-1}x}{dt^{n-1}},\ \frac{dV}{dt} = \cdots \ \mapsto use\ c_1\ and\ c_2\ instead\ of\ just\ c$$

$$\text{Newton's Law of Cooling (or Heating)}: \left\langle \begin{array}{l} \dfrac{dx}{dt} = k(x-m),\ x(t_1) = x_1 \mapsto \text{for finding } k \\ x(t_0) = x_0 \mapsto \text{for finding } c \end{array} \right.$$

3 methods to solving: 1) D: Let $x - m = U \to x' = U'$

$$(U') = kU \to (D-k)U = 0 \quad U = x - m = ce^{kt} \to x = ce^{kt} + m$$

2) linear

3) separable (strongly not recommended!)

$$\text{Mixture Problem}: \left\langle \begin{array}{l} \dfrac{dx}{dt} = \text{rate IN} - \text{rate OUT} = r_i c_i - r_o c_o,\ x = x(t) = \text{amount},\ r = \left[{}^{vol}\!/_{time}\right] \\ x(t_0) = x_0 \\ c_o = \dfrac{x}{V} = \dfrac{x}{V_0 + (r_i - r_o)t} \mapsto \therefore V = V_0 + (r_i - r_o)t \end{array} \right.$$

$-$ linear, n^{th} order o.d.e.: $a_0(t)\dfrac{d^n x}{dt^n} + a_1(t)\dfrac{d^{n-1}x}{dt^{n-1}} + \ldots + a_{n-1}(t)\dfrac{dx}{dt} + a_n(t)x = E(t)$

$-$ if $a_0, a_1, \cdots, a_n \in \mathbb{R}$ constants, then is special case: $L = P(D)$, $P(D)x = E(t)$

$-$ homogeneous differential equation if $E(t) = 0$

$-$ diffential operator "L": $Lx = E(t)$, $L = a_0 D^n + a_1 D^{n-1} + \ldots + a_{n-1}D + a_n$, $D = \dfrac{d}{dt}, \ldots$

$-$ linear combination, Wronskian, fundamental set $P(D) \to P(r)$ (characteristic polynomial)

$-$ $L = P(D)$ if $a_0, a_1, \ldots, a_n$ in L are constant coefficients

$-$ linear independence: $c_1 = c_2 = \ldots = c_n = 0$ (trivial solution) <u>or</u> $W(h_1, h_2, \ldots, h_n)(t_0) \neq 0\ \forall\, t_0 \in \mathbb{Z}$

$\mapsto$ if $W = 0$, may or may not be independent!!

$-$ General solution: $P(D)x = E(t)$, $E(t) = 0$

1. n distinct roots $r_1, r_2, \ldots, r_n$: $h_1(t) = e^{r_1 t}, h_2(t) = e^{r_2 t}, \ldots$
2. root α with multiplicity k : $h_1(t) = e^{\alpha t} t^0$, $h_2(t) = e^{\alpha t} t^1, \ldots, h_k(t) = e^{\alpha t} t^k$
3. complex conjugate roots $\alpha \pm i\beta$: $h_1(t) = e^{\alpha t}\cos\beta t$, $h_2(t) = e^{\alpha t}\sin\beta t$

$-$ Exponential shift: $P(D)e^{\alpha t}x(t) = e^{\alpha t}P(D+\alpha)x(t)$

Euler's identity: $e^{\pm i\theta} = \cos\theta \pm i\sin\theta,\ \theta \in \mathbb{R}$

$-$ Particular solution:

#	<u>UC function</u>	<u>UC set</u>
1.	$e^{\alpha t}, \alpha \neq 0$	$\{e^{\alpha t}\}$
2.	$\sin\beta t, \cos\beta t,\ \beta \neq 0$	$\{\sin\beta t, \cos\beta t\}$
3.	$poly.$ deg n	$\{1, t, t^2, \ldots, t^n\}$

<u>Steps</u>

1. $x_h(t) \to H = \{\cdots\}$
2. $S_1, S_2, S_3, \ldots$ $-$ revise
3. $x_p =$ l.c. of S's using k's
4. solve $x_p, x'_p, \ldots$ sub into orig DE solve k's as const coeff.
5. $x(t) = x_h(t) + x_p(t)$

($-$ see Cauchy $-$ Euler's DE of 2^{nd} order next page)

$-$ Method of Variation of Parameters: <u>2^{nd}</u> order DE $a_0 x'' + a_1 x' + a_2 x = E(t)$

use when h_1, h_2 are known, $E(t)$ not UC function (formulae on next page) *method of variation of parameters cont'd*:

$$x(t) = c_1(t)h_1(t) + c_2(t)h_2(t);\ c_1'(t) = \frac{\begin{vmatrix} 0 & h_2 \\ \frac{E(t)}{a_0} & h_2' \end{vmatrix}}{W(h_1, h_2)},\ c_2'(t) = \frac{\begin{vmatrix} h_1 & 0 \\ h_1' & \frac{E(t)}{a_0} \end{vmatrix}}{W(h_1, h_2)}$$

− *Cauchy − Euler's DE of* 2^{nd} *order*: $a_0 t^2 \frac{d^2x}{dt^2} + a_1 t^1 \frac{d^1x}{dt^1} + a_2 x = 0,\ t > 0 \cup t < 0 \Leftrightarrow t \neq 0$

$\mapsto$ *power of t matches derivative* $\mapsto$ *may have to *t to achieve form of CEDE*

Use $x(t) = t^\alpha \to$ *get indecial equation (quadratic)*

General solution:

1. α_1, α_2 *real and distinct*: $x(t) = c_1 t^{\alpha_1} + c_2 t^{\alpha_2},\ c_1, c_2 \in \mathbb{R}$
2. α_1, α_2 *real equal roots*: $h_1(t) = t^\alpha,\ h_2(t) = t^\alpha \ln t,\ t \in (0, \infty),\ c_1, c_2 \in \mathbb{R}$
3. $\alpha_{1,2} = \alpha \pm i\beta : x(t) = c_1 t^\alpha \cos(\beta \ln t) + c_2 t^\alpha \sin(\beta \ln t)$

− $D\vec{x} = A\vec{x} + \vec{E}(t)$

− *Eigenvalues, Eigenvectors* $\det(A - \lambda I) = 0,\ (A - \lambda I)\vec{v} = \vec{0} \to$ *non − trivial*

$\to$ *if pick more than one, use s, t…!* $\to$ *solve* $\vec{v}$ *using cross product of 2 rows (for* $m = 1$ *only)*

− *General solution*: $D\vec{x} = A\vec{x},\ A \in$ *const* $\mathbb{R}$

1. $\lambda \ni m = 1 : \overrightarrow{h(t)} = e^{\lambda t}\vec{v}$
2. *conjugate pair* $\lambda = \alpha \pm i\beta : \vec{h}_1(t) = \mathrm{Re}(e^{\lambda t}\vec{v}),\ \vec{h}_2(t) = \mathrm{Im}(e^{\lambda t}\vec{v})$
   - $\mapsto$ *use/pick one of eigenvalue and corresponding eigenvector*
   - $\mapsto$ *apply Euler's identity*

3a) $\lambda \ni m > 1$ *but with different* $\vec{v}$*'s*: $\vec{h}_i(t) = e^{\lambda t}\vec{v}_i,\ i = 1, 2, \ldots, m$ $\mapsto$ *no *t!!*

b) $\lambda \ni m = 2 \cap$ *one* $\vec{v} = \vec{\alpha} : \vec{h}_1(t) = e^{\lambda t}\vec{\alpha},\quad \vec{h}_2(t) = e^{\lambda t}(t\vec{\alpha} + \vec{\beta}),\ (A - \lambda I)\vec{\beta} = \vec{\alpha}$

− *Non − homogeneous system* $D\vec{x} = A\vec{x} + \vec{E}(t)$: *Method of variation of parameters*

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t),\ B = \begin{bmatrix} \vec{h}_1 & \vec{h}_2 & \cdots & \vec{h}_n \end{bmatrix},\ \vec{C}'(t) = B^{-1}\vec{E}(t),\qquad K = \begin{bmatrix} a & b \\ c & d \end{bmatrix},\ K^{-1} = \frac{1}{\det K}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

− $\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

− *PFD by inspection*: $\frac{1}{(s+\alpha)(s+\beta)} = \frac{1}{\beta - \alpha}\left(\frac{1}{s+\alpha} - \frac{1}{s+\beta}\right)$

− *see* $\mathscr{L}$ *formula sheet*

− *2 − piece function* $f(t) : f(t) =$ *first piece* $+ (2^{nd}$ *piece* $- 1^{st}$ *piece*$) \cdot$ *switch*

$-$ *Sequences*

Test for monotonic sequence:

1. *Defference Test* : $a_{n+1} - a_n \geq 0, \forall n \geq n_0, \{a_n\}$ *is increasing*

2. *Quotient Test* : $\dfrac{a_{n+1}}{a_n} \geq 1, n \geq n_0, \{a_n\}$ *is increasing* $\mapsto \dfrac{a_{n+1}}{a_n} \leq 1 \Rightarrow$ *decreasing* $\mapsto$ *use with factorial*

3. *Derivative Test* : $f'(x) \geq 0, \{a_n\}$ *increasing* $\mapsto f'(x) \leq 0 \Rightarrow$ *decreasing*

$$\boxed{\begin{array}{l} \lim\limits_{n\to\infty}\left(1+\dfrac{\alpha}{n}\right)^n = e^{\alpha} \\ \lim\limits_{n\to\infty}\sqrt[n]{n^p} = 1,\ p \in R \end{array}}$$

$-$ *Infinite series* $\mapsto$ *must write determined by which method*! (*by* $\cdots$ *test*)

1. *Divergence Test* (n^{th} *term test*) : *Failpoint* : $\lim\limits_{n\to\infty} a_n = 0$

2. *Ratio Test* : *Failpoint* : $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = 1 \mapsto$ *use for factorial*

3. *Root Test* : *Failpoint* : $\lim\limits_{n\leftarrow\infty} \sqrt[n]{a_n = 1}$

4. *Integral Test* : *Conditions* : $f(x) > 0 \cap f(x)$ *continuous* $\cap$ *decreasing*

5. *Comparison Test* :
6. *Limit Comparison Test* :

$\rangle$ *usually use for geometric or* $p-$ *series*

7. *Leibnitz Test* : *Conditions* : $\lim\limits_{n\to\infty} a_n = 0 \cap$ *decreasing*

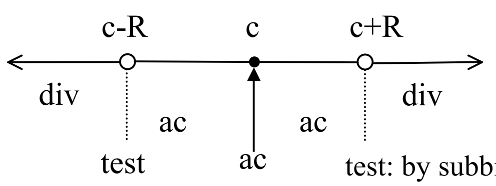
$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}}$$

$-$ *Absolute / conditional convergence*

$$\sum_{n=1}^{\infty} a_n \to \text{series of arbitrary terms} \left\langle \begin{array}{l} \text{if } \sum\limits_{n=1}^{\infty} |a_n| \text{ converges} \Rightarrow \text{absolutely convergent} \\ \text{if } \sum\limits_{n=1}^{\infty} |a_n| \text{ diverges whereas } \sum\limits_{n=1}^{\infty} a_n \text{ converges} \Rightarrow \text{conditionally conv.} \end{array} \right.$$

$-$ *Power series* : $\sum\limits_{n=0}^{\infty} a_n (t-c)^n$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|, \; R = \frac{1}{L}$$



test: by subbing $t = c \pm R$ into original power series

$-$ *Power series solution to a DE at an ordinary point* : $x(t) = \sum\limits_{n=0}^{\infty} a_n (t-c)^n, \; a_0 \neq 0$

<u>*Steps*</u> : 1. $c = 0$ (*unless otherwise stated*), *told to solve up to what power*.

2. *Expand power series up to that power*.

3. $x = \cdots, x' = \cdots, x'' = \cdots, \ldots,$ *apply IVP* (*if any*)

4. *Sub step* 3 *into original DE*. *Solve* $a_0, a_1, \ldots$

5. *Write* $x = a_0 + a_1 t + a_2 t^2 + \ldots = \cdots$

$-$ *Taylor's series* (*a special power series*) : *if* $f(x)$ *is infinitely differentiable at some* $t = c$, *then the Taylor's series of* f *centred at* "c" *is the power series*

$$f(x) = \sum_{n=0}^{\infty} a_n (t-c)^n, \;\; a_n = \frac{f^{(n)}(c)}{n!} \quad n = 0, 1, 2, \ldots$$