AMAT 219

SUPPORT MATERIAL

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Good Luck!

Y. E

Support Material

Techniques of Integration: Six. Given $I = \int f(x)\,dx$.

(1) **Integration by the Basic Table**: Just match the integral given with one or more basic Integral appearing in Table.

Ex $\int x^{\frac{3}{4}}\,dx = \frac{4}{7}x^{\frac{7}{4}} + C$, $\int \frac{dx}{\cos^2 x} = \int \sec^2 x\,dx = \tan x + C$

$$\int \left(\frac{u^2+3u+1}{u^2}\right)du = \int\left(1+\frac{3}{u}+\frac{1}{u^2}\right)du = u + 3\ln|u| - \frac{1}{u} + C$$

Important property: If each "x" in Table is replaced by the Linear quantity $ax+b$, $a \neq 0$, the Corresponding formula is divided by "a" which represents the derivative of $ax+b$!

Ex: Know $\int \frac{1}{x}\,dx = \ln|x| + C$ .. hence $\int \frac{dx}{7x+3} = \frac{1}{7}\ln|7x+3| + C$

Know $\int e^x\,dx = e^x + C$ .. hence $\int e^{-3x}\,dx = -\frac{1}{3}e^{-3x} + C$

Know $\int \frac{dx}{4+x^2} = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$ .. hence $\int \frac{dx}{4+(1-3x)^2} = \frac{\frac{1}{2}\tan^{-1}\left(\frac{1-3x}{2}\right)}{-3} + C$

Another Important property: The integral of a fractional expression whose numerator is the Derivative of its denominator is

$$\ln|\text{The Denominator}| + C \text{ !}$$

Ex $\int \frac{3x^2+10x}{x^3+5x^2-2}\,dx$ .. we notice that the numerator is indeed the Derivative of denominator!

$$\therefore \quad I = \ln|x^3+5x^2-2| + C$$

Ex $\int \frac{x}{1+x^2}\,dx$. Here numerator is not exactly derivative of denominator .. a minor adjustment needed!

Indeed $I = \int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$ ← Now it is ready!

$$= \frac{1}{2}\ln|1+x^2| + C$$

You may add this Integral to Table:

$$\int \sec^3 x\,dx = \frac{1}{2}\{\text{Derivative of } \sec x + \text{Integral of } \sec x\}$$

$$= \frac{1}{2}\{\sec x \tan x + \ln|\sec x + \tan x|\} + C$$

Ex $\int \sec^3(\underbrace{4x}_{\text{Linear!}})\,dx = \frac{1}{2}\cdot\frac{1}{4}\{\sec 4x \tan 4x + \ln|\sec 4x + \tan 4x|\} + C$

(2) Integration by substitution

If $I = \int f(x)\,dx$ can not be evaluated directly by the table, one may attempt a substitution $u = u(x)$. Here is the outlines of the method of Integration by substitution.

(1) Make a suitable choice of a substitution $u = u(x)$.
(2) Compute du. Note $\frac{du}{dx} = u'(x) \Rightarrow du = u'\,dx$. Hence $dx = \frac{du}{u'}$
(3) Substitute u, and dx into integral I. Make sure that the new integrand is a function of only u. If it is mixed, you must find a way to get rid of original variable "x" or otherwise your substitution is unsuccessful!
(4) Integrate result of step (3) by The Table!
(5) Recall u and substitute it back into result of step (4).

Remark: How can I choose a successful substitution?

Indeed, we usually choose a substitute u where $du = u'\,dx$ is a multiplicative part of integrand (upto a constant multiple).

Ex $\int \frac{\cos x}{\sqrt{4+3\sin x}}\,dx$

let $u = 4+3\sin x$, $du = \underbrace{3\cos x\,dx}_{\text{part of Integrand}} \Rightarrow \cos x\,dx = \frac{du}{3}$

$\therefore I = \frac{1}{3}\int \frac{du}{\sqrt{u}} = \frac{2}{3}\sqrt{u} + C = \frac{2}{3}\sqrt{4+3\sin x} + C$

Ex $\int \frac{e^{x}\,dx}{16+e^{2x}}$. let $u = e^{x} \Rightarrow du = e^{x}\,dx$

$\therefore I = \int \frac{e^{x}\,dx}{16+(e^{x})^{2}} = \int \frac{du}{16+u^{2}} = \frac{1}{4}\tan^{-1}\left(\frac{u}{4}\right) + C = \frac{1}{4}\tan^{-1}\left(\frac{e^{x}}{4}\right) + C.$

Ex $\int \frac{\ln^{4} x}{x}\,dx$ .. write as $\int (\ln x)^{4}\left(\frac{1}{x}\,dx\right)$

let $u = \ln x$, $du = \left(\frac{1}{x}\,dx\right)$

$\therefore I = \int u^{4}\,du = \frac{1}{5}u^{5} + C = \frac{1}{5}(\ln x)^{5} + C.$

special Integrals:

① $\int \sin^{2}\alpha x\,dx$ .. use double angle formula $\sin^{2}\alpha x = \frac{1}{2}[1 - \cos 2\alpha x]$

$\therefore I = \frac{1}{2}\int (1-\cos 2\alpha x)\,dx = \frac{1}{2}\left[x - \frac{\sin 2\alpha x}{2\alpha}\right] + C$

② Similarly $\int \cos^{2}\alpha x\,dx = \frac{1}{2}\left[x + \frac{\sin 2\alpha x}{2\alpha}\right] + C$

③ $\int \sin^{3} x\,dx = \int \sin^{2} x \sin x\,dx = \int (1-\cos^{2} x)\sin x\,dx$ .. Now use $u = \cos x$

④ Similarly $\int \cos^{3} x\,dx = \int (1-\sin^{2} x)\cos x\,dx$ ... Use $u = \sin x$!

(3) Integration by parts :

The idea of this method is to evaluate $\int u\,dV$ by converting it to a much easier integral $\int V\,du$. In fact

$$\int u\,dV = uV - \int V\,du$$

$$\begin{array}{ccc} u & & dV \\ du & \xleftarrow{\int} & V \end{array}$$

The following Table summarizes the types of integrals that can be computed by "parts".
In this table $P_n(x)$ represents a polynomial of degree "n",

$$P_n(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n, \quad a_0 \neq 0$$

Note Also that "x" may be replaced by the linear quantity $ax+b$, $a \neq 0$!

	Integral Type	choice of U	choice of dV	How many Times	An Example
1	$\int e^{\alpha x}P_n(x)\,dx,\ \alpha\neq 0$	$P_n(x)$	$e^{\alpha}dx$	n	$\int x^4e^{-2x}dx$
2	$\int P_n(x)\sin\beta x\,dx,\ \beta\neq 0$	$P_n(x)$	$\sin\beta x\,dx$	n	$\int (x+1)^2\sin 3x\,dx$
3	$\int P_n(x)\cos\beta x\,dx,\ \beta\neq 0$	$P_n(x)$	$\cos\beta x\,dx$	n	$\int x^3\cos(4x+3)\,dx$
4	$\int x^m\ln x\,dx,\ m\in\mathbb{R},\ m\neq -1$	$\ln x$	$x^m dx$	1	$\int x^{-7}\ln x\,dx$
5	$\int$ Inverse Trig/Hyp. dx	The inverse function	$1\,dx$	1	$\int \sin^{-1}(x)\,dx$ $\int \cosh^{-1}(2x)\,dx$
6	$\int (\ln x)^n\,dx$	$(\ln x)^n$	$1\,dx$	n	$\int (\ln x)^5dx$
7	$\int x^n$. An inverse function dx $n=1,2,3,\dots$	The inverse function	$x^n\,dx$	1	$\int x^2\tan^{-1}(3x)\,dx$, $\int x\sinh^{-1}x\,dx$
8	$\int \sec^n(\alpha x)\,dx,\ \alpha\neq 0$ n odd : 3, 5, 7, ...	$\sec^{n-2}(\alpha x)$	$\sec^2(\alpha x)\,dx$	$\frac{n-1}{2}$	$\int \sec^3(5x)\,dx$
9	Other Types : $\int x\sec^2x\,dx$ $\int x\csc^2x\,dx$	 x x	 $\sec^2x\,dx$ $\csc^2x\,dx$	 1 1	 $\int (3x-1)\sec^2(4x)\,dx$ $\int x\csc^2(2x-1)\,dx$

EX: $\int (x+2)\,e^{3x-1}dx$

$$\begin{array}{ccc} u = x+2 & & dV = e^{3x-1}dx \\ du = 1.dx & \xleftarrow{\int} & V = \frac{1}{3}e^{3x-1} \end{array}$$

$$I = \tfrac{1}{3}(x+2)e^{3x-1} - \int \tfrac{1}{3}e^{3x-1}dx$$
$$= \tfrac{1}{3}(x+2)e^{3x-1} - \tfrac{1}{3}\cdot\frac{e^{3x-1}}{3} + C$$

EX $\int \cosh^{-1}(3x)\,dx$

$$\begin{array}{ccc} u = \cosh^{-1}(3x) & & dV = 1\,dx \\ du = \dfrac{3\,dx}{\sqrt{9x^2-1}} & \xleftarrow{\int} & V = x \end{array}$$

$$= x\cosh^{-1}(3x) - 3\int \frac{x}{\sqrt{9x^2-1}}\,dx$$

let $u = 9x^2-1$

$$= x\cosh^{-1}(3x) - \tfrac{1}{3}\sqrt{9x^2-1} + C$$

EX $\int (x+2)^3 \sin x \, dx$ -- by parts three times!

$u = (x+2)^3 \qquad dV = \sin x \, dx$

$du = 3(x+2)^2 dx \quad \int \quad V = -\cos x$

$$I = -(x+2)^3 \cos x + 3\int (x+2)^2 \cos x \, dx$$

$$= -(x+2)^3 \cos x + 3\left[(x+2)^2 \sin x - 2\int (x+2)\sin x\, dx\right]$$

$u = (x+2)^2 \qquad dV = \cos x \, dx$

$du = 2(x+2)dx \quad \int \quad V = \sin x$

$$= -(x+2)^3 \cos x + 3(x+2)^2 \sin x - 6\int (x+2)\sin x \, dx$$

$$= -(x+2)^3 \cos x + 3(x+2)^2 \sin x - 6\left[-(x+2)\cos x + \int \cos x \, dx\right]$$

$u = x+2 \qquad dV = \sin x \, dx$

$du = 1 . dx \quad \int \quad V = -\cos x$

$$= -(x+2)^3 \cos x + 3(x+2)^2 \sin x + 6(x+2)\cos x - 6\sin x + C$$

Too long!

<u>A short cut!</u> $I = \int (x+2)^3 \sin x \, dx$

Successive Derivatives		Successive Integrals
$(x+2)^3$	$\oplus$	$\sin x$
$3(x+2)^2$	$\ominus$	$-\cos x$
$6(x+2)$	$\oplus$	$-\sin x$
6	$\ominus$	$\cos x$
0		$\sin x$

$$I = -(x+2)^3 \cos x + 3(x+2)^2 \sin x + 6(x+2)\cos x - 6\sin x + C$$

very short method !!

EX $\int x^9 e^{x^5} dx$ -- This integral may be Evaluated by 1st. making the substitution $t = x^5 \Rightarrow dt = 5x^4 dx$ or $x^4 dx = \frac{1}{5} dt$

$$I = \int x^9 e^{x^5} dx = \int x^5 e^{x^5} (x^4 dx) = \frac{1}{5}\int t e^t dt \leftarrow \text{by parts once!}$$

$$= \frac{1}{5}\left[te^t - e^t\right] + C \quad \text{.. check it..!}$$

$$= \frac{1}{5}\left[x^5 e^{x^5} - e^{x^5}\right] + C$$

EX: $\int \sec^3 x \, dx \rightarrow$ by parts $\frac{n-1}{2} = \frac{3-1}{2} = 1$ time!

$u = \sec x \qquad dV = \sec^2 x \, dx$

$du = \sec x \tan x \, dx \quad \int \quad V = \tan x$

$$I = \int \sec x \sec^2 x \, dx$$

$$= \sec x \tan x - \int \sec x \tan^2 x \, dx$$

But $\tan^2 x = \sec^2 x - 1$ .. hence

$$I = \sec x \tan x - \int \sec x(\sec^2 x - 1)\, dx$$

$$= \sec x \tan x - \underbrace{\int \sec^3 x \, dx}_{\substack{\text{This is } I \\ \text{(move to L.H.S)}}} + \int \sec x \, dx \leftarrow \text{Table!}$$

$$\therefore I + I = \sec x \tan x + \ln|\sec x + \tan x| + C$$

$$2I = \sec x \tan x + \ln|\sec x + \tan x| + C \text{ or}$$

$$I = \frac{1}{2}\{\sec x \tan x + \ln|\sec x + \tan x|\} + C$$

EX $\int \sin\sqrt{x}\, dx$ .. Let $x = t^2$ .. Then by parts. Ans. $-2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$

EX $\int 2x \tan^{-1} x \, dx$ -- by parts once!

$u = \tan^{-1} x \qquad dV = 2x \, dx$

$du = \frac{1}{1+x^2} dx \quad \int \quad V = x^2$

$$I = x^2 \tan^{-1} x - \int \frac{x^2}{1+x^2} dx \leftarrow \text{use long division}$$

$$x^2 \tan^{-1} x - \int \left\{1 - \frac{1}{1+x^2}\right\} dx = x^2 \tan^{-1} x - \left[x - \tan^{-1} x\right] + C$$

(4) Integration by Partial Fractions DeComposition (PFD)

Recall: A rational function $f(x)$ is a quotient of two polynomials $p(x)$, $q(x)$. Now, let $f(x) = \frac{p(x)}{q(x)}$. Then f is a proper rational function if degree of its numerator is Less than that of its denominator. If $f(x)$ is not proper, we say f is an Improper Rational function. Using long division, we can write an Improper rational function as a sum of a polynomial and a proper rational function.

Ex $f = \dfrac{x^3-2x^2+5x+1}{x^2-x+7}$ is an Improper rational function.

$$\begin{array}{r} x-1 \\ x^2-x+7\ \overline{)\, x^3-2x^2+5x+1} \\ \underline{x^3-x^2+7x} \\ -x^2-2x+1 \\ \underline{-x^2+x-7} \\ -3x+8 \end{array}$$

Dividing, we get $f(x) = \underbrace{x-1}_{\text{poly.}} + \underbrace{\dfrac{-3x+8}{x^2-x+7}}_{\text{proper Rational}}$

The method of PFD: let $f(x) = \frac{p(x)}{q(x)}$ be a proper Rational Function

step1: Factor $q(x)$ completely into Linear or quadratic factors.

step2: ① For each Linear factor $ax+b$, $a \neq 0$ (non-repeated) in $q(x)$, there Corresponds one fraction $\dfrac{A}{ax+b}$, A is a constant to be found

② For each linear factor repeated K-times, namely $(ax+b)^K$, there Corresponds the sum of K fractions: $\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_K}{(ax+b)^K}$

(3) For each quadratic factor ax^2+bx+c, $a \neq 0$ (non-repeated), there Corresponds one fraction $\dfrac{Ax+B}{ax^2+bx+c}$

(4) For each quadratic factors repeated K times, namely $(ax^2+bx+c)^K$, there Corresponds the sum of K fractions

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_Kx+B_K}{(ax^2+bx+c)^K}.$$

step3: Multiply each side of Expression resulting from step(2) by the factored denominator. Then assign various values of "x" to find all unknown constant. The most recommended values of "x" to be used are the values in which "Denominator of f becomes zero".

Now, let $I = \int f(x)\,dx$ where $f(x)$ is a proper Rational function. Then I may be Evaluated by PFD. The following two special Integrals appear often in the PFD methods.

(1) $\displaystyle\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C$, $a \neq 0$

(2) $\displaystyle\int \frac{dx}{ax^2+bx+c}$ Can be done by Completing the square. This produces a standard Integral!

Completing the square: Consider the quadratic expression Ax^2+Bx+C where $A>0$. Then $Ax^2+Bx+C = \left(\sqrt{A}\,x + \frac{B}{2\sqrt{A}}\right)^2 + \left(C - \frac{B^2}{4A}\right)$ --- (*)

Note: If Ax^2+Bx+C is not factorable over $\mathbb{R}$, the R.H.S of (*) above becomes a sum of squares. Completing the squares is often used to evaluate integrals involving a quadratic expression!

Ex Find $\int \frac{dx}{x^2+6x+34}$.

Soln. $x^2+6x+34 = (x+3)^2+25$ (check!) $\Rightarrow I = \int \frac{dx}{(x+3)^2+25} = \frac{1}{5}\tan^{-1}\left(\frac{x+3}{5}\right)+C$

Ex Find $\int \frac{dx}{\sqrt{10x-x^2}}$

Soln. warning $\sqrt{10x-x^2} \neq -\sqrt{x^2-10x}$!! Instead:

$10x-x^2 = -[x^2-10x] = -[(x-5)^2-25] = 25-(x-5)^2$

$\therefore I = \int \frac{dx}{\sqrt{25-(x-5)^2}}$ .. let $t = x-5$, $dt = dx$

$\therefore I = \int \frac{dt}{\sqrt{25-t^2}} = \sin^{-1}\left(\frac{t}{5}\right)+C = \sin^{-1}\left(\frac{x-5}{5}\right)+C$

Ex $\int \frac{24}{x^3-2x^2-8x}\,dx$ — use PFD

$\frac{24}{x^3-2x^2-8x} = \frac{24}{x(x^2-2x-8)} = \boxed{\frac{24}{x(x+2)(x-4)} = \frac{A}{x}+\frac{B}{x+2}+\frac{C}{x-4}}$

Multiply each side by $x(x+2)(x-4)$:

$24 = A(x+2)(x-4)+Bx(x-4)+Cx(x+2)$

special values $x(x+2)(x-4)=0$, $x=0,-2,4$

$x=0$: $24 = -8A$, $A=-3$

$x=-2$: $24 = 12B$, $B=2$

$x=4$: $24 = 24C$, $C=1$

$\therefore \frac{24}{x(x+2)(x-4)} = -\frac{3}{x}+2\cdot\frac{1}{x+2}+\frac{1}{x-4}$

$I = -3\int\frac{dx}{x}+2\int\frac{dx}{x+2}+\int\frac{1}{x-4}\,dx = -3\ln|x|+2\ln|x+2|+\ln|x-4|+C.$

Ex $\int \frac{dx}{(x-1)^2x}$ → use PFD

$\frac{1}{(x-1)^2x} = \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x}$ $\Rightarrow$ $1 = Ax(x-1)+Bx+C(x-1)^2$

special values are $x=0,1$; Non special Say: $x=-1$

$x=0$: $1=C$

$x=1$: $1=B$

$x=-1$: $1=2A-B+4C \Rightarrow A=-1$

$\therefore \frac{1}{(x-1)^2x} = \frac{-1}{x-1}+\frac{1}{(x-1)^2}+\frac{1}{x}$

$\therefore I = -\int\frac{dx}{x-1}+\int\frac{dx}{(x-1)^2}+\int\frac{1}{x}\,dx = -\ln|x-1|+\int(x-1)^{-2}dx+\ln|x|+C$

$= -\ln|x-1|+\frac{(x-1)^{-1}}{-1}+\ln|x|+C$

Ex $\int\frac{2-x}{x(x^2+1)}\,dx$ .. PFD

$\frac{1-x}{x(x^2+1)} = \frac{A}{x}+\frac{Bx+C}{x^2+1}$ $\Rightarrow$ $2-x = A(x^2+1)+(Bx+C)x$ .. use $x=0,1,-1$

to get $A=2$, $B=-2$ and $C=-1$ "check!"

$$\frac{1-x}{x(x^2+1)} = \frac{2}{x} + \frac{-2x-1}{x^2+1}$$ Trick: write $\frac{-2x-1}{x^2+1}$ as $\frac{-2x}{x^2+1} - \frac{1}{x^2+1}$

$$\therefore I = \int \left(\frac{2}{x} - \frac{2x}{x^2+1} - \frac{1}{x^2+1} \right) dx = 2\ln|x| - \ln|x^2+1| - \tan^{-1}x + C$$

EX $\int \frac{2e^{3x}}{e^{6x}-9}\,dx$ .. let $u = e^{3x}$, $du = 3e^{3x}dx \Rightarrow e^{3x}dx = \frac{1}{3}du$

$$= \frac{2}{3}\int \frac{du}{u^2-9} \leftarrow \text{PFD}$$

Consider $\frac{1}{u^2-9} = \frac{1}{(u-3)(u+3)} = \frac{A}{u-3} + \frac{B}{u+3}$

check that $A = \frac{1}{6}$, $B = -\frac{1}{6}$

$$\therefore \frac{1}{u^2-9} = \frac{1}{6}\cdot\frac{1}{u-3} - \frac{1}{6}\cdot\frac{1}{u+3} \Rightarrow I = \frac{2}{3}\left\{ \frac{1}{6}\int \frac{du}{u-3} - \frac{1}{6}\int \frac{du}{u+3} \right\}$$

$$I = \frac{2}{3}\left\{ \frac{1}{6}\ln|u-3| - \frac{1}{6}\ln|u+3| \right\} + C, \quad u = e^{3x}$$

$$= \frac{1}{9}\ln|u-3| - \frac{1}{9}\ln|u+3| + C$$

$$= \frac{1}{9}\ln\left|\frac{e^{3x}-3}{e^{3x}+3}\right| + C$$

EX: $\int \frac{x^3}{x^2-25}\,dx$

Note $\frac{x^3}{x^2-25}$ is an Improper rational function. Use long division 1st. to get,

$$\begin{array}{r|l} & x \\ x^2-25 & x^3 \\ & x^3-25 \\ \hline & 25 \end{array}$$

$$\frac{x^3}{x^2-25} = x + \frac{25}{x^2-25}$$

$$\therefore I = \int \left(x + \frac{25}{x^2-25} \right) dx = \frac{1}{2}x^2 + 25\int \frac{dx}{x^2-25}$$ ← use PFD or may use a standard formula from Table!

Note $\int \frac{dx}{x^2-25} = -\int \frac{dx}{25-x^2}$

$\underset{\text{(see *)}}{\underline{\text{Table}}}$ $-\frac{1}{10}\ln\left|\frac{5+x}{5-x}\right| + C$

$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + \quad \text{---- (*)}$

EX $\int \frac{x+1}{2x+3}\,dx$. Note $\frac{x+1}{2x+3}$ is Improper! Use long division!

$$\begin{array}{r|l} & \frac{1}{2} \\ 2x+3 & x+1 \\ & x+\frac{3}{2} \\ \hline & -\frac{1}{2} \end{array}$$

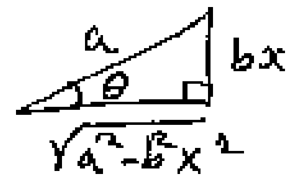
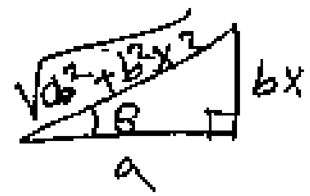
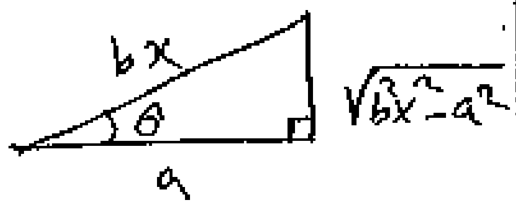
$$\frac{x+1}{2x+3} = \frac{1}{2} + \frac{-\frac{1}{2}}{2x+3} = \frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2x+3}$$

$$\therefore I = \int \left(\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2x+3} \right) dx$$

$$= \frac{1}{2}x - \frac{1}{2}\cdot\frac{\ln|2x+3|}{2} + C = \frac{1}{2}x - \frac{1}{4}\ln|2x+3| + C$$

INTEGRATION BY A TRIGONOMETRIC SUBSTITUTION

Consider the integral $I = \int f(x)\,dx$. If $f(x)$ involves a special quantity, a trigonometric substitution may be used to evaluate the integral. The following Table summarizes the special quantities and the suggested substitutions. Here a, and b are assumed <u>positive</u> real numbers.

The special quantity	Suggested substitution x, dx	The special quantity Becomes	Associated Δ	Example
$a^2 - b^2x^2$	$x = \frac{a}{b}\sin\theta$, $dx = \frac{a}{b}\cos\theta\,d\theta$	$a^2 - b^2x^2 = a^2\cos^2\theta$	$\sin\theta = \frac{bx}{a}$ 	$I = \int \frac{dx}{(16-25x^2)^{\frac{3}{2}}}$ $16 - 25x^2 = a^2 - b^2x^2$ $\Rightarrow a = 4,\ b = 5$ $\therefore$ use $x = \frac{4}{5}\sin\theta$
$a^2 + b^2x^2$	$x = \frac{a}{b}\tan\theta$, $dx = \frac{a}{b}\sec^2\theta\,d\theta$	$a^2 + b^2x^2 = a^2\sec^2\theta$	$\tan\theta = \frac{bx}{a}$ 	$I = \int \frac{x^2}{(x^2+1)^{\frac{3}{2}}}\,dx$ $1 + x^2 = a^2 + b^2x^2 \Rightarrow a = b = 1$ $\therefore$ use $x = \tan\theta$
$b^2x^2 - a^2$	$x = \frac{a}{b}\sec\theta$, $dx = \frac{a}{b}\sec\theta\tan\theta\,d\theta$	$b^2x^2 - a^2 = a^2\tan^2\theta$	$\sec\theta = \frac{bx}{a}$ 	$I = \int \frac{dx}{x^2\sqrt{x^2-4}}$ $x^2 - 4 = b^2x^2 - a^2 \Rightarrow a = 2, b = 1$ $\therefore$ use $x = \frac{2}{1}\sec\theta$

5 Integration by a special Trigonometric substitution:

Given $I = \int f(x)\,dx$. Three cases will be examined:

(a) If integrand has the special quantity $a^2 - b^2x^2$, Let $x = \frac{a}{b}\sin\theta$

(b) If integrand has the special quantity $a^2 + b^2x^2$, let $x = \frac{a}{b}\tan\theta$ $\quad a,b>0$

(c) If integrand has the special quantity $b^2x^2 - a^2$, let $x = \frac{a}{b}\sec\theta$

Refer to Details in table of page (8)! Here is an Example:

EX: $\int \frac{dx}{x\sqrt{x^2-9}}$. let $x^2-9 = b^2x^2 - a^2$ ... hence $b=1$, $a=3$

$\therefore$ use $x = \frac{a}{b}\sec\theta = \frac{3}{1}\sec\theta$

$dx = 3\sec\theta\tan\theta\,d\theta$, and

$x^2 - 9 = (3\sec\theta)^2 - 9 = 9(\sec^2\theta - 1) = 9\tan^2\theta$

$$I = \int \frac{3\sec\theta\tan\theta\,d\theta}{3\sec\theta\sqrt{9\tan^2\theta}} = \int d\theta \;!!$$

$= \theta + C$, but $x = 3\sec\theta$

$\Rightarrow \theta = \sec^{-1}\frac{x}{3}$

$= \sec^{-1}\left(\frac{x}{3}\right) + C$

More Examples in page (10).

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6 Misc. substitutions: Two:

(a) The half-angle substitution: If $I = \int f(\theta)\,d\theta$ where $f(\theta)$ is a rational function of $\sin\theta$, and $\cos\theta$, one may use the so called $\frac{1}{2}$ angle substitution: $\boxed{z = \tan\frac{\theta}{2}}$ ... It follows that

$\boxed{d\theta = \frac{2dz}{1+z^2}}$, $\boxed{\sin\theta = \frac{2z}{1+z^2}}$, $\boxed{\cos\theta = \frac{1-z^2}{1+z^2}}$

we recommend that these formulae be memorized!

EX: $\int \frac{d\theta}{1+\sin\theta+\cos\theta}$ .. use $\frac{1}{2}$ angle substitution.

From above, $I = \int \frac{\frac{2dz}{1+z^2}}{\left\{1 + \frac{2z}{1+z^2} + \frac{1-z^2}{1+z^2}\right\}}$ To simplify multiply each by $1+z^2$

$$= \int \frac{2\,dz}{1+z^2+2z+1-z^2} = \int \frac{2\,dz}{2+2z} = \int \frac{dz}{1+z}$$

$$= \ln|1+z| + C = \ln\left|1+\tan\frac{\theta}{2}\right| + C$$

(b) Integrals involving radicals $\sqrt[n_1]{ax+b}, \sqrt[n_2]{ax+b}, \ldots \sqrt[n_k]{ax+b}$:

let $ax + b = u^n$ where n is the least common multiple of $n_1, n_2, \ldots$, and n_k.

EX Find $I = \int \frac{\sqrt{x}}{\sqrt[6]{x} + \sqrt[3]{x}}\,dx$

Here let $x = u^n$, where n = least Common multiple of 6, 3, and 2 $= 6$

$\therefore x = u^6$, $dx = 6u^5du$

$$I = \int \frac{\sqrt{u^6}}{\sqrt[6]{u^6} + \sqrt[3]{u^6}}\cdot(6u^5du) = 6\int \frac{u^6}{u+u^2}\,du = 6\int \frac{u^5}{1+u}\,du = \ldots$$

Finish !!

(END)

Illustrated Examples: Do at Home: $\int \frac{dx}{x\sqrt{x^2-1}}$

Ans: $\sec^{-1}x + C$.

(1) $\int \frac{dx}{(49-x^2)^{\frac{3}{2}}}$

The Integral involves the quantity

$a^2 - b^2x^2 = 49 - 1x^2$

$\therefore a = 7,\ b = 1$

$\therefore$ let $x = \frac{a}{b}\sin\theta = \frac{7}{1}\sin\theta$

$dx = 7\cos\theta\, d\theta$

and $49 - x^2 = 49 - 49\sin^2\theta$

$= 49(1-\sin^2\theta) = 49\cos^2\theta$

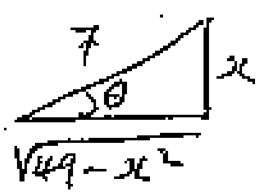
Hence $I = \int \frac{7\cos\theta\, d\theta}{(7^2\cos^2\theta)^{\frac{3}{2}}}$

$= \int \frac{7\cos\theta\, d\theta}{7^3\cos^3\theta} = \frac{1}{49}\int \frac{d\theta}{\cos^2\theta}$

$= \frac{1}{49}\int \sec^2\theta\, d\theta = \frac{1}{49}\tan\theta + C$

But $x = 7\sin\theta \Rightarrow \frac{x}{7} = \sin\theta$

$I \xrightarrow[\Delta]{\text{From}} \frac{1}{49}\cdot\frac{x}{\sqrt{49-x^2}} + C$



(2) $\int \frac{dx}{x^2\sqrt{1+x^2}}$

Integral involves the quantity

$a^2 + b^2x^2 = 1 + 1\cdot x^2$

$\therefore a = b = 1$

$\therefore$ Use $x = \frac{a}{b}\tan\theta = \tan\theta$

$dx = \sec^2\theta\, d\theta$

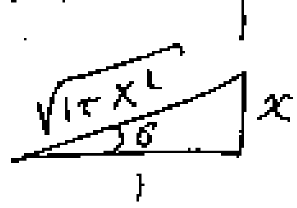
and $1 + x^2 = 1 + \tan^2\theta = \sec^2\theta$

$\therefore I = \int \frac{\sec^2\theta\, d\theta}{\tan^2\theta\sqrt{\sec^2\theta}} = \int \frac{\sec\theta}{\tan^2\theta}d\theta$

$= \int \csc\theta\cot\theta\, d\theta$ (check!)

$= -\csc\theta + C$

But $x = \tan\theta$ or $\tan\theta = \frac{x}{1}$



$\therefore I = -\csc\theta + C$

$\xrightarrow[\Delta]{\text{From}} -\frac{\sqrt{1+x^2}}{x} + C$

(3) $I = \int \frac{dx}{(1+4x^2)^{\frac{3}{2}}}$

The integral has the quantity

$a^2 + b^2x^2 = 1 + 4x^2$

$\therefore a = 1,\ b = 2$

Use $x = \frac{a}{b}\tan\theta = \frac{1}{2}\tan\theta$

$dx = \frac{1}{2}\sec^2\theta\, d\theta$

and that

$1 + 4x^2 = 1 + 4(\frac{1}{4}\tan^2\theta)$

$= 1 + \tan^2\theta = \sec^2\theta$

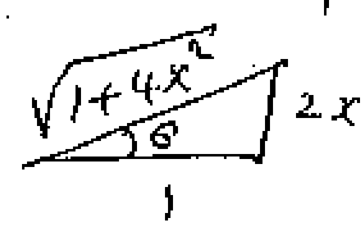
$I = \int \frac{\frac{1}{2}\sec^2\theta\, d\theta}{(\sec^2\theta)^{\frac{3}{2}}} = \frac{1}{2}\int \frac{\sec^2\theta\, d\theta}{\sec^3\theta}$

$= \frac{1}{2}\int \frac{d\theta}{\sec\theta} = \frac{1}{2}\int \cos\theta\, d\theta$

$= \frac{1}{2}\sin\theta + C$

But $x = \frac{1}{2}\tan\theta$ or $\tan\theta = \frac{2x}{1}$

$\therefore I \xrightarrow[\Delta]{\text{From}} \frac{1}{2}\cdot\frac{2x}{\sqrt{1+4x^2}} + C$



(4) Find arc length of $y = \frac{1}{2}x^2$ from $x = 0$ to $x = \sqrt{3}$.

Solution: Arc length

$L = \int_a^b \sqrt{1+y'^2}\, dx$

Here $y = \frac{1}{2}x^2$, $y' = x$, $a = 0$, $b = \sqrt{3}$

$\therefore L = \int_0^{\sqrt{3}} \sqrt{1+x^2}\, dx$

let $x = \frac{a}{b}\tan\theta = \frac{1}{1}\tan\theta$

$\therefore dx = \sec^2\theta\, d\theta$,

$\sqrt{1+x^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = \sec\theta$

$\therefore L = \int_0^{\frac{\pi}{3}} \sec\theta\cdot\sec^2\theta\, d\theta$

$= \int_0^{\frac{\pi}{3}} \sec^3\theta\, d\theta \leftarrow$ by parts (Done in class)

$= \frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]\Big|_0^{\frac{\pi}{3}}$

$= \frac{1}{2}\left[2\sqrt{3} + \ln|2+\sqrt{3}|\right]$

Note: $x = \tan\theta$

when $x = 0$, $\theta = 0$

when $x = \sqrt{3}$, $\theta = \frac{\pi}{3}$

INTEGRALS OF POWERS OF TRIGONOMETRIC FUNCTIONS

#	INTEGRAL(S) TYPE	CONDITION(S)	WHAT TO DO FIRST	SUBSTITUTION W AND dW	REQUIRED IDENTITY	REMARK
1	$\int \sin^m x\,dx$	$m=3,5,7,\dots$ (odd)	Isolate $\sin x$	$w=\cos x$, $dx=-\sin x\,dx$	$\sin^2 x = 1-\cos^2 x = 1-w^2$	
2	$\int \cos^n x\,dx$	$n=3,5,7,\dots$ (odd)	Isolate $\cos x$	$w=\sin x$, $dx=\cos x\,dx$	$\cos^2 x = 1-\sin^2 x = 1-w^2$	
3	$\int \sin^m x\cos^n x\,dx$	$m=1,3,5,\dots$ (odd), $n\in\mathbb{R}$	Isolate $\sin x$	$w=\cos x$, $dx=-\sin x\,dx$	$\sin^2 x = 1-\cos^2 x = 1-w^2$	SAME AS #(1)
4	$\int \sin^m x\cos^n x\,dx$	$n=1,3,5,\dots$ (odd), $m\in\mathbb{R}$	Isolate $\cos x$	$w=\sin x$, $dx=\cos x\,dx$	$\cos^2 x = 1-\sin^2 x = 1-w^2$	SAME AS #(2)
5	$\int \sin^m x\,dx$, $\int \cos^n x\,dx$, $\int \sin^m x\cos^n x\,dx$	$m,n = 2,4,6,\dots$ (Even)	If $m=2$, $n=2$ see Example			USE A REDUCTION FORMULA
6	$\int \tan^m x\,dx$, $\int \sec^m x\,dx$, $\int \sec^m x\tan^r x\,dx$	$m=2,4,6,\dots$ (Even), $r\in\mathbb{R}$	Isolate $\sec^2 x$ (see remark)	$w=\tan x$, $dw=\sec^2 x\,dx$	$\sec^2 x = 1+\tan^2 x = 1+w^2$	For $\int\tan^m x\,dx$, write $\int \frac{\tan^m x}{\sec^2 x}\cdot\sec^2 x\,dx$!!
7	$\int \cot^m x\,dx$, $\int \csc^m x\,dx$, $\int \csc^m x\cot^r x\,dx$	$m=2,4,6,\dots$ (Even), $r\in\mathbb{R}$	Isolate $\csc^2 x$ (see remark)	$w=\cot x$, $dw=-\csc^2 x\,dx$	$\csc^2 x = 1+\cot^2 x = 1+w^2$	For $\int\cot^m x\,dx$, write $\int \frac{\cot^m x}{\csc^2 x}\csc^2 x\,dx$!!
8	$\int \tan^n x\,dx$, $\int \sec^m x\tan^n x\,dx$	m,n are both odd	Isolate $\sec x\tan x$ (see remark)	$w=\sec x$, $dw=\sec x\tan x\,dx$	$\tan^2 x = \sec^2 x - 1 = w^2-1$	For $\int\tan^n x\,dx$, write $\int \frac{\tan^{n-1}x}{\sec x}\cdot\sec x\tan x\,dx$
9	$\int \cot^n x\,dx$, $\int \csc^m x\cot^n x\,dx$	m,n are both odd	Isolate $\csc x\cot x$ (see remark)	$w=\csc x$, $dw=-\csc x\cot x\,dx$	$\cot^2 x = \csc^2 x - 1 = w^2-1$	For $\int\cot^n dx$, write $\int \frac{\cot^{n-1}x}{\csc x}\csc x\cot x\,dx$
10	$\int \sec^m x\,dx$, $\int \csc^m x\,dx$, $\int \sec^m x\tan^n x\,dx$, $\int \csc^m x\cot^n x\,dx$	m odd, n Even	Replace every $\tan^2 x$ by $\boxed{\sec^2 x - 1}$ and Every $\cot^2 x$ by $\boxed{\csc^2 x - 1}$			USE A REDUCTION FORMULA
11	Types (1) – (10) In Hyperbolic					Exactly same as TRIG !!

Examples:

Type (1): $I = \int \sin^5 x\,dx = \int \sin^4 x \sin x\,dx = \int (\sin^2 x)^2 \sin x\,dx$

Now, let $w = \cos x$, $dw = -\sin x\,dx$, $\sin^2 x = 1-\cos^2 x = 1-w^2$

$$I = -\int (1-w^2)^2\,dw = -\int (1-2w^2+w^4)\,dw = -\left[w - \tfrac{2}{3}w^3 + \tfrac{1}{5}w^5\right] + C$$

$$= -\left[\sin x - \tfrac{2}{3}\sin^3 x + \tfrac{1}{5}\sin^5 x\right] + C$$

Type (2): Hyperbolic example

$$I = \int \cosh^3 x\,dx = \int \cosh^2 x \cosh x\,dx$$

Now, let $w = \sinh x$, $dw = \cosh x\,dx$, $\cosh^2 x = 1+\sinh^2 x = 1+w^2$

$$I = \int (1+w^2)\,dw = w + \tfrac{1}{3}w^3 + C = \sinh x + \tfrac{1}{3}\sinh^3 x + C$$

Type (3): $I = \int \sin^3 x \cos^{\frac{7}{3}} x\,dx = \int \sin^2 x \cos^{\frac{7}{3}} x \cdot \sin x\,dx$

Now, let $w = \cos x$, $dw = -\sin x\,dx$, $\sin^2 x = 1-\cos^2 x = 1-w^2$

$$I = -\int (1-w^2)\,w^{\frac{7}{3}}\,dw = -\int \left(w^{\frac{7}{3}} - w^{\frac{13}{3}}\right) dw$$

$$= -\left[\tfrac{3}{10} w^{\frac{10}{3}} - \tfrac{3}{16} w^{\frac{16}{3}}\right] + C$$

$$= -\left[\tfrac{3}{10} \cos^{\frac{10}{3}} x - \tfrac{3}{16} \cos^{\frac{16}{3}} x\right] + C$$

Type (4): Hyperbolic example

$$I = \int \sinh^{10}(3x) \cosh^3(3x)\,dx = \int \sinh^{10}(3x)\cosh^2(3x)\cosh(3x)\,dx$$

let $w = \sinh(3x)$, $dw = 3\cosh(3x)\,dx$, $\cosh^2(3x) = 1+\sinh^2(3x) = 1+w^2$

$$\therefore I = \int w^{10}(1+w^2)\cdot\frac{dw}{3} = \frac{1}{3}\int (w^{10}+w^{12})\,dw = \frac{1}{3}\left[\frac{1}{11}w^{11} + \frac{1}{13}w^{13}\right] + C$$

$$= \frac{1}{3}\left[\frac{1}{11}\sinh^{11}(3x) + \frac{1}{13}\sinh^{13}(3x)\right] + C$$

Type (6): $I = \int \sec^6 x \tan^{\frac{3}{5}} x\,dx = \int \sec^4 x \tan^{\frac{3}{5}} x \sec^2 x\,dx$

$$= \int (\sec^2 x)^2 \tan^{\frac{3}{5}} x \sec^2 x\,dx$$

Now let $w = \tan x$, $dw = \sec^2 x\,dx$, $\sec^2 x = 1+\tan^2 x = 1+w^2$

$$\therefore I = \int (1+w^2)^2\, w^{\frac{3}{5}}\,dw = \int (1+2w^2+w^4)\,w^{\frac{3}{5}}\,dw$$

$$= \int \left(w^{\frac{3}{5}} + 2w^{\frac{13}{5}} + w^{\frac{23}{5}}\right) dw = \frac{5}{8}w^{\frac{8}{5}} + \frac{5}{18}w^{\frac{18}{5}} + \frac{5}{28}w^{\frac{28}{5}} + C$$

$$= \frac{5}{8}\tan^{\frac{8}{5}} x + \frac{5}{18}\tan^{\frac{18}{5}} x + \frac{5}{28}\tan^{\frac{28}{5}} x + C$$

Type (7): Hyperbolic Example

$I = \int \coth^6 x\,dx$ .. 1st. rewrite as $\int \dfrac{\coth^6 x}{\operatorname{csch}^2 x}\operatorname{csch}^2 x\,dx$

Now, let $w = \coth x$, $dw = -\operatorname{csch}^2 x\,dx$, $\operatorname{csch}^2 x = \coth^2 x - 1 = w^2 - 1$

$\therefore I = -\int \dfrac{w^6}{(w^2-1)}\,dw$ ←(use long division)→

$$\begin{array}{r|l} & w^4+w^2+1 \\ \hline w^2-1 & w^6 \\ & w^6-w^4 \\ \hline & w^4 \\ & w^4-w^2 \\ \hline & w^2 \\ & w^2-1 \\ \hline & +1 \end{array}$$

$$= -\int \left(w^4 + w^2 + 1 + \frac{1}{w^2-1}\right) dw$$

$$= -\frac{1}{5}w^5 - \frac{1}{3}w^3 - w + \coth^{-1}(w) + C$$

$$= -\frac{1}{5}\coth^5 x - \frac{1}{3}\coth^3 w - \coth w + x + C$$

Type (8) $I = \int \sec^{11}x \tan^3 x\, dx = \int \sec^{10}x \tan^2 x (\sec x \tan x)\, dx$

Now, let $w = \sec x$, $dw = \sec x \tan x\, dx$, $\tan^2 x = \sec^2 x - 1 = w^2 - 1$.

$\therefore I = \int w^{10}(w^2-1)\, dw = \int (w^{12} - w^{10})\, dw = \frac{1}{13} w^{13} - \frac{1}{11} w^{11} + C$

$= \frac{1}{13} \sec^{13}x - \frac{1}{11} \sec^{11}x + C$

Type (9): Hyperbolic Example

$I = \int \coth^3 x\, dx$ ... 1st. rewrite as $\int \frac{\coth^2 x}{\operatorname{csch} x} \operatorname{csch} x \coth x\, dx$

Now, let $w = \operatorname{csch} x$, $dw = -\operatorname{csch} x \coth x$, $\coth^2 x = \operatorname{csch}^2 x + 1 = w^2 + 1$.

$\therefore I = -\int \frac{w^2+1}{w}\, dw = -\int (w + \frac{1}{w})\, dw = -\frac{1}{2} w^2 - \ln|w| + C$

$= -\frac{1}{2} \operatorname{csch}^2 x - \ln|\operatorname{csch} x| + C$

Type (5): Find (1) $\int \sin^2 x\, dx$, (2) $\int \cos^2(10x)\, dx$

Note: we shall use the identities: $\sin^2 A = \frac{1}{2}[1 - \cos 2A]$, $\cos^2 A = \frac{1}{2}[1 + \cos 2A]$

(1) $\int \sin^2 x\, dx \overset{id.}{=} \frac{1}{2} \int (1 - \cos 2x)\, dx = \frac{1}{2}[x - \frac{\sin 2x}{2}] + C$.

(2) $\int \cos^2(10x)\, dx \overset{id.}{=} \frac{1}{2} \int \{1 + \cos(20x)\}\, dx = \frac{1}{2}[x + \frac{\sin(20x)}{20}] + C$

Type (10): Find $\int \sec^7 x\, dx$. We shall use a reduction formula:

let $I_n = \int \sec^n x\, dx$, $n \geq 2$

$= \int \sec^{n-2}x \sec^2 x\, dx \to$ by parts once:

$u = \sec^{n-2}x$ $\qquad dv = \sec^2 x\, dx$

$du = (n-2) \sec^{n-3}x \cdot \sec x \tan x\, dx$ $\qquad v = \tan x$

$I_n = \tan x \sec^{n-2}x - (n-2) \int \sec^{n-2}x \tan^2 x\, dx$ ... but $\tan^2 x = \sec^2 x - 1$

$= \tan x \sec^{n-2}x - (n-2) \int \sec^{n-2}x (\sec^2 x - 1)\, dx$

$= \tan x \sec^{n-2}x - (n-2) \int (\sec^n x - \sec^{n-2}x)\, dx$

$= \tan x \sec^{n-2}x - (n-2)[I_n - I_{n-2}]$

$I_n = \tan x \sec^{n-2}x - (n-2) I_n + (n-2) I_{n-2}$

$(n-2) I_n + I_n = \tan x \sec^{n-2}x + (n-2) I_{n-2} \Rightarrow$

$I_n = \frac{1}{n-1} \tan x \sec^{n-2}x + \frac{n-2}{n-1} I_{n-2}$, $n = 2, 3, 4, \ldots$

Since we need $\int \sec^7 x\, dx = I_7$, we put $n = 3, 5,$ and 7 to get

$I_7 = \int \sec^7 x\, dx = \frac{1}{6} \tan x \sec^5 x + \frac{5}{24} \tan x \sec^3 x + \frac{5}{16} \tan x \sec x + \frac{5}{16} \ln|\sec x + \tan x| + C$ (verify!!)

AMAT 219

Support Material

Improper Integrals

Def.: The integral $I = \int_a^b f(x)\,dx$, $a, b \in \mathbb{R}$ is said to be proper if $f(x)$ is continuous on $[a, b]$.

Def.: The integral $I = \int_\alpha^\beta f(x)\,dx$ is said to be an Improper integral of type (I) if either $\alpha = -\infty$ or $\beta = +\infty$ and $f(x)$ is continuous over the integration range.

Def.: The integral $I = \int_\alpha^\beta f(x)\,dx$, $\alpha, \beta \in \mathbb{R}$ is said to be Improper integral of type (II) if $f(x)$ is continuous except at either lower limit α, upper limit β, or at some γ between α, and β where f becomes unbounded $\left(\frac{\text{a non-zero \#}}{\text{zero}}\right)$.

EX: $\int_0^\infty x e^{-2x}\,dx$, $\int_{-\infty}^2 \frac{dx}{(3-x)^2}$, $\int_{-\infty}^\infty \frac{dx}{1+x^2}$ are Improper integrals of type (I).

EX $\int_1^e \frac{dx}{\ln x}$, $\int_2^{18} \frac{dx}{\sqrt[3]{x-12}}$, $\int_0^{2\pi} \tan\left(\frac{\pi}{4}\right) dx$ are improper integrals of type (II).

Indeed the integrand $\frac{1}{\ln x}$ is unbounded at lower limit $x=1$, the integrand $\frac{1}{\sqrt[3]{x-12}}$ becomes unbounded at $x = 12 \in [2, 18]$, and $\tan\left(\frac{\pi}{4}\right)$ is infinite at upper limit $x = 2\pi$.

Def.: Consider the Improper integral of type (I):

$$I = \int_a^{+\infty} f(x)\,dx, \quad f \text{ continuous on } [a, \infty)$$

is said to converge to real number "L" if

$$\lim_{R \to +\infty} \int_a^R f(x)\,dx \quad \text{exists and equal to } L.$$

However if the limit above does not exist or equal to $\pm\infty$, one says the integral diverges.

The convergence or divergence of $\int_{-\infty}^b f(x)\,dx$ is defined similarly.

<u>Def.</u> : let $I = \int_a^b f(x)\,dx$ be an improper Integral type (II) due to the unboundedness of f at lower limit "a". The integral is said to Converge to real number "L" if

$$\lim_{c \to a^+} \int_c^b f(x)\,dx \quad \text{exists and Equal to "L".}$$

However if limit does not exist or is $+\infty$ or $-\infty$, we say integral diverges.

<u>Note</u> : If Integral is improper due to the unboundedness of $f(x)$ at upper limit "b", we say $\int_a^b f(x)\,dx$ Converge to L if

$$\lim_{c \to b^-} \int_a^c f(x)\,dx \quad \text{exists and Equal to } L.$$

Next : If $f(x)$ is unbounded only at d, $a < d < b$, we write

$$I = \int_a^d f(x)\,dx + \int_d^b f(x)\,dx = I_1 + I_2$$ and examine Convergence or divergence of I_1 and I_2 separately!

<u>Examples</u>: In each case show whether integral Converges or diverges.

(1) $\int_0^\infty x e^{-x^2}\,dx$

<u>Solution</u> : Integral is improper of type (I).

$\therefore \; I = \lim_{R \to \infty} \int_0^R x e^{-x^2}\,dx$ — let $u = -x^2$, $du = -2x\,dx$, we get

$$I = -\frac{1}{2} \lim_{R \to \infty} e^{-x^2}\Big|_0^R = -\frac{1}{2} \lim_{R \to \infty} \left[e^{-R^2} - 1\right] = -\frac{1}{2}(0-1) = \frac{1}{2}$$

$\therefore$ Integral Converges to $\frac{1}{2}$.

(2) $\int_0^\infty \frac{dx}{(x+1)^4}$

Soln. $I = \lim_{R \to \infty} \int_0^R \frac{dx}{(x+1)^4} = \lim_{R \to \infty} \int_0^R (x+1)^{-4}\,dx$

$$= \lim_{R \to \infty} \frac{(x+1)^{-3}}{-3}\Big|_0^R = -\frac{1}{3} \lim_{R \to \infty} \left[\frac{1}{(R+1)^3} - 1\right] = -\frac{1}{3}[0-1] = \frac{1}{3}$$

$\therefore$ Integral Converges to $\frac{1}{3}$.

(3) $\int_{-\infty}^0 x^2 e^{3x}\,dx$

Soln $I = \lim_{R \to -\infty} \int_R^0 x^2 e^{3x}\,dx$ ← by parts Twice.

$$= \lim_{R \to -\infty} e^{3x}\left[\frac{x^2}{3} - \frac{2}{9}x + \frac{2}{27}\right]\Big|_R^0$$

$$= \lim_{R \to -\infty} \left\{ \frac{2}{27} - e^{3R}\left(\frac{R^2}{3} - \frac{2}{9}R + \frac{2}{27}\right) \right\} = \frac{2}{27}.$$

$$\begin{array}{ccc} x^2 & \oplus & e^{3x} \\ 2x & \ominus & \frac{e^{3x}}{3} \\ 2 & \oplus & \frac{e^{3x}}{9} \\ 0 & & \frac{e^{3x}}{27} \end{array}$$

(4) $\displaystyle\int_{-1}^{7} \frac{dx}{\sqrt[3]{x+1}}$

Sol. Note that the integrand $\dfrac{1}{\sqrt[3]{x+1}}$ is unbounded at $x=-1$ (it becomes $\frac{1}{0}$!!). Hence integral is Improper – type (II).

Now $I = \displaystyle\lim_{c\to -1^+} \int_{-1}^{7} \frac{dx}{\sqrt[3]{x+1}} = \lim_{c\to -1^+} \int_{c}^{7} (x+1)^{-\frac{1}{3}}\, dx$

$= \frac{3}{2} \displaystyle\lim_{c\to -1^+} (x+1)^{\frac{2}{3}}\Big|_c^7 = \frac{3}{2} \lim_{c\to -1^+} \left[8^{\frac{2}{3}} - (c+1)^{\frac{2}{3}} \right]$

$= \frac{3}{2}\left[8^{\frac{2}{3}} - 0\right] = \frac{3}{2}(2)^{\cancel{3}\cdot\frac{2}{\cancel{3}}} = \frac{3}{2}\cdot 4 = 6$

$\therefore$ It Converges to 6.

(5) $\displaystyle\int_0^{6\pi} \tan\left(\frac{x}{12}\right) dx$. observe that at $x = 6\pi$, integrand becomes $\tan\left(\frac{6\pi}{12}\right) = \tan\frac{\pi}{2}$ which is unbounded (infinite).

$\therefore$ $I = \displaystyle\int_0^{6\pi} \tan\left(\frac{x}{12}\right) dx$ is improper of type (II).

Now, $I = \displaystyle\lim_{c\to (6\pi)^-} \int_0^{c} \tan\left(\tfrac{1}{12}x\right) dx = \lim_{c\to(6\pi)^-} \frac{\ln|\sec\frac{x}{12}|}{\left(\frac{1}{12}\right)}\Bigg|_0^c$

$= 12 \displaystyle\lim_{c\to(6\pi)^-} \left[\ln|\sec\tfrac{c}{12}| - \ln|\sec 0| \right]$

$= 12\left[\infty - 0\right] = \infty$ .. Integral diverges to $+\infty$.

(6) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d}{dx}\left\{\frac{\sin x}{x}\right\} dx$. Integral is improper of type (II). It is unbounded at lower limit $x = 0$.

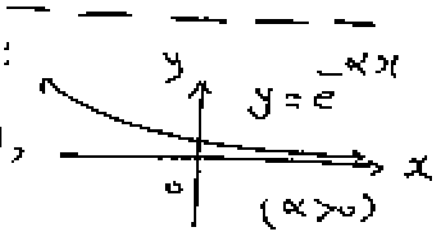
write $I = \displaystyle\lim_{c\to 0^+} \int_c^{\frac{\pi}{2}} \frac{d}{dx}\left(\frac{\sin x}{x}\right) dx = \lim_{c\to 0^+} \frac{\sin x}{x}\Bigg|_c^{\frac{\pi}{2}}$

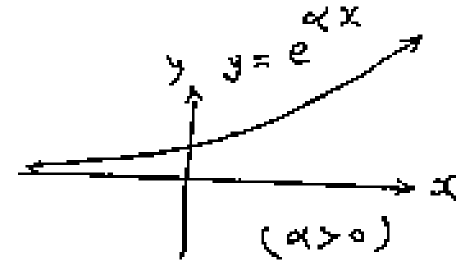
$= \displaystyle\lim_{c\to 0^+}\left[\frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} - \frac{\sin c}{c}\right] = \frac{2}{\pi} - \lim_{c\to 0^+}\frac{\sin c}{c}$ ← Apply L'H Rule.

$= \frac{2}{\pi} - \displaystyle\lim_{c\to 0^+} \frac{\cos c}{1} = \frac{2}{\pi} - 1$.

Special Limits / Values:

$\boxed{1}$ Refer to figures: ($e^0 = 1$),

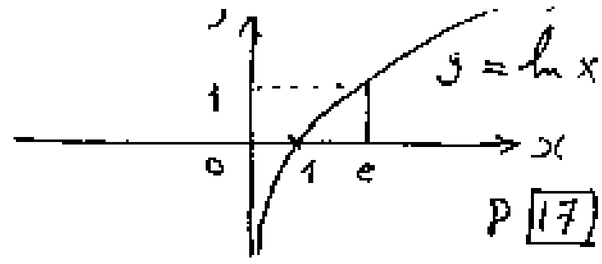




$\displaystyle\lim_{x\to -\infty} e^{\alpha x} = 0$, $\displaystyle\lim_{x\to +\infty} e^{\alpha x} = \infty$, $\displaystyle\lim_{x\to -\infty} e^{-\alpha x} = +\infty$, $\displaystyle\lim_{x\to \infty} e^{-\alpha x} = 0$

Important: Limits above remain true if exponential function is multiplied by x^n for some $n > 0$

Ex $\displaystyle\lim_{x\to -\infty} x^3 e^{7x} = 0$, $\displaystyle\lim_{x\to \infty} x^2 e^{4x} = \infty$, $\displaystyle\lim_{x\to -\infty} e^{-2x}\cdot x^4 = \infty$, $\displaystyle\lim_{x\to\infty} x^9 e^{-2x} = 0$



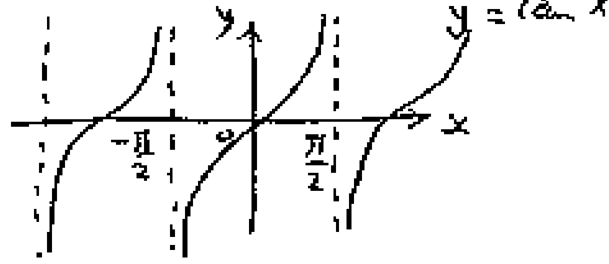
2 Refer to figure:

$\ln 1 = 0$, $\ln e = 1$.,

$\lim_{x\to+\infty} \ln x = +\infty$, $\lim_{x\to 0^+} \ln x = -\infty$

3 $\lim_{x\to 0^-} \frac{1}{x} = -\infty$ (or $\frac{1}{0^-} = -\infty$), $\lim_{x\to 0^+} \frac{1}{x} = +\infty$ (or $\frac{1}{0^+} = +\infty$)

$\lim_{x\to+\infty} \frac{1}{x} = 0$ (or $\frac{1}{\infty} = 0$), $\lim_{x\to-\infty} \frac{1}{x} = 0$ (or $\frac{1}{-\infty} = 0$)



4 $\lim_{\theta\to\frac{\pi}{2}^-} \tan\theta = +\infty$, $\lim_{\theta\to\frac{\pi}{2}^+} \tan\theta = -\infty$

$\lim_{\theta\to-\frac{\pi}{2}^-} \tan\theta = +\infty$ $\lim_{\theta\to-\frac{\pi}{2}^+} \tan\theta = -\infty$

5 $\lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$ (or $\tan^{-1} -\infty = -\frac{\pi}{2}$), $\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$ (or $\tan^{-1}\infty = \frac{\pi}{2}$).

6 $\sin 0 = 0$, $\sinh 0 = 0$, $\cos 0 = 1$, and $\cosh 0 = 1$.

L'Hospital Rule: If $\lim_{x\to c} \frac{f(x)}{g(x)}$ has the form $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

The rule may be repeated several time if necessary!
also the rule is true if $x\to c$ is replaced by $x\to c^-$, $x\to c^+$, $x\to+\infty$, or $x\to-\infty$.

EX: Find $\lim_{x\to+\infty} \frac{\ln x}{x^2}$.

Soln. $\lim_{x\to+\infty} \frac{\ln x}{x^2}$ (form $\frac{\infty}{\infty}$) $\Rightarrow \lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{(\ln x)'}{(x^2)'}$

$= \lim_{x\to\infty} \frac{\frac{1}{x}}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0$

EX Find $\lim_{x\to 0^+} \left(\frac{\cos 3x - 1}{x^2}\right)$

Soln. $\lim_{x\to 0^+} \frac{\cos 3x - 1}{x^2}$ (form $\frac{0}{0}$)

$\therefore \lim_{x\to 0^+} \frac{(\cos 3x - 1)'}{(x^2)'} = \lim_{x\to 0^+} \frac{-3\sin 3x}{2x}$ (form $\frac{0}{0}$!!)

$= \lim_{x\to 0^+} \frac{-9\cos 3x}{2} = -\frac{9\cos 0}{2} = -\frac{9(1)}{2} = -\frac{9}{2}$

EX show whether $I = \int_{-\infty}^{\infty} \frac{dx}{x^2+9}$ Converges!

Soln. 1st. note that $\frac{1}{x^2+9}$ is an Even function.. hence $I = 2\int_0^{\infty} \frac{dx}{x^2+9}$

$I = 2\lim_{R\to\infty} \int_0^R \frac{dx}{x^2+9} = \frac{2}{3}\lim_{R\to\infty} \tan^{-1}\left(\frac{x}{3}\right)\Big|_0^R = \frac{2}{3}\lim_{R\to\infty}\left[\tan^{-1}\frac{R}{3} - \tan^{-1} 0\right]$

$= \frac{2}{3}\left[\tan^{-1}\infty\right] = \frac{2}{3}\frac{\pi}{2} = \frac{\pi}{3}$.

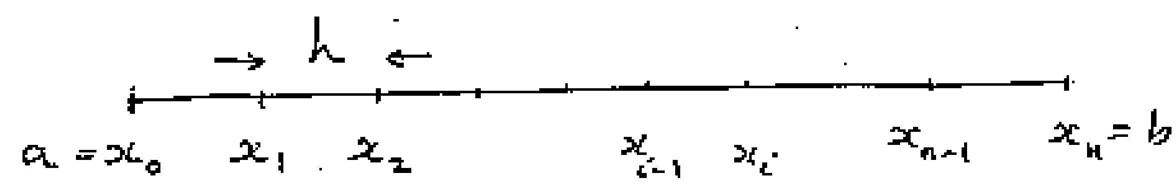
Converges!

AMAT 219

Support Material

Numerical Integration

Let $I = \int_a^b f(x)\,dx$, $a, b \in \mathbb{R}$ and $f(x)$ is assumed continuous on $[a,b]$. Subdivide the interval $[a,b]$ into n subintervals of equal length using the points $x_0 = a, x_1, x_2, \ldots, x_{n-1}$, and $x_n = b$ as shown.



Clearly: The length of each interval $\boxed{h = \frac{b-a}{n}}$ and that

$$x_1 = x_0 + h, \quad x_2 = x_1 + h, \quad x_3 = x_2 + h, \ \ldots \text{ and so on!}$$

There are three methods for approximating the integral I.

1 The Trapezoidal Rule: "T_n"

Let T_n be an approximation to I using n subintervals above. Then

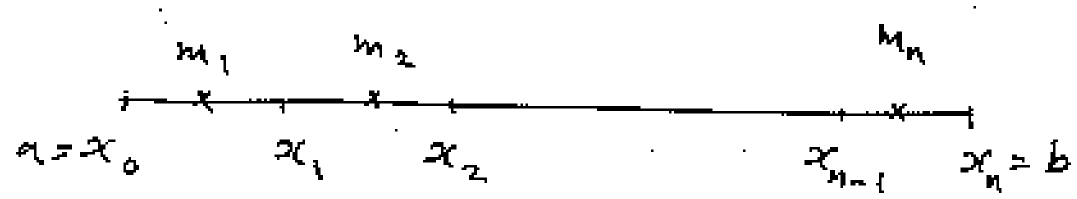
$$T_n = \frac{h}{2}\left\{ y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n \right\}$$

$$\text{or } = \frac{h}{2}\left\{ y_{\text{Ends}} + 2y_{\text{middle}} \right\}, \text{ where } y_i = f(x_i),\ i = 0, 1, 2, \ldots n.$$

The Error in the approximation is estimated by $|I - T_n| \leq \frac{K(b-a)^3}{12n^2}$ where K is a constant positive real number satisfying $|f''(x)| \leq K$, $x \in [a,b]$.

2 The Midpoint Rule: "M_n"

Let $m_1, m_2, \ldots,$ and m_n be the mid points of the n subintervals as shown



Note: $m_1 = \frac{x_0 + x_1}{2}$, $m_2 = \frac{x_1 + x_2}{2}$, ... and so on.

Then an approximation to I is thus given by

$$M_n = h\left\{ y_1 + y_2 + \cdots + y_n \right\} \text{ where } y_i = f(m_i) \quad i = 1, 2, \ldots n$$

The error in the approximation is estimated by $|I - M_n| \leq \frac{K(b-a)^3}{24n^2}$ where K is as defined above!

3 The Simpson Rule: "S_n"

Let "n" be a positive Even integer. Then the integral I may be approximated by

$$S_n = \frac{h}{3}\left\{ y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 4y_{n-1} + y_n \right\}, \quad y_i = f(x_i), \; i = 1, 2, \ldots n$$

$$\text{or } \frac{h}{3}\left\{ y_{\text{Ends}} + 4y_{\text{odds}} + 2y_{\text{Evens}} \right\}$$

The Error in the approximation is estimated by $|I - S_n| \leq \frac{K(b-a)^5}{180\, n^4}$

where K is a positive real number such that $|f^{(4)}(x)| \leq K$, $x \in [a,b]$.

Important conclusions

(1) If $f(x)$ is a function whose Second Derivative is zero (that is $f(x)$ is a polynomial in x of degree one or less), then

$$T_n = M_n = \text{Exact value of } I$$

In other words: Errors in estimation are zeros!

(2) If $f(x)$ is a function whose Fourth Derivative is zero (that is $f(x)$ is a polynomial of degree three or less), then

$$S_n = \text{Exact value of } I$$

Again: This means No Error involved in approximation!

EX: Find approximate value of $\int_0^5 2x^3\, dx$ using Simpson rule with $n = 20$.

solution: need S_{20}. Indeed $f(x) = 2x^3 \Rightarrow f^{(4)}(x) = 0$

$$\therefore S_{20} = \text{Exact value of } I = \int_0^5 2x^3 dx = \left.\frac{x^4}{2}\right|_0^5 = \frac{625}{2}$$

Important Relations: (1) $T_{2n} = \frac{T_n + M_n}{2}$ (2) $S_{2n} = \frac{T_n + 2M_n}{3}$

(3) $S_{2n} = \frac{2T_{2n} + M_n}{3}$ (4) $S_{2n} = \frac{4T_{2n} - T_n}{3}$

EX Let $I = \int_1^5 x^2 dx$. Find T_4, M_4, T_8, and S_8. Also find $3S_8 - 4T_8 + T_4$.

soln. we need only to calculate T_4, and M_4, then use relations above!

$$T_4 = \frac{h}{2}\left\{ y_{\text{Ends}} + 2y_{\text{mid}} \right\} = 42 \quad \text{check!}$$

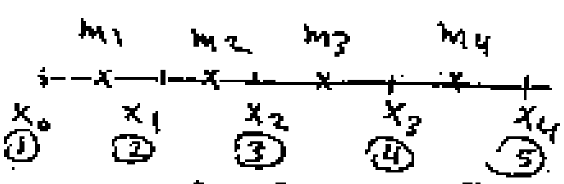
$$M_4 = h\left\{ y_0 + y_1 + y_2 + y_3 + y_4 \right\} = 1\left[\left(\tfrac{3}{2}\right)^2 + \left(\tfrac{5}{2}\right)^2 + \left(\tfrac{7}{2}\right)^2 + \left(\tfrac{9}{2}\right)^2 \right] = 41$$

$h = \frac{b-a}{n} = \frac{5-1}{4} = 1$

clearly: $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$, and $x_4 = 5$.

From (1): $T_8 = \frac{T_4 + M_4}{2} = \frac{42 + 41}{2} = \frac{83}{2} = 41.5$

From (2) $S_8 = \frac{T_4 + 2M_4}{3} = \frac{42 + 2(41)}{3} = \frac{124}{3}$ (*)



$m_1 = \frac{1+2}{2} = \frac{3}{2}$, $m_2 = \frac{5}{2}$, $m_3 = \frac{7}{2}$, $m_4 = \frac{9}{2}$

From (4): $S_{2n} = \frac{4T_{2n} - T_n}{3} \Rightarrow 3S_{2n} = 4T_{2n} - T_n$

or $3S_{2n} - 4T_{2n} + T_n = 0$

put $n = 4$: $3S_8 - 4T_8 + T_4 = 0$ #

Note: $f(x) = x^2 \Rightarrow f^{(4)}(x) = 0 \Rightarrow S_8 = \text{Exact value of } I = \int_1^5 x^2 dx = \left.\frac{x^3}{3}\right|_1^5 = \frac{124}{3}$ as shown in (*)!!

AMAT 219

Support Material

Double Integrals

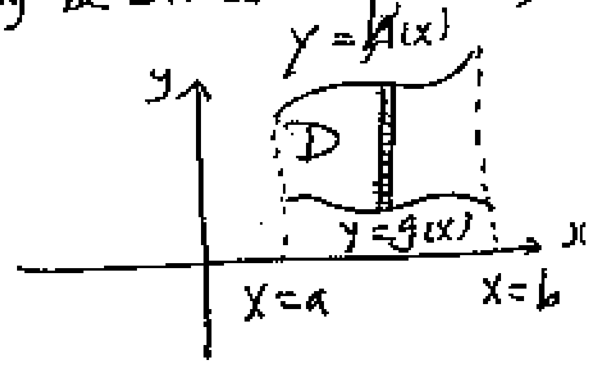
Let $z = f(x,y)$ be a function of the two independent variables x, and y and let D be a closed region in xy-plane. The double integral of f over D is denoted by $I = \iint_D f(x,y)\, dA$ where $dA = dx\,dy$

There are two types of regions in $\mathbb{R}^2$. These are

(a) The y - Simple region

A region D in $\mathbb{R}^2$ is said to be y-Simple if it is bounded from below and above by the continuous curves $y = g(x)$, $y = h(x)$ respectively and bounded from sides by the vertical lines $x = a$, and $x = b$.

It follows that a region of y-Simple type may be sliced vertically and thus is described by the inequalities:



$$D: \quad g(x) \le y \le h(x)$$
$$a \le x \le b$$

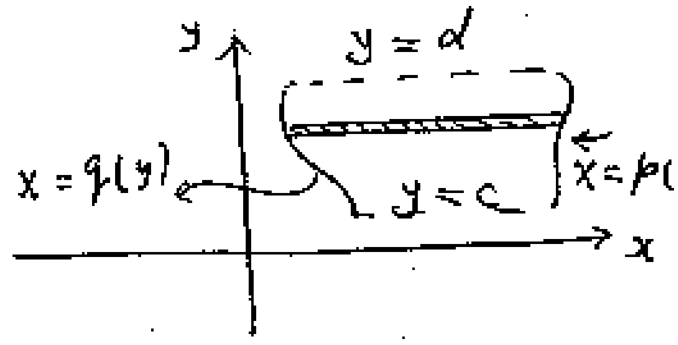
The double integral is equivalent to the iterated integral

$$I = \int_a^b \left\{ \int_{\text{From } y = g(x)}^{\text{to } y = h(x)} f(x,y)\, dy \right\} dx \quad \text{--- (1)}$$

(b) The x - Simple region

A region D in $\mathbb{R}^2$ is said to be x-Simple if it bounded from the left and the right by the continuous curves $x = q(y)$, $x = p(y)$ respectively and bounded from bottom and top by the horizontal lines $y = c$, $y = d$.

It follows that an x-Simple region may be sliced horizontally and thus is described by



$$D: \quad q(y) \le x \le p(y),$$
$$c \le y \le d$$

The double integral is equivalent to the iterated integral

$$I = \int_c^d \left\{ \int_{\text{From } x = q(y)}^{\text{to } x = p(y)} f(x,y)\, dx \right\} dy \quad \text{--- (2)}$$

Remarks

(1) Let $f(x,y)$ be a function of two variables. The integral $\int f(x,y)\,dx$ is called "a partial integral". This means we integrate w.r.to "x" considering "y" a constant! Similarly $\int f(x,y)\,dy$ means we integrat w.r.to y holding "x" fixed!

EX $I = \int (x^2 + 2x\sin y)\,dy = \int x^2\,dy + \int 2x\sin y\,dy$ .. But "x" Constant

$\therefore \quad I = x^2 \int dy + 2x\int \sin y\,dy = x^2 y - 2x\cos y$

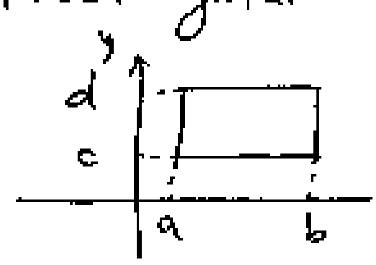
(2) An iterated integral consists of two partial integrations.

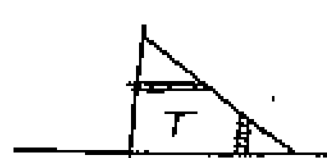
(3) For y-Simple region we slice vertically and hence integrate w.r.to y-first. For x-Simple region we slice horizontally and hence integrate w.r.to x-first.

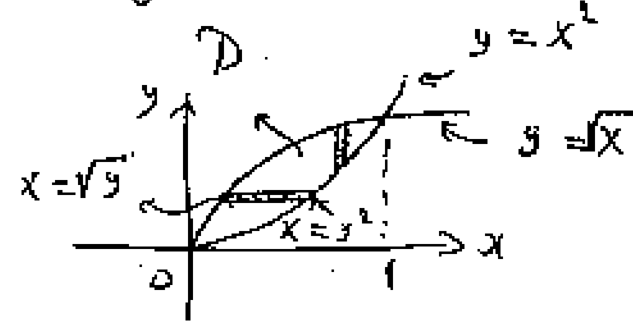
(4) Read limits always from Left to Right or bottom to Top.

(5) For integrals (1), (2) in page #(1), the OUTER LIMITS are always constant real numbers !!

(6) Some regions D in $\mathbb{R}^2$ are both x-Simple and y-Simple. For Example: A circular region, a Triangular region, a rectangular region, .. etc.







$R: \; a \le x \le b$
$\quad\;\; c \le y \le d$

(Both x & y-Simple)

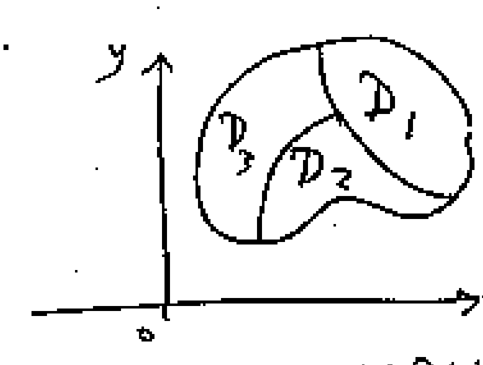
T: Both x & y Simple

$D: \; \sqrt{x} \le y \le x^2$ (y-Simple)
$\quad\;\; 0 \le x \le 1$

OR

$D: \; \sqrt{y} \le x \le y^2$ (x-Simple)
$\quad\;\; 0 \le y \le 1$

(7) A region that is neither x or y Simple may be divided into regions that are x-Simple or y-Simple.
Let D be a region. Then D may be expressed as a union of regions $D_1, D_2, \dots$ that are x or y-Simple.



In this Case

$$I = \iint_D f(x,y)\,dA = \iint_{D_1} + \iint_{D_2} + \cdots$$

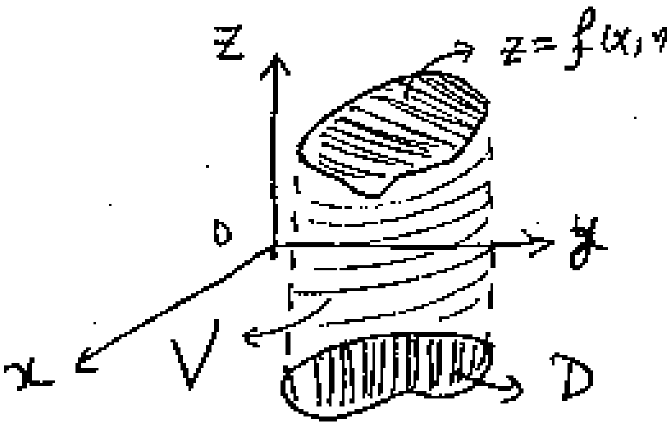
$D = D_1 \cup D_2 \cup D_3 \cup \dots$

(8) Geometric Interpretation of a double Integral

Let $z = f(x,y)$ be a surface in $\mathbb{R}^3$ and D be a closed region in $\mathbb{R}^2$ as shown. Then if $f(x,y) \ge 0$ for all $(x,y) \in D$, the double integral

$$I = \iint_D f(x,y)\, dA = \text{volume of solid which lies vertically above } D \text{ and below the surface } z=f(x,y)$$

$= \text{say } V$



(4) Let $I = \iint_D f(x,y)\, dA$. If $f(x,y) = 1$, we get:

$$I = \iint_D dA = \text{Area of region } D \text{ !!}$$

Double Integrals in polar coordinates

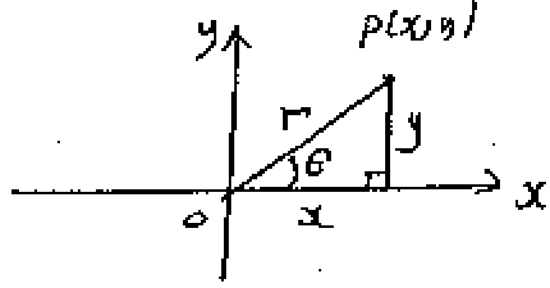
The polar coordinates: let $P(x,y)$ be a point in xy-plane and define:

(1) r = distance from O to P

(2) θ = Angle made by $\vec{OP}$ and the positive half of x-axis measured in radians in the c.c.w sense!

Note that (r,θ) is called the polar coordinates of point P.

and that $r \geq 0$, $0 \leq \theta \leq 2\pi$.



From Δ: $\frac{x}{r} = \cos\theta$, or $\boxed{x = r\cos\theta}$

$\frac{y}{r} = \sin\theta$, or $\boxed{y = r\sin\theta}$

and $\boxed{x^2+y^2=r^2}$

Important Relation: $dA = dx\,dy = r\,dr\,d\theta$

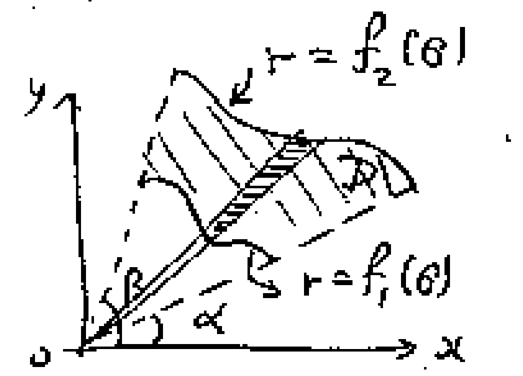
$\therefore$ In polar coordinates $\boxed{dA = r\,dr\,d\theta}$.

Setting up Integrals in polar coordinates

The double integral

$$I = \iint_D f(x,y)\, dA$$

may be computed in polar coordinates as follows:



Consider region D

$f_1(\theta) \leq r \leq f_2(\theta)$

$\alpha \leq \theta \leq \beta$

(1) Replace x, y by $r\cos\theta$, $r\sin\theta$ in $f(x,y)$ to get $g(r,\theta)$

(2) Replace dA by $r\,dr\,d\theta$

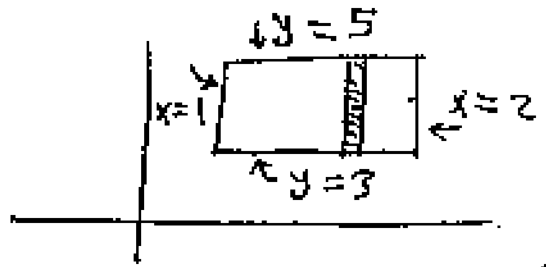
(3) slice region D is expressed in polar form as well, as shown

$$\therefore I = \int_\alpha^\beta \left\{ \int_{f_1(\theta)}^{f_2(\theta)} g(r,\theta)\, dr \right\} d\theta.$$

Note: usually we integrate w.r.t. r first!

Illustrated examples:

EX1: Find $\iint_R xy\,dx\,dy$ where R is the rectangular region $1 \le x \le 2$, $3 \le y \le 5$.

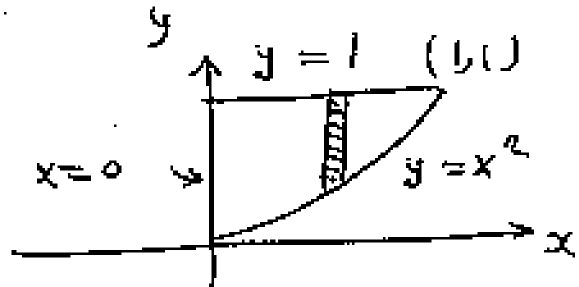


solution
$$I = \iint_R xy\,dA = \int_1^2 \left\{ \int_3^5 xy\,dy \right\} dx$$
$$= \int_1^2 x \cdot \frac{y^2}{2}\Big|_3^5 dx$$
$$= \frac{1}{2}[25-9]\int_1^2 x\,dx$$
$$= 8\left[\frac{x^2}{2}\right]_1^2 = 12$$

R: Both x, y simple. Let us treat as y-simple

$\therefore R: (3 \le y \le 5)$ Inner limits
$(1 \le x \le 2)$ outer limits

EX2: Find $\iint_R x\,dA$ where R is the region enclosed by $x=0$, $y=1$, and $y=x^2$.



soln.
$$I = \int_0^1 \left\{ \int_{x^2}^1 x\,dy \right\} dx$$
$$= \int_0^1 x \left\{ \int_{x^2}^1 dy \right\} dx$$
$$= \int_0^1 x\,[y]_{x^2}^1\,dx = \int_0^1 x[1-x^2]\,dx$$
$$= \int_0^1 (x-x^3)\,dx = \frac{x^2}{2} - \frac{x^4}{4}\Big|_0^1 = \frac{1}{4}.$$

R . Treated as y-simple.
$x^2 \le y \le 1$
$0 \le x \le 1$

EX3: Find the volume of the solid region in the first octant $(x \ge 0, y \ge 0, z \ge 0)$ below the plane $x+y+z=1$

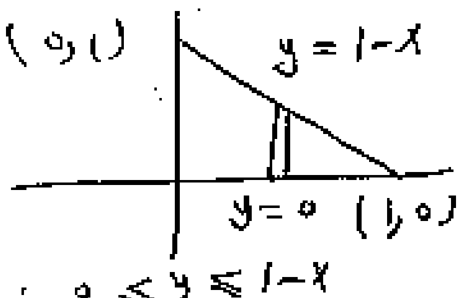
soln: Volume $V = \iint_R f(x,y)\,dA$

Need Two things: (1) The function $f(x,y)$: This is "z" from the surface given: $x+y+z=1$
$\Rightarrow z = 1-x-y \Rightarrow \boxed{f(x,y) = 1-x-y}$

(2) The region R. This is the projection onto xy-plane of $z = f(x,y) = 1-x-y$.

This projection can be found by simply putting $z=0$ into $z = 1-x-y$ to get $0 = 1-x-y$ or $x+y=1$

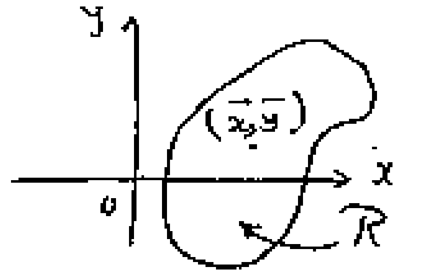
$$\therefore V = \iint_R f\,dA = \int_0^1 \left\{ \int_0^{1-x} (1-x-y)\,dy \right\} dx$$
$$= \int_0^1 \frac{(1-x-y)^2}{(-1)2}\Big|_0^{1-x} dx = -\frac{1}{2}\int_0^1 \left[0-(1-x)^2\right] dx$$
$$= +\frac{1}{2}\int (1-x)^2\,dx = \frac{1}{6}$$



R: $0 \le y \le 1-x$
$0 \le x \le 1$

Moment and Centre of mass of a plate occupying a plane region

Let R be a plane region occupied by a plate and let $\delta(x,y)$ be the areal density of the plate. Therefore its centre of mass $(\bar{x},\bar{y})$ is given by $\bar{x} = \dfrac{M_{x=0}}{m}$, $\bar{y} = \dfrac{M_{y=0}}{m}$



where m = Total mass $= \iint_R dm$,

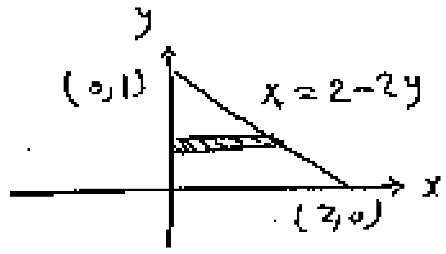
$M_{x=0}$ = moment about y-axis $= \iint_R x\,dm$, $\quad$, $\boxed{dm = \delta\, dA}$

$M_{y=0}$ = moment about x-axis $= \iint_R y\,dm$

In particular, if the plate is <u>uniform</u>, the density δ = a constant = say 1

and the centre of mass $(\bar{x},\bar{y})$ is then called "The Centroid" of the plane region R.

<u>Examples:</u> (1) Find the centre of mass of the triangle enclosed by $y=0$, $x=0$, and $x+2y=2$ where the density $\delta(x,y)=x$.



<u>solution</u> $dm = \delta\, dA = x\, dA$

$\therefore\ m = \iint_R dm = \iint_R x\, dA = \int_0^1 \left\{\int_0^{2-2y} x\,dx\right\} dy$

$= \int_0^1 \frac{1}{2}x^2\Big|_0^{2-2y} dy = \frac{1}{2}\int_0^1 (2-2y)^2\, dy$

$= \frac{1}{2}\ \frac{(2-2y)^3}{(-2)(3)}\Big|_0^1 = \frac{2}{3}$

$M_{x=0} = \iint_R x\,dm = \iint_R x\cdot x\, dA = \int_0^1\int_0^{2-2y} x^2\,dx\,dy = \frac{1}{3}\int_0^1 x^3\Big|_0^{2-2y} dy$

$= \frac{1}{3}\int_0^1 (2-2y)^3\, dy = \frac{1}{3}\ \frac{(2-2y)^4}{(-2)(4)}\Big|_0^1 = 2/3$

$M_{y=0} = \iint_R y\,dm = \iint_R y\,x\,dA = \int_0^1\int_0^{2-2y} y\ x\,dx\,dy$

$= \frac{1}{2}\int_0^1 y\,(2-2y)^2\,dy = \frac{1}{2}\int_0^1 (4y-8y^2+4y^3)\,dy = \frac{1}{6}$

$\therefore\ \bar{x} = \dfrac{M_{x=0}}{m} = \dfrac{2/3}{2/3} = 1, \quad \bar{y} = \dfrac{M_{y=0}}{m} = \dfrac{1/6}{2/3} = \dfrac{1}{4}$

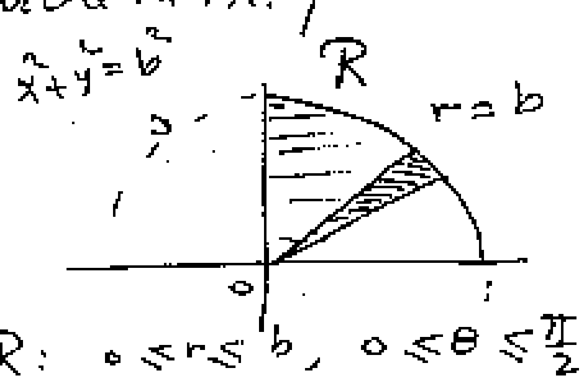
$\therefore\ (\bar{x},\bar{y}) = (1, \frac{1}{4})$.

---

(2) Find the centre of mass of the region $x^2+y^2 \le b^2$, $x \ge 0$, $y \ge 0$ with density $\delta(x,y) = \sqrt{x^2+y^2}$

sol. $x^2+y^2 \le b^2$, $x\ge 0$, $y \ge 0$ is the part of circle in 1st. quadrant!

Because of the quantity x^2+y^2, we suggest the usage of <u>polar Coordinates</u>.



$R:\ 0 \le r \le b,\ 0 \le \theta \le \frac{\pi}{2}$

$$dm = \delta\, dA = \sqrt{x^2+y^2}\; r\,dr\,d\theta = \sqrt{r^2}\cdot r\,dr\,d\theta = r^2\,dr\,d\theta$$

$$\therefore\; m = \iint_R dm = \int_0^{\frac{\pi}{2}}\int_0^b r^2\,dr\,d\theta = \frac{1}{3}b^3\cdot\frac{\pi}{2} = \frac{\pi}{6}b^3$$

$$M_{x=0} = \iint_R x\,dm = \int_0^{\frac{\pi}{2}}\int_0^b r\cos\theta\cdot r^2\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\cos\theta\,d\theta\cdot\int_0^b r^3\,dr = \frac{1}{4}b^4$$

$$M_{y=0} = \iint_R y\,dm = \int_0^{\frac{\pi}{2}}\int_0^b r\sin\theta\cdot r^2\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\sin\theta\,d\theta\int_0^b r^3\,dr = \frac{1}{4}b^4$$

$$\therefore\; \bar{x} = \bar{y} = \frac{\frac{1}{4}b^4}{\frac{\pi}{6}b^3} = \frac{3b}{2\pi}.$$

---

(3) Use double integrals to find the volume bounded by the cone $z = 9 - \sqrt{x^2+y^2}$ and the plane $z = 0$.

<u>Solution</u>: Volume $V = \iint_R f(x,y)\,dA$

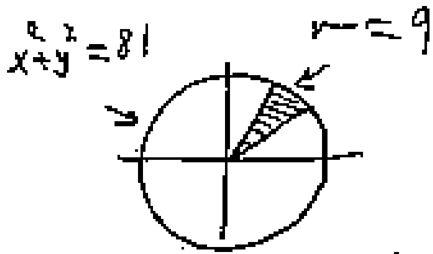
here $f(x,y) = z = 9 - \sqrt{x^2+y^2}$

R: Projection of $z = 9 - \sqrt{x^2+y^2}$ onto xy-plane (or $z = 0$)

Eliminating z between $z = 9 - \sqrt{x^2+y^2}$, $z = 0$

we get $0 = 9 - \sqrt{x^2+y^2}$ or $9 = \sqrt{x^2+y^2}$

$\Rightarrow 81 = x^2+y^2$



R: $0 \le r \le 9$, $0 \le \theta \le 2\pi$

$$\therefore\; V = \iint_R \{9 - \sqrt{x^2+y^2}\}\,dA$$

use polar coordinates:

$$V = \int_0^{2\pi}\int_0^9 (9-\sqrt{r^2})\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^9 (9-r)\,r\,dr\,d\theta = \int_0^{2\pi} d\theta\int_0^9 (9-r)\,r\,dr$$

$$= 2\pi\int_0^9 (9r - r^2)\,dr = 2\pi\left[9\frac{r^2}{2} - \frac{r^3}{3}\right]_0^9$$

$$= 2\pi\left[\frac{9^3}{2} - \frac{9^3}{3}\right] = 2\pi(9^3)\left(\frac{1}{2}-\frac{1}{3}\right)$$

$$= 2\pi(729)\left(\frac{1}{6}\right) = 243\pi$$

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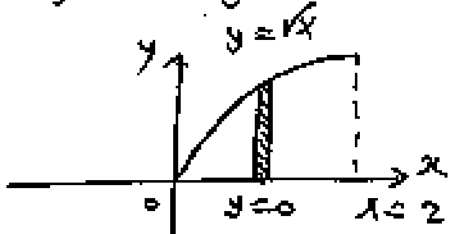
(4) Use double integral to find the $\bar{y}$ of the centroid of the region bounded by $x = 2$, $y = 0$, $y = \sqrt{x}$.



<u>soln</u>: For centroid $\delta = 1 \Rightarrow dm = \delta\,dA = dA$

$$\therefore\; m = \iint_R dA = \int_0^2\int_0^{\sqrt{x}} dy\,dx = \int_0^2 y\Big|_0^{\sqrt{x}} dx$$

$$= \int_0^2 \sqrt{x}\,dx = \frac{2}{3}x^{\frac{3}{2}}\Big|_0^2 = \frac{2}{3}\,x\sqrt{x}\Big|_0^2 = \frac{4}{3}\sqrt{2}.$$

$$M_{y=0} = \iint_R y\,dm = \iint_R y\,dA = \int_0^2\left(\int_0^{\sqrt{x}} y\cdot dy\right)dx = \int_0^2 \frac{1}{2}y^2\Big|_0^{\sqrt{x}} dx$$

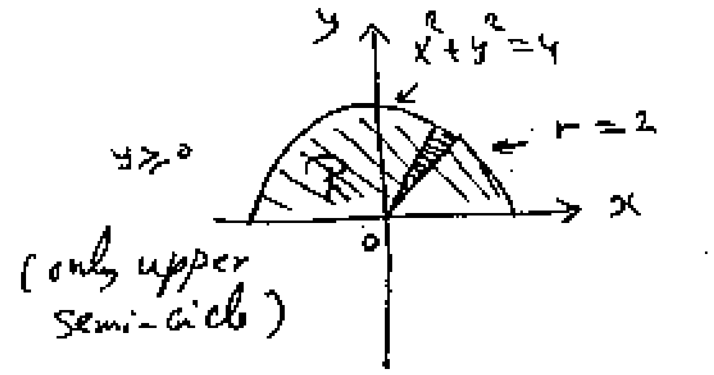
$$= \frac{1}{2}\int_0^2 (\sqrt{x})^2\,dx = \frac{1}{2}\int_0^2 x\,dx = \frac{1}{2}\cdot\frac{1}{2}x^2\Big|_0^2 = 1$$

$$\therefore\; \bar{y} = \frac{M_{y=0}}{m} = \frac{1}{\frac{4}{3}\sqrt{2}} = \frac{3}{4\sqrt{2}}.$$

---

(5) Use double integrals to find centre of mass of the planar region $x^2+y^2 \le 4$, $y \ge 0$, when the density is given by $\delta = \sqrt{x^2+y^2}$

Soln. We shall use polar coordinate



$$dm = \delta\, dA = \sqrt{x^2+y^2}\, dA$$
$$= r \cdot r\,dr\,d\theta = r^2\,dr\,d\theta$$

$$\therefore m = \iint_R dm = \int_0^{\pi}\int_0^{2} r^2\,dr\,d\theta = \frac{8}{3}\pi$$

R: $0 \le r \le 2$
$0 \le \theta \le \pi$

$$M_{x=0} = \iint_R x\,dm = \int_0^{\pi}\int_0^{2} r\cos\theta \cdot r^2\,dr\,d\theta$$
$$= \int_0^{\pi}\cos\theta\,d\theta \int_0^{2} r^3\,dr = \sin\theta\Big|_0^{\pi} \cdot \frac{1}{4}r^4\Big|_0^{2} = 0$$

$$M_{y=0} = \iint_R y\,dm = \int_0^{\pi}\int_0^{2} r\sin\theta \cdot r^2\,dr\,d\theta = \int_0^{\pi}\sin\theta\,d\theta \int_0^{2} r^3\,dr$$
$$= -\cos\theta\Big|_0^{\pi}\ \frac{1}{4}r^4\Big|_0^{2} = -[\cos\pi - \cos 0]\cdot\frac{1}{4}(2)^4$$
$$= -[-1-1]\cdot 4 = 8$$

$$\therefore \bar{x} = \frac{M_{x=0}}{m} = \frac{0}{\frac{8}{3}\pi} = 0, \quad \bar{y} = \frac{M_{y=0}}{m} = \frac{8}{\frac{8}{3}\pi} = \frac{3}{\pi}.$$

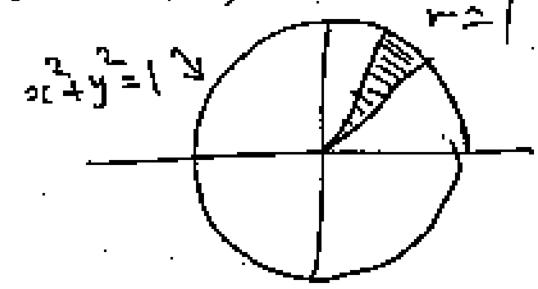
<u>Note</u>: we could have used symmetry to predict $\bar{x} = 0$!

(6) Use double integrals to find the volume of that part of the sphere $x^2+y^2+z^2 \le 4$ inside the cylinder $x^2+y^2 \le 1$

Soln. $V = 2\iint_R f(x,y)\,dA$. Here: $z^2 = 4 - x^2 - y^2 \Rightarrow z = \sqrt{4-(x^2+y^2)} = f(x,y)$

R: Base of cylinder: $x^2+y^2=1$

$$\therefore V = 2\iint_R \sqrt{4-(x^2+y^2)}\,dA \quad \text{.. use polar coordinate}$$

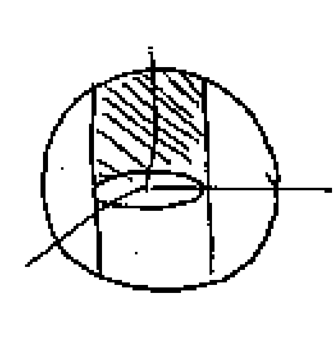


$$V = 2\int_0^{2\pi}\int_0^{1} \sqrt{4-r^2}\,r\,dr\,d\theta = 2\int_0^{2\pi} d\theta \int_0^{1} r\sqrt{4-r^2}\,dr$$
$$= 2(2\pi)\int_0^{1} r\sqrt{4-r^2}\,dr \leftarrow \text{Let } w = 4-r^2$$

R: $0 \le r \le 1$
$0 \le \theta \le 2\pi$

$$= 4\pi\left(-\frac{1}{3}(4-r^2)^{\frac{3}{2}}\right)\Big|_0^{1} = -\frac{4}{3}\left[3^{\frac{3}{2}} - 4^{\frac{3}{2}}\right]$$
$$= -\frac{4}{3}\left[3\sqrt{3} - 4\sqrt{4}\right] = -\frac{4}{3}\left[3\sqrt{3} - 8\right] = \frac{4}{3}(8 - 3\sqrt{3})$$

<u>Note</u>: From symmetry: Total volume V
$= 2 \times$ top half
$= 2\iint_R f(x,y)\,dA$



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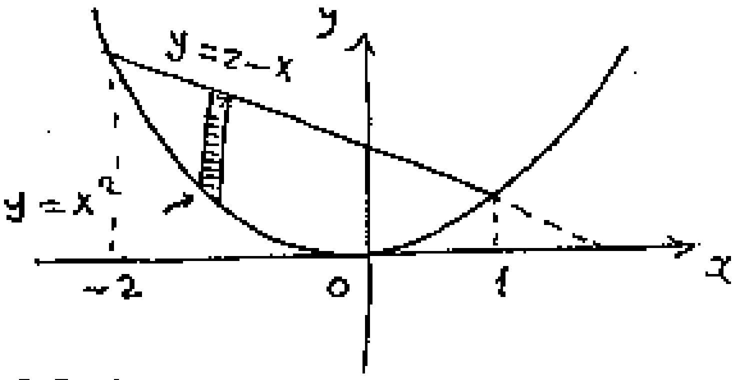
(7) show that the y-coordinate of the centroid of the planar region enclosed by $y = x^2$, and $y = 2 - x$ is $\bar{y} = \frac{8}{5}$

Soln. 1st. let us find where curves intersect?



$$y = x^2 \quad \cdots \quad (1)$$
$$y = 2 - x \quad \cdots \quad (2)$$

From (1), (2) : $x^2 = 2 - x \Rightarrow x^2 + x - 2 = 0$

$$(x+2)(x-1) = 0$$
$$x = -2, 1$$

For the centroid $\delta = 1$

$$\therefore \quad dm = \delta\, dA = dA$$

$$m = \iint_R dm = \iint_R dA = \int_{-2}^{1} \left\{ \int_{x^2}^{2-x} dy \right\} dx = \int_{-2}^{1} y \Big|_{x^2}^{2-x} dx$$

$$= \int_{-2}^{1} (2 - x - x^2)\, dx = 2x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 \Big|_{-2}^{1} = (2 - \tfrac{1}{2} - \tfrac{1}{3}) - (-4 - 2 + \tfrac{8}{3})$$
$$= \frac{9}{2}$$

$$M_{y=0} = \iint_R y\, dm = \int_{-2}^{1} \left\{ \int_{x^2}^{2-x} y\, dy \right\} dx$$

$$= \int_{-2}^{1} \tfrac{1}{2} y^2 \Big|_{x^2}^{2-x} dx = \tfrac{1}{2} \int_{-2}^{1} \left\{ (2-x)^2 - x^4 \right\} dx$$

$$= \frac{1}{2} \left[\frac{(2-x)^3}{(-1)(3)} - \frac{x^5}{5} \right]_{-2}^{1} = \frac{1}{2} \left[-\frac{1}{3} + \frac{64}{3} - \frac{1}{5} - \frac{32}{5} \right]$$

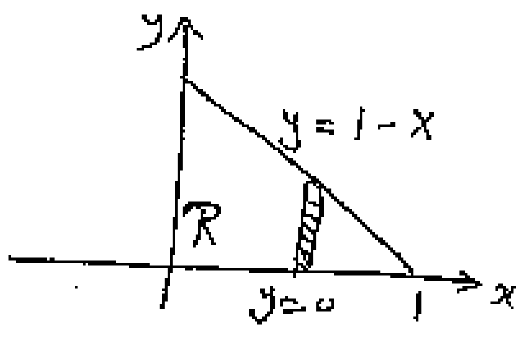
$$= \frac{1}{2} \left[\frac{63}{3} - \frac{33}{5} \right] = \frac{1}{2} \left[21 - \frac{33}{5} \right]$$

$$= \frac{1}{2} \left(\frac{105 - 33}{5} \right) = \frac{36}{5}$$

$$\therefore \quad \bar{y} = \frac{M_{y=0}}{m} = \frac{\frac{36}{5}}{\frac{9}{2}} = \frac{36}{5} \cdot \frac{2}{9} = \frac{8}{5}.$$

(8) Use double integrals to find the volume under the surface $z = 1 - (x^2 + y^2)$ and above the region $x \geq 0$, $y \geq 0$, $x + y \leq 1$

Solution $V = \iint_R f(x,y)\, dA$ .. Here



$f(x,y) = z = 1 - (x^2 + y^2)$, R is as shown

$$V = \iint \left[1 - (x^2 + y^2) \right] dA = \int_0^1 \int_0^{1-x} \left[1 - (x^2 + y^2) \right] dy\, dx$$

R: $0 \leq y \leq 1 - x$
$0 \leq x \leq 1$

$$= \int_0^1 y - x^2 y - \tfrac{1}{3} y^3 \Big|_0^{1-x} dx = \int_0^1 \left[(1-x) - x^2(1-x) - \tfrac{1}{3}(1-x)^3 \right] dx$$

$$= \frac{(1-x)^2}{(-1)2} - \left(\frac{x^3}{3} - \frac{x^4}{4} \right) - \frac{1}{3} \frac{(1-x)^4}{(-1)(4)} \Big|_0^1$$

$$= \frac{1}{2} - \left(\frac{1}{3} - \frac{1}{4} \right) - \frac{1}{12} = \frac{1}{3}$$

Vectors in $\mathbb{R}^3$: Let $P(x,y,z)$ be a point in $\mathbb{R}^3$. The directed line from O to P is called a position vector and is denoted by $\vec{OP}$ or $\vec{u}$ and we write $\vec{u} = (x,y,z)$.

More generally if $P(x_1,y_1,z_1)$, $Q(x_2,y_2,z_2)$ are points in $\mathbb{R}^3$, the directed line from P to Q is called a vector in $\mathbb{R}^3$ and is denoted and given by $\vec{PQ} \overset{\text{or}}{=} \vec{u} = (x_2-x_1, y_2-y_1, z_2-z_1)$.
The points P and Q are respectively the initial and terminal points of $\vec{u}$. Note that $\vec{u}$ = Terminal − Initial.

Defn.: If $\bar{u} = (x,y,z)$. we call x, y, and z the Components of $\bar{u}$.

Arithmetics of vectors: Let $\bar{u} = (x_1,y_1,z_1)$, $\bar{v} = (x_2,y_2,z_2)$.

[1] $\bar{u} = \bar{v}$ if and only if: Corresponding Components are equal. That is:
$$x_1 = x_2 \;,\; y_1 = y_2 \;, \text{ and } z_1 = z_2$$

[2] The sum $\bar{u}+\bar{v}$ is defined by $\bar{u}+\bar{v} = (x_1+x_2, y_1+y_2, z_1+z_2)$

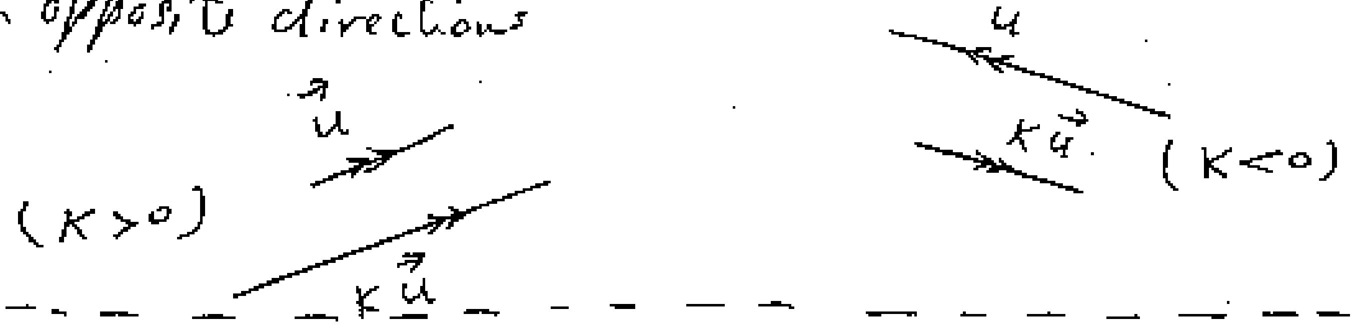
[3] The difference $\bar{u}-\bar{v}$ is defined by $\bar{u}-\bar{v} = (x_1-x_2, y_1-y_2, z_1-z_2)$

[4] If $K \in \mathbb{R}$ is a scalar, we define the vector $K\bar{u}$ by
$$K\bar{u} = K(x,y,z) = (Kx, Ky, Kz)$$

The zero vector is the vector $(0,0,0)$ and is denoted by $\vec{0}$.

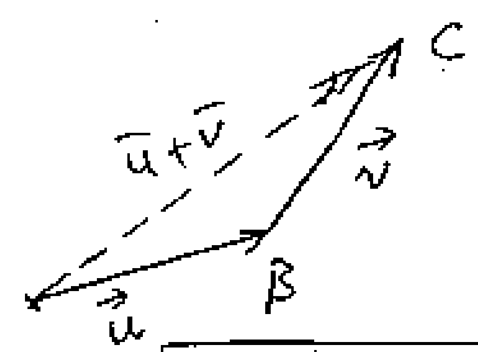
Geometric Interpretation: Two

(a) The vector $K\vec{u}$ is a vector parallel to $\vec{u}$. If $K>0$, they are parallel in the same direction and if $K<0$, they are parallel but in opposite directions



$\therefore$ vectors $\bar{u}$, $\bar{v}$ are parallel iff $\bar{u} = K\bar{v}$, $K \in \mathbb{R}$

(b) Let $\bar{u}$, and $\bar{v}$ be vectors where initial point of $\vec{v}$ coincide with the terminal point of $\vec{u}$ as shown. Then $\bar{u}+\bar{v}$ lies along the dotted line shown.
This rule may be easily memorized as follows:
If you are at point A and you want to reach point C. Then: Either go from A to B, then B to C or go directly from A to C!!

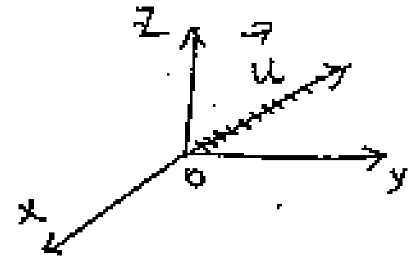


$\boxed{\overline{AB} + \overline{BC} = \overline{AC}}$

Def. : Norm (magnitude) of a vector :

Let $\bar{u} = (x, y, z)$. The norm of $\bar{u}$ is denoted and given by

$$\|\bar{u}\| = \sqrt{x^2 + y^2 + z^2}$$



Geometrically $\|\bar{u}\|$ = length of the vector $\vec{u}$

The vector $\vec{u}$ is a unit vector if $\|\bar{u}\| = 1$

Note that if $\bar{v}$ is a non-zero vector, then $\dfrac{\bar{v}}{\|\bar{v}\|}$ is a unit vector in the direction of $\bar{v}$ whereas $-\dfrac{\bar{v}}{\|\bar{v}\|}$ is a unit vector in opposite direction

EX : let $\bar{v} = (1, -2, 2)$. Find a unit vector in opposite direction of $\bar{v}$.

sol. The vector is $$-\frac{\bar{v}}{\|\bar{v}\|} = -\frac{(1,-2,2)}{\sqrt{(1)^2+(-2)^2+(2)^2}} = -\frac{1}{3}(1,-2,2) = \left(-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)$$

Two special products :

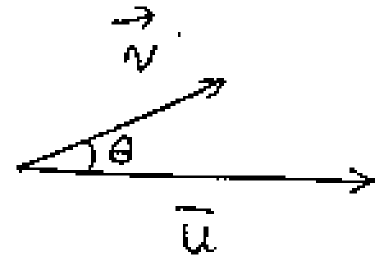
The Dot product : let $\bar{u} = (x_1, y_1, z_1)$, $\bar{v} = (x_2, y_2, z_2)$. The dot product of $\bar{u}$, and $\bar{v}$ is denoted and defined by

$$\bar{u} \cdot \bar{v} = x_1 x_2 + y_1 y_2 + z_1 z_2 = \text{a scalar quantity.}$$

Note : $\bar{u} \cdot \bar{v} = \bar{v} \cdot \bar{u}$, $\bar{u} \cdot (\bar{v} + \bar{w}) = \bar{u} \cdot \bar{v} + \bar{u} \cdot \bar{w}$, $\bar{u} \cdot \bar{u} = \|\bar{u}\|^2$

Angle between vectors : let θ be angle between $\bar{u}, \bar{v}$. Then

$$\boxed{\cos\theta = \frac{\bar{u} \cdot \bar{v}}{\|\bar{u}\| \, \|\bar{v}\|}}$$



$0^\circ \leq \theta \leq 180^\circ$

or $\boxed{0 \leq \theta \leq \pi}$

special case : If $\theta = 90^\circ$, we say vectors $\bar{u}$, and $\bar{v}$ are perpendicular or Orthogonal and we write $\bar{u} \perp \bar{v}$.

From Eq. above $\boxed{\bar{u} \perp \bar{v} \text{ if and only if } \bar{u} \cdot \bar{v} = 0}$

EX : let $\bar{u} = (7, 1, -3)$, $\bar{v} = (4, -1, 9)$. Then

$$\bar{u} \cdot \bar{v} = 28 - 1 - 27 = 0 \Rightarrow \bar{u} \perp \bar{v}$$

scalar and vector projection

Let $\bar{u}$, and $\bar{v}$ be vectors.

(1) The scalar projection of $\vec{u}$ on $\vec{v}$ is given by $$S = \frac{\bar{u} \cdot \bar{v}}{\|\bar{v}\|^2}$$

(2) The vector projection of $\vec{u}$ on $\vec{v}$ is denoted and given by $$\bar{u}_{\bar{v}} = S\,\bar{v}.$$

$\vec{AB}$ = vector projection of $\vec{u}$ on $\vec{v}$

S = scalar projection

Ex: let $\bar{u} = (4,1,-2)$, $\bar{v} = (1,-2,-2)$. Then

$$s = \frac{\bar{u}\cdot\bar{v}}{\|\bar{v}\|^2} = \frac{(4,1,-2)\cdot(1,-2,-2)}{\left(\sqrt{(1)^2+(-2)^2+(-1)^2}\right)^2} = \frac{4-2+4}{3^2} = \frac{6}{9} = \frac{2}{3}$$

and vector projection $\bar{u}_{\bar{v}} = s\bar{v} = \frac{2}{3}(1,-2,-2)$.

The cross product of vectors in 3-space:

The cross product of vectors $\bar{u}$, and $\bar{v}$ in $\mathbb{R}^3$ is denoted by $\vec{u}\times\vec{v}$.
This vector is orthogonal to both $\bar{u}$, and $\bar{v}$ and is given by

$$\bar{u}\times\bar{v} = \left(+\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, -\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}, +\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\right)$$

obtained from (*) by deleting c_1 ; ~~~ ~~ c_2 ; ~~~ ~~ c_3

$\vec{u} = (x_1, y_1, z_1)$
$\bar{v} = (x_2, y_2, z_2)$
Consider matrix
$$\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix} \quad (*)$$

Note: $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ = a determinant of order 2 $= ad - bc$.

properties of cross product:

$\bar{u}\times\bar{v} = \ominus\, \bar{v}\times\bar{u}$ $\qquad \bar{u}\times\bar{u} = \bar{0}$ $\qquad (\bar{u}+\bar{v})\times\bar{w} = \bar{u}\times\bar{w} + \bar{v}\times\bar{w}$

Ex: Find a vector orthogonal to both $(1,-1,3)$, $(5,4,-1)$.

solution: The vector may be given by the cross product of $\bar{u} = (1,-1,3)$, and $\bar{v} = (5,4,-1)$.

$$\therefore\quad \bar{w} = \bar{u}\times\bar{v} = \left(+\begin{vmatrix} -1 & 3 \\ 4 & -1 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ 5 & -1 \end{vmatrix}, +\begin{vmatrix} 1 & -1 \\ 5 & 4 \end{vmatrix}\right) = (-11, 16, 9)$$

is the required vector

$$\begin{array}{c|ccc} & \oplus & \ominus & \oplus \\ \vec{u} & 1 & -1 & 3 \\ \bar{v} & 5 & 4 & -1 \end{array}$$

Ex: Find two vectors among the set $\{\bar{a} = (4,1,-1), \bar{b} = (2,0,1), \bar{c} = (3,9,21)\}$ which are orthogonal

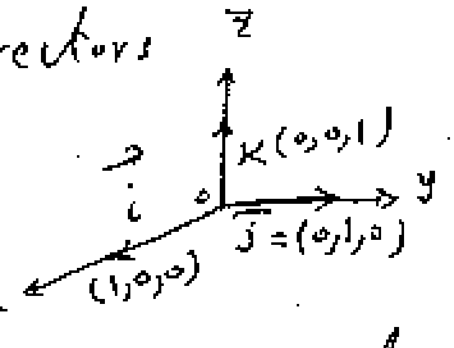
solution: we want the two vectors whose dot product is zero.
Indeed, by inspection, try $\bar{a}$, and $\bar{c}$:

$$\bar{a}\cdot\bar{c} = (4,1,-1)\cdot(3,9,21) = 12+9-21 = 0 \quad \text{(Lucky Guess!)}$$

$$\therefore\quad \bar{a} \perp \bar{c}.$$

---

standard basis vectors in $\mathbb{R}^3$: These are the unit vectors
$\bar{i} = (1,0,0)$, $\bar{j} = (0,1,0)$, and $\bar{k} = (0,0,1)$



Note that any vector (x,y,z) can be written as
$(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$
$= \bar{i}x + \bar{j}y + \bar{k}z$

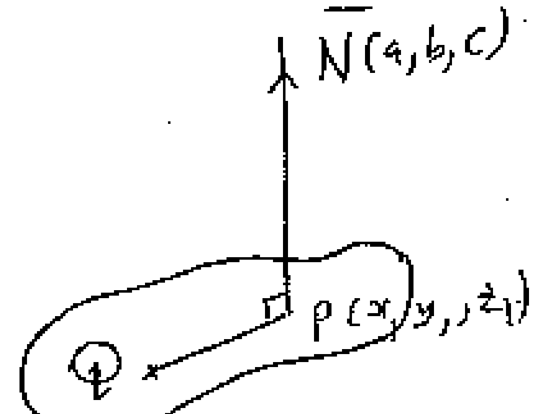
Therefore The vector $5\bar{i} - 2\bar{j} + 7\bar{k}$ is $(5,-2,7)$ ← preferred!

Lines and planes in $\mathbb{R}^3$:

[1] The plane in $\mathbb{R}^3$: A plane is an infinite sheet in $\mathbb{R}^3$.

The point - normal form

Equation of a plane passing through a point P with position vector $\bar{r}_0$ and has normal vector $\bar{N}$ is



$$\boxed{\bar{r}\cdot\bar{N} = \bar{r}_0\cdot\bar{N}}$$

This is the so called point-normal form!

Note, $\bar{r}=(x,y,z)$, $\bar{N}=(a,b,c)$, $\bar{r}_0=(x_1,y_1,z_1)$

A vector $\bar{N}\perp$ to every vector in plane is called a normal vector

If we expand eq. above we get:

$$ax+by+cz+d=0$$

which is the standard eq. of a plane having a normal vector $(a,b,c)=\bar{N}$.

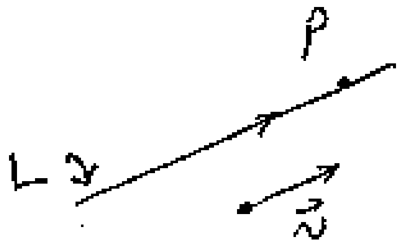
EX The equation $5x-3y=10$ in $\mathbb{R}^3$ represents a plane.

In fact, we write it as $5x-3y+0\cdot z-10=0$

$\therefore$ A normal vector to this plane is $\bar{N}=(5,-3,0)$.

[2] The st. line: Let L be a st. line which passes through the point $P(x_1,y_1,z_1)$ and has a direction vector $\bar{v}=(a,b,c)$.

The vector eq. of "L" is



$$\bar{r}=\bar{r}_0+t\bar{v}$$

where $\bar{r}_0$ = position vector of $P=(x_1,y_1,z_1)$

$\bar{v}=(a,b,c)$, $t\in\mathbb{R}$ is a parameter.

"L" parallel to $\vec{v}$

The equation above is equivalent to

$$(x,y,z)=(x_1,y_1,z_1)+t(a,b,c)$$

$$\Rightarrow \begin{cases} x=x_1+at \\ y=y_1+bt \\ z=z_1+ct \end{cases} \quad t\in\mathbb{R}$$

These are the so called parametric equations of line "L". They are the most useful.

EX: Find parametric equations of line passing through the origin and is parallel to the vector $\vec{PQ}$ where $P=(1,4,1)$, $Q(7,1,-2)$.

solution: Need: (1) A point: The origin $(0,0,0)$

(2) A direction vector $\bar{v}$. This is $\bar{v}=\overline{PQ}$

$\therefore \bar{v}=(7,1,-2)-(1,4,1)=(6,-3,-3)$

parametric equations are $\begin{cases} x=0+6t \\ y=0-3t \\ z=0-3t \end{cases}$ $t\in\mathbb{R}$

EX: Find equation of a plane passing through $(0,0,0)$ and has a normal vector $\bar{N}=(1,1,-2)$.

Soln $\bar{r}\cdot\bar{N}=\bar{r}_0\cdot\bar{N} \Rightarrow (x,y,z)\cdot(1,1,-2)=(0,0,0)\cdot(1,1,-2)$

$\Rightarrow \boxed{x+y-2z=0}$

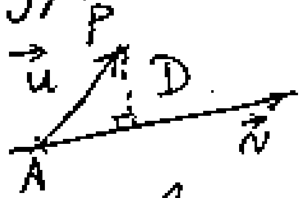
Distance formulae

(a) Distance from the point $P(x_1, y_1, z_1)$ to the plane $ax+by+cz+d=0$

is $$D = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

(b) Distance from the point $P(x_1, y_1, z_1)$ to the line $L: r = r_0 + t\vec{v},\ t \in \mathbb{R}$

is $$D = \frac{|\vec{u}\cdot\vec{v}|}{\|\vec{v}\|}$$ where $\vec{u} = \vec{AP}$ where A is any point on line and P is the given point



EX: Find the distance from the point $(1, 4, -3)$ and the plane

$2x - y - 2z = -5$.

soln. Distance $D = \dfrac{|2(1) - (4) - 2(-3) + 5|}{\sqrt{2^2+(-1)^2+(-2)^2}} = \dfrac{9}{\sqrt{9}} = 3$ units.

EX: Find the distance from the point $(2, 7, 3)$ to the line $\begin{cases} x = 3 + 6t \\ y = 7 - 3t \\ z = -6t \end{cases}$

soln. A point on line is $A(3,7,0)$ and a direction vector is $\vec{v} = (6,-3,-6)$

$\therefore\ \vec{u} = \vec{AP} = (2,7,3) - (3,7,0) = (-1, 0, 3)$

$\therefore$ distance $D = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|} = \dfrac{|(-1,0,3)\cdot(6,-3,-6)|}{\sqrt{(6)^2+(-3)^2+(-6)^2}} = \dfrac{|-6-18|}{\sqrt{81}} = \dfrac{24}{9}$

$= \dfrac{8}{3}$ units.

EX: Find equation of plane passing through the points $(1,1,-1)$, $(2,4,3)$, and $(7,1,-3)$.

soln. Need: (1) A point: choose say $P(1,1,-1) \Rightarrow \vec{r_0} = (1,1,-1)$

(2) A normal vector. From figure

$\vec{N} = \vec{u} \times \vec{v}$

where $\vec{u} = \vec{PQ} = (1,3,4)$, $\vec{v} = \vec{PR} = (6,0,-2)$

$\therefore \vec{N} = \vec{u}\times\vec{v} = (-6, 26, -18)$

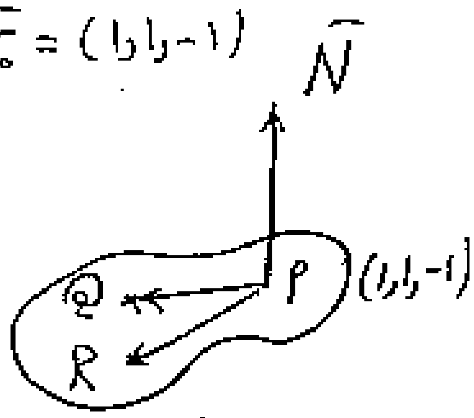
$\therefore$ Eq. of plane is $\vec{r}\cdot\vec{N} = \vec{r_0}\cdot\vec{N}$

$(x,y,z)\cdot(-6,26,-18) = (1,1,-1)\cdot(-6,26,-18)$

$-6x + 26y - 18z = -6 + 26 + 18$

$\Rightarrow -6x + 26y - 18z = 38 \qquad (\div -2)$

$3x - 13y + 9z + 19 = 0$



$Q(2,4,3)$
$R(7,1,-3)$

$$\begin{array}{c|ccc} & \oplus & \ominus & \oplus \\ \vec{u} & 1 & 3 & 4 \\ \vec{v} & 6 & 0 & -2 \end{array}$$

EX: Find an equation of plane passing through the point $(0,1,4)$ and contains the line $\vec{r} = (1,5,-3) + t(1,0,-2)$ $t \in \mathbb{R}$.

Soln. Need (1) A point: Given as $(0,1,4) \Rightarrow \vec{r_0} = (0,1,4)$

(2) A "normal" vector $\vec{N}$.

From figure $\vec{N} = \vec{u}\times\vec{v}$ where $\vec{u} = \vec{AP} = (-1,-4,7)$,

$\vec{v} = (1,0,-2)$

For line: $\vec{v} = (1,0,-2)$
$A =$ pt. $(1,5,-3)$

Therefore $\quad \bar{u}\times\bar{v} = (8,5,4)$

Eq. of plane is $\quad (x,y,z)\cdot(8,5,4) = (0,1,4)\cdot(8,5,4)$

$\therefore \quad 8x+5y+4z = 21$

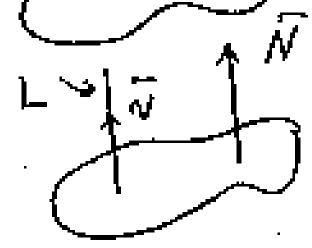
<u>Relationships</u>: Three:

(A) <u>Relation between</u> <u>A plane and a line</u>

let L be a line with direction vector $\bar{v}$ and P be a plane with normal vector $\bar{N}$. Then

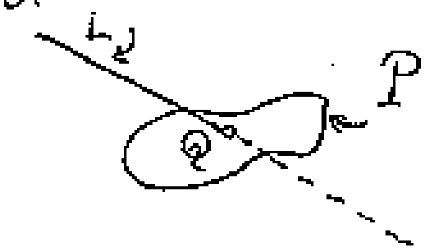
(1) The plane P is <u>parallel</u> to line L if $\bar{v} \perp \bar{N}$

(2) The plane P is <u>perpendicular</u> to line L if $\bar{v} \parallel \bar{N}$



<u>Note</u>: In this case $\bar{v} = K\bar{N}$ and we may take $K=1$ and get $\bar{v} = \bar{N}$!!

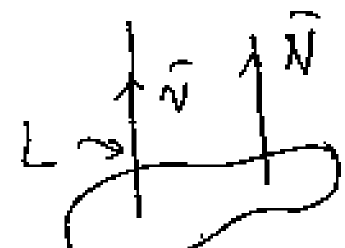
(3) If line L is not parallel to plane P, they intersect at one point Q



<u>EX</u>: Find eq. of plane passing through $(1,4,-2)$ and is perpendicular to line $\begin{cases} x = 2-t \\ y = 7 \\ z = 3+2t \end{cases}$ $t\in\mathbb{R}$

Soln. Need: (1) A point : $(1,4,-2) \Rightarrow \bar{r}_0 = (1,4,-2)$

(2) A normal vector $\bar{N}$!



From figure $\quad \bar{N} = \bar{v} = (-1,0,2)$

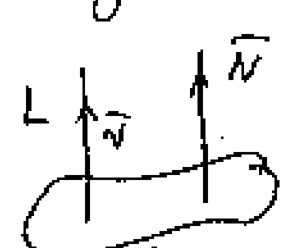
From eq. of L: $\bar{v} = (-1,0,2)$

Eq. is $\quad \bar{r}\cdot\bar{N} = \bar{r}_0\cdot\bar{N}$

$\Rightarrow (x,y,z)\cdot(-1,0,2) = (1,4,-2)\cdot(-1,0,2)$

$\therefore -x+2z = -5$

EX: Find parametric equations of line passing through the origin and is perpendicular to plane $3x-7y-16=0$



Soln. Need: (1) A pt. : $(0,0,0)$

(2) A direction vector $\bar{v}$!

From figure: $\quad \bar{v} = \bar{N} = (3,-7,0)$

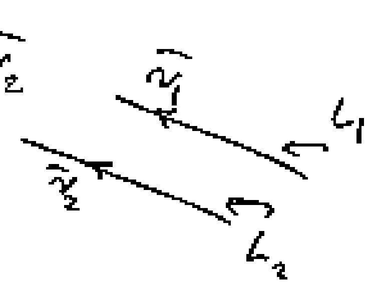
From plane: $3x-7y+0\cdot z-16=0$

$\therefore \bar{N} = (3,-7,0)$

parametric equations are $\quad \begin{aligned} x &= 0+3t \\ y &= 0-7t \\ z &= 0+0\cdot t \end{aligned} \quad t\in\mathbb{R}$

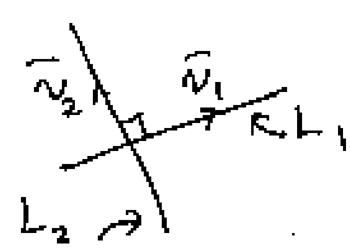
(B) <u>Relation between two lines</u>:

let L_1, and L_2 be lines with direction vectors $\bar{v}_1$ and $\bar{v}_2$



(1) lines L_1 and L_2 are parallel if $\bar{v}_1 \parallel \bar{v}_2$, that is if $\bar{v}_1$ is a scalar multiple of $\bar{v}_2$ (or vise versa)

(2) lines L_1 and L_2 are perpendicular if $\bar{v}_1 \perp \bar{v}_2$, that is if $\bar{v}_1\cdot\bar{v}_2 = 0$



<u>EX</u> Are the lines $\bar{r} = (2,1,5)+t(7,1,-3)$, $\quad t\in\mathbb{R}$

$\bar{r} = (0,1,4)+s(3,-6,5)$ $\quad s\in\mathbb{R}$

perpendicular? Ans. yes! $\quad \bar{v}_1\cdot\bar{v}_2 = (7,1,-3)\cdot(3,-6,5) = 0$

Ex: show that the planes $5x-2y+3z-17=0$, $-15x+6y-9z=1$ are parallel.

solution: Normal vectors are $\bar{N}_1=(5,-2,3)$, $\bar{N}_2=(-15,6,-9)$

Indeed $\bar{N}_2=(-15,6,-9)=-3(5,-2,3)=-3\bar{N}_1 \Rightarrow$ planes are parallel

Ex: show that the planes $2x-y+3z-1=0$ and $5x+31y+7z=0$ are perpendicular!

soln. Again $\bar{N}_1\cdot\bar{N}_2=(2,-1,3).(5,31,7)=10-31+21=0$

$\Rightarrow$ planes are orthogonal!

Ex: Find point of intersection of the plane $2x+3y-5z+33=0$ and the line $\bar{r}=(1,1,-2)+t(5,3,-1)$

soln. $2x+3y-5z \quad =0 \quad ---(1)$

Parametric eqs. of line are
$$\begin{aligned} x&=1+5t\\ y&=1+3t \quad ----(2)\\ z&=-2-t \end{aligned}$$

sub. (2) into (1), we get: $2(1+5t)+3(1+3t)-5(-2-t)+33=0$

$\Rightarrow 24t+48=0 \Rightarrow t=-2$

Now put $t=-2$ into Eqs. (2) we get the point
$$\begin{aligned} x&=1+5(-2)=-9\\ y&=1+3(-2)=-5\\ z&=-2-(-2)=0 \end{aligned}$$

$\therefore$ pt. of intersection is $(-9,-5,0)$

Ex: Find parametric Equations of the line of intersection of the planes $4x+9y+2z=1$, $x+2y-z-2=0$

soln. Consider the system of two Equations:
$$\begin{aligned} 4x+9y+2z&=1\\ x+2y-z&=2 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 4 & 9 & 2 & 1\\ 1 & 2 & -1 & 2 \end{array}\right) \xrightarrow{R_1\leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2\\ 4 & 9 & 2 & 1 \end{array}\right) \xrightarrow{R_2\to R_2-4R_1} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2\\ 0 & 1 & 6 & -7 \end{array}\right)$$

$$\xrightarrow{R_1\to R_1-2R_2} \left(\begin{array}{ccc|c} 1 & 0 & -13 & 16\\ 0 & 1 & 6 & -7 \end{array}\right)$$

system reduces to:
$$\begin{aligned} x \quad -13z&=16\\ y+6z&=-7 \end{aligned}$$

let $z=t$ (parameter), $t\in\mathbb{R}$ .. hence $x=16+13t$, $y=-7-6t$

parametric equations are $x=16+13t$, $y=-7-6t$, $z=t$, $t\in\mathbb{R}$

Another method: High school!
$$\begin{aligned} 4x+9y+2z&=1 \quad ---(1)\\ x+2y-z&=2 \quad ---(2) \end{aligned}$$

let us eliminate say x as follows:

$(4x+9y+2z=1)\times 1$
$(x+2y-z=2)\times -4$
$\Rightarrow$
$$\begin{aligned} 4x+9y+2z&=1\\ -4x-8y+4z&=-8 \quad \text{add}\\ \hline y+6z&=-7 \end{aligned}$$

$\Rightarrow \boxed{y=-7-6z}$

Now put $y=-7-6z$ into either Eq. (1) or (2). If we use (2), we get: $x+2(-7-6z)-z=2 \Rightarrow \boxed{x=16+13z}$

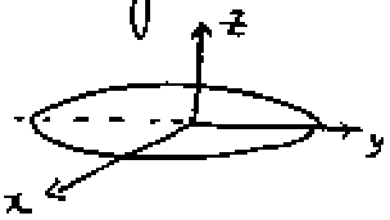
Now let $z=$ a free unknown t, $t\in\mathbb{R}$. Hence x,y becomes

$x=16+13t$, $y=-7-6t$.

Parametric eqs. are therefore given by
$$\left\{\begin{aligned} x&=16+3t\\ y&=-7-6t \quad t\in\mathbb{R}\\ z&=t \quad \text{(as above!)} \end{aligned}\right.$$

AMAT 219
Support Material
QUADRIC SURFACES

A second Degree equation in the three variables $x, y,$ and z is called a quadric Surface. Here is a list of the most popular quadric Surfaces.



(A) The ellipsoid Family : Two members :

(1) The Ellipsoid : $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, $a, b, c > 0$

The ellipsoid is centred at the origin and has semi-axes of length a, b, c.

(2) The sphere : $x^2 + y^2 + z^2 = r^2$, $r > 0$

The sphere is Centred at the origin and has radius r. Note that, if we solve for z we get either $z = +\sqrt{r^2 - x^2 - y^2}$ or $z = -\sqrt{r^2 - x^2 - y^2}$. These are respectively equations of upper and lower hemi-spheres!

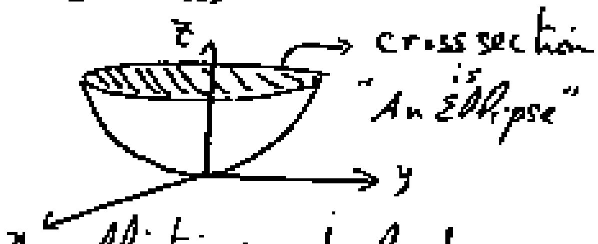
(B) The paraboloid Family : Three members

(1) The elliptic paraboloid $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$, $a, b, c > 0$

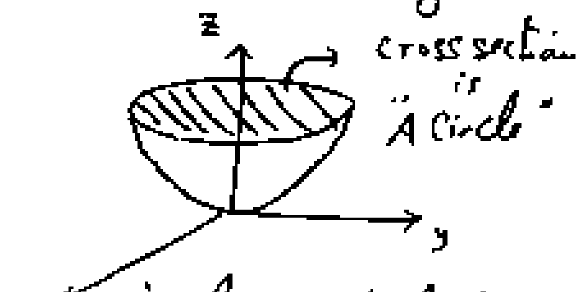
(2) The circular paraboloid $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{a^2}$, $a, c > 0$

(3) The Hyperbolic paraboloid $\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ or $\frac{z}{c} = \frac{y^2}{b^2} - \frac{x^2}{a^2}$, $a, b, c > 0$

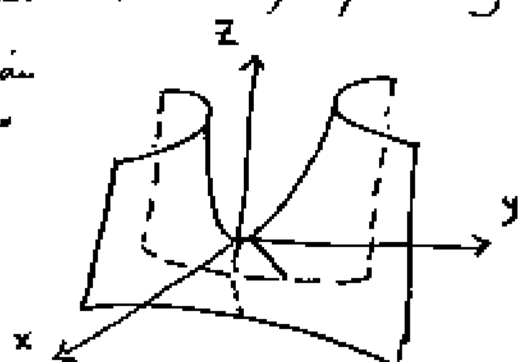
Note that each paraboloid has vertex at the origin, and its axis of symmetry is the z-axis.



elliptic paraboloid



circular paraboloid



Hyperbolic paraboloid

(C) The Hyperboloid Family : Three members

(1) The Hyperboloid of one sheet : $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = +1$, $a, b, c > 0$

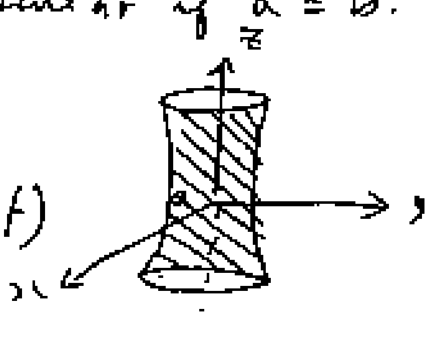
(2) The Hyperboloid of Two sheets : $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$ $a, b, c > 0$

(3) The Cone : $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ $a, b, c > 0$

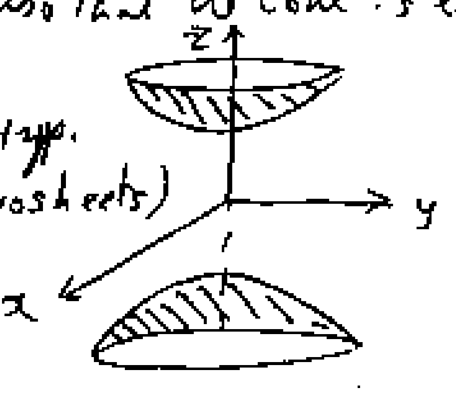
Notes: (1) The Hyperboloids of one and two sheets has Centre at O and z-axis being axis of symmetry.

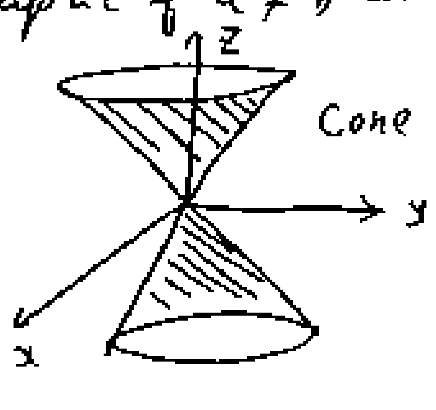
(2) The cone has vertex at O and axis being z-axis. If we solve for z we get $\frac{z}{c} = +\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}$ "upper Semi-Cone" or $\frac{z}{c} = -\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}$ "the lower semi-cone". Note also that the Cone is elliptic if $a \neq b$ and is circular if $a = b$.

Hyp. (one sheet)



Hyp. (Two sheets)





Remarks (1) In equations above if we replace x, y, and z by $x-h$, $y-K$, and $z-l$, we get a translated quadric surface "with centre or vertex at (h, K, l) instead of the origin".

(2) Similar Equations hold for quadric surfaces with axis of symmetry being x or y - axis.

Example : Identify and sketch each of the following quadric surfaces:

(a) $x^2 = y^2 + z^2$. This is a circular Cone with axis being x - axis

(b) $z = \sqrt{9 - x^2 - y^2}$. This is an equation of the upper hemi-sphere centred at O and has radius 3.

(c) $z = 2 - 4x^2 - 4y^2$. This eq. can be written in the form

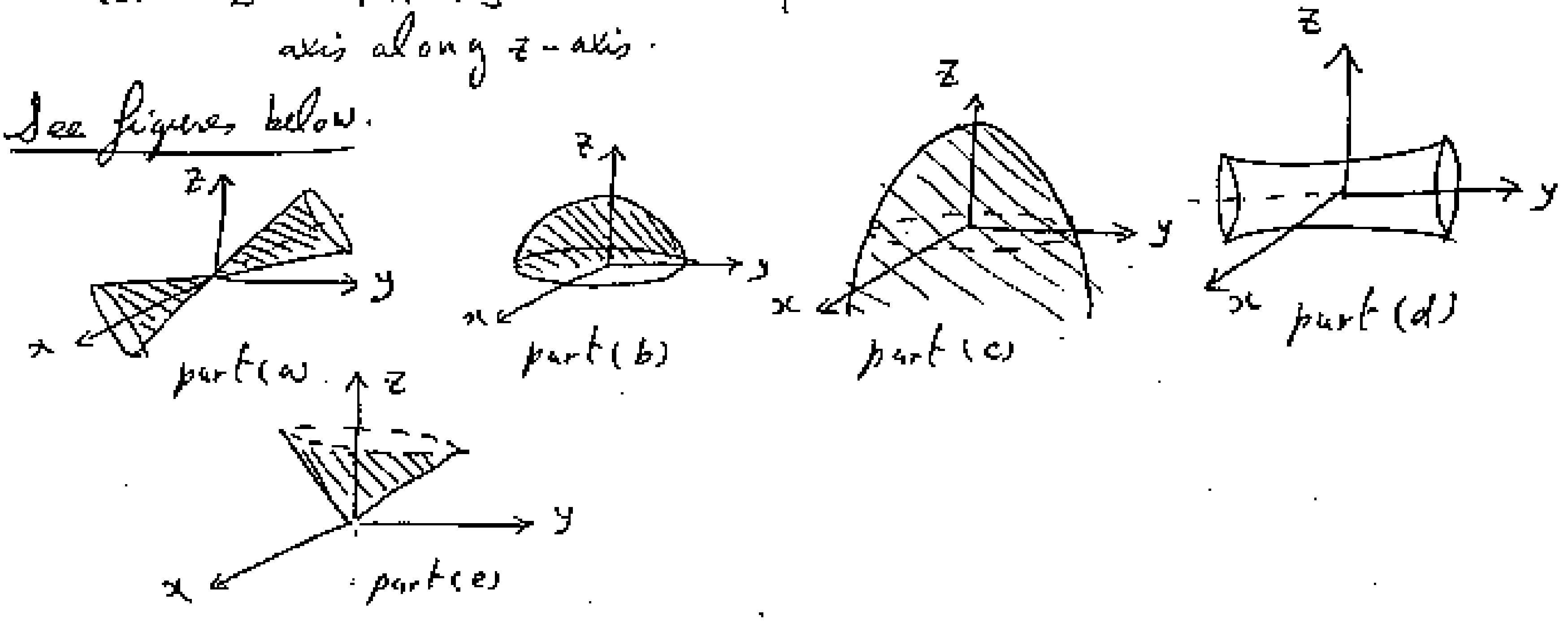
$$z - 2 = -(4x^2 + 4y^2)$$

Therefore it represents a translated paraboloid with vertex at $(h, K, l) = (0, 0, 2)$, and which opens downwards!

(d) $\frac{x^2}{4} - \frac{y^2}{9} + \frac{z^2}{25} = 1$ is a Hyperboloid of one sheet with axis along y - axis.

(e) $z = \sqrt{x^2 + y^2}$. This is an equation of the upper semi-cone with axis along z - axis.

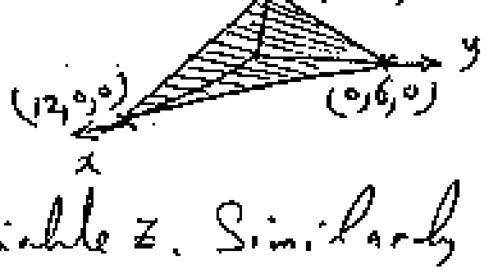
See figures below.



Other Important surfaces : Two

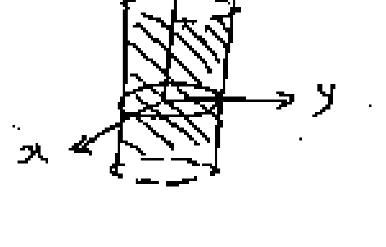
(a) The plane : $ax + by + cz + d = 0$ represents an equation of a plane. If $d = 0$, the plane passes through the origin.

EX: $x + 2y + 4z = 12$ represents a plane in $\mathbb{R}^3$ (infinite sheet). The x, y, and z - intercepts are respectively 12, 6, and 3 which we may used to sketch the plane as shown!



(b) An equation of two variables, say $F(x,y) = 0$ in $\mathbb{R}^3$ represents a cylinder with axis parallel to the 3rd variable z. Similarly $g(x,z) = 0$ represents a cylinder with axis parallel to y - axis, ...etc.

EX The equation $x^2 + y^2 = 4$ in $\mathbb{R}^3$ is a cylinder with axis // to z - axis and whose base in the circle $x^2 + y^2 = 4$ in xy-plane.



EX: sketch $y = x^2$ in $\mathbb{R}^3$.
(cylinder - axis. z-axis)

AMAT 219

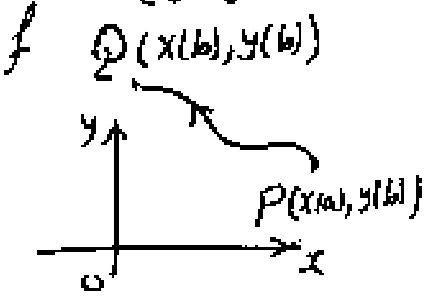
Support Material

parametric Curves

Let $f(t)$, $g(t)$ be continuous functions for all t in some interval I. The equations $x(t) = f(t)$, $y(t) = g(t)$, $t \in I$ are said to be the parametric equations of a plane curve C and the independent variable "t" is called "The parameter". Indeed the plane curve "C" may be thought of as the path of a particle P moving in xy-plane and that $(x(t), y(t))$ is its position at time "t". The plane curve C is said to be smooth at $t = t_0$ provided that $f'(t_0)$ and $g'(t_0)$ are not both zeros.

Let C be the plane curve given parametrically by $C: \begin{cases} x = f \\ y = g \end{cases}$ $t \in I$.

In particular if $I = [a, b]$ we define the end points of C as the points $P(x(a), y(a))$, and $Q(x(b), y(b))$ and the orientation of C - indicated by arrow heads - is the direction of motion from initial point P to terminal point Q. Note that if end points P and Q coincide, the curve C is said to be a <u>closed</u> curve.



<u>Some Important parametric Curves:</u>

(a) A st. line passing through the points (x_1, y_1), (x_2, y_2) is given parametrically by

$$\begin{aligned} x &= x_1 + (x_2 - x_1)t \\ y &= y_1 + (y_2 - y_1)t \end{aligned} \qquad t \in \mathbb{R}$$

(b) A circle centred at (α, β) and has radius "a" is given parametrically by

$$x = \alpha + a\cos t, \quad y = \beta + a\sin t, \quad 0 \le t \le 2\pi$$

observe that the <u>cartesian</u> equation of a plane curve $C: \begin{cases} x = f(t) \\ y = g(t) \end{cases}$ $t \in I$ can be obtained by simply eliminating "t" between the pair of equations. Observe also that there are infinitely-many ways to parametrize a curve C!

EX Given $C: \begin{cases} x = 2t^2 \\ y = 16t^4 \end{cases} \quad -2 \leq t \leq 0$

Find Cartesian equation of C. Identify and sketch the curve!

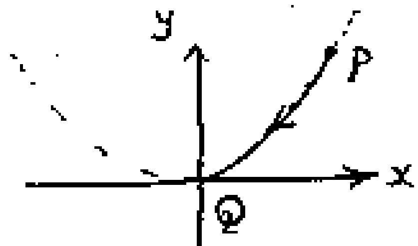
<u>solution:</u> $x = 2t^2 \cdots (1)$, $y = 16t^4 \cdots (2)$

From (1) $t^2 = \frac{x}{2}$ .. substitute into (2), $y = 16\left(\frac{x}{2}\right)^2 = 4x^2$

$\therefore$ $y = 4x^2$ .. an equation of a parabola!

<u>End points:</u> at $t = -2$, $x = 2(-2)^2 = 8$, $y = 16(-2)^4 = 256$ $\therefore P(8, 256)$

at $t = 0$, $x = 0$, $y = 0$; $Q(0,0)$



C : part of parabola from P to Q as shown.

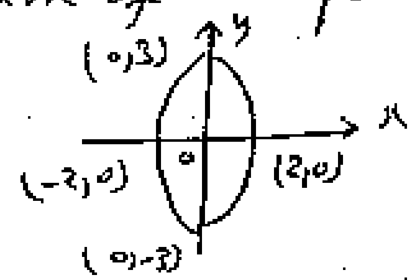
EX: Find Cartesian eq. of $x = 2\cos t$, $y = 3\sin t$, $0 \leq t \leq 2\pi$

<u>solution:</u> $x = 2\cos t \Rightarrow \cos t = \frac{x}{2}$, $y = 3\sin t \Rightarrow \sin t = \frac{y}{3}$

Now Since $\cos^2 t + \sin^2 t = 1$, we get: $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

$\Rightarrow \frac{x^2}{4} + \frac{y^2}{9} = 1$ .. an equation of an ellipse

with semi-axis of length 2, and 3



<u>Tangent and normal lines to a parametric curve:</u>

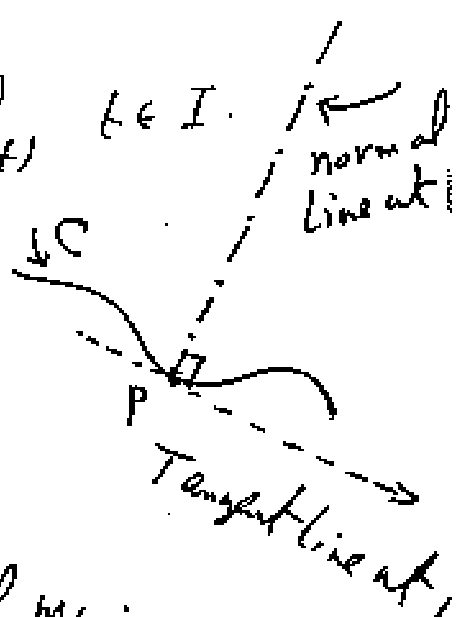
Let C be a parametric curve given by $C: \begin{cases} x = f(t) \\ y = g(t) \end{cases}$ $t \in I$.

If C is smooth at some t_0, then

(1) The slope of tangent line at P is

$$M_{tan} = \left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=t_0} = \left.\frac{\dot{y}}{\dot{x}}\right|_{t=t_0} \quad \text{provided } \dot{x} \neq 0$$

(2) slope of normal line at P is: Negative-reciprocal of M_{tan}



observe that if at some value of t you find $\frac{dy}{dt}$ and $\frac{dx}{dt}$ are both zeros, you may use limits to compute the slope.

EX: Find Cartesian equation of tangent and normal line to the curve
$$x = t^2+5, \quad y = t^6+2t^2+3 \quad \text{at } t=1$$

Solution: At "$t=1$", $\quad x = 1+5 = 6, \quad y = 1+2+3 = 6$; pt. $(6,6)$ ✓

Now, $\frac{dx}{dt} \overset{\text{or}}{=} \dot{x} = 2t, \quad \frac{dy}{dt} \overset{\text{or}}{=} \dot{y} = 6t^5+4t$

$$\therefore \quad m_{tan} = \frac{\dot{y}}{\dot{x}} = \left.\frac{6t^5+4t}{2t}\right|_{t=1} = \frac{6+4}{2} = 5 \checkmark$$

$\therefore$ Eq. of T.L is $\quad y-6 = 5(x-6)$,

Eq. of normal line is $\quad y-6 = -\frac{1}{5}(x-6)$

EX: Find parametric equations of tangent line to the curve
$$x = t^3+2t^2+1, \quad y = t^3+6t^2+1 \quad \text{at } t=0.$$

Solution: At $t=0$, $\quad x=1, y=1$; $(1,1)$

$$\text{slope } m_{tan} = \left.\frac{\dot{y}}{\dot{x}}\right|_{t=0} = \left.\frac{3t^2+12t}{3t^2+4t}\right|_{t=0} = \frac{0}{0} \text{ !! use limits as } t\to 0$$

$$= \lim_{t\to 0} \frac{\cancel{t}(3t+12)}{\cancel{t}(3t+4)} = \frac{12}{4} = 3$$

$\therefore$ Cartesian Eq. is $\quad y-1 = 3(x-1)$.

To find parametric equations: let say $x = t$ (or anything you want)

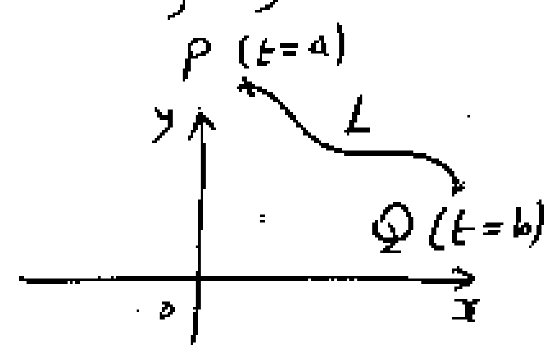
$\therefore \quad y-1 = 3(t-1) \quad$ or $\quad y = 3t-2$

$\therefore$ parametric Eqs. are $\quad \begin{array}{l} x = t \\ y = 3t-2 \end{array} \quad t \in \mathbb{R}$

Arc length of a Smooth Curve:

Let C be a plane curve given parametrically by

$$C: \begin{cases} x(t) = f(t) \\ y(t) = g(t) \end{cases} \quad t \in [a,b]$$



The arc length of C from P to Q is given by $L = \int_{t=a}^{t=b} ds$,
where $ds = \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$.

Surface area: Assume that the arc of plane curve C is rotated about x-axis. The area of the surface generated is thus given by $S_x = 2\pi \int_a^b |y|\, ds$, $ds = \sqrt{\dot{x}^2+\dot{y}^2}\, dt$

Similarly $S_y = 2\pi \int_a^b |x|\, ds$, $ds = \sqrt{\dot{x}^2+\dot{y}^2}\, dt$

Ex: Find the arc length of the curve $\bar{r} = (t^3, 2t^2-1)$, $0 \le t \le 1$

solution: Here $\bar{r} = (x,y) = (t^3, 2t^2-1)$ .. hence $x = t^3$, $y = 2t^2-1$

Now $\dot{x} = 3t^2$, $\dot{y} = 4t$ $\Rightarrow$ $ds = \sqrt{(3t^2)^2+(4t)^2}\, dt = t\sqrt{9t^2+16}\, dt$

$\therefore L = \int_0^1 t\sqrt{9t^2+16}\, dt$ .. let $w = 9t^2+16$, we get,

$L = \frac{1}{27}(9t^2+16)^{\frac{3}{2}}\Big|_0^1 = \frac{1}{27}(5^3-4^3) = \frac{61}{27}$

Ex: Find the area of the surface generated by rotating the arc of the curve $C: \begin{cases} x = 3t^2 \\ y = 2t^3 \end{cases}$ $t \in [0,1]$ about y-axis.

solution $S_y = \int_a^b |x|\, ds$, $ds = \sqrt{\dot{x}^2+\dot{y}^2}\, dt$

here $\dot{x} = 6t$, $\dot{y} = 6t^2$ $\Rightarrow$ $ds = \sqrt{36t^2+36t^4}\, dt = 6t\sqrt{1+t^2}\, dt$

$\therefore S_y = \int_0^1 3t^2 \cdot 6t\sqrt{1+t^2}\, dt = 18\int_0^1 t^3\sqrt{1+t^2}\, dt$

let $w = 1+t^2$, we get (check!),

$S_y = 9\left[\frac{2}{5}(t^2+1)^{\frac{5}{2}} - \frac{2}{3}(t^2+1)^{\frac{3}{2}}\right]_0^1$

$= \frac{12}{5}(\sqrt{2}+1)$

AMAT 219
support Material
Partial Derivatives

let $z=f(x,y)$ be a function of two independent variables x and y. we define: The partial derivative of f w.r.to "x" as the Derivative of f w.r.to x but treating y as a constant. we denote this derivative by $\frac{\partial z}{\partial x}$, or $\frac{\partial f}{\partial x}$, or f_x, or f_1. Similarly we define the partial derivative of f w.r.to y as the Derivative of f w.r.to y but treating "x" as a Constant. Again we denote this derivative by $\frac{\partial z}{\partial y}$, $\frac{\partial f}{\partial y}$, ... etc.

Ex: let $z = x^3 y + \ln(x^3+2y)$. Find $\frac{\partial z}{\partial x}$, and $\frac{\partial z}{\partial y}$.

solution: $\frac{\partial z}{\partial x} = 3x^2 y + \frac{3x^2}{x^3+2y}$, $\frac{\partial z}{\partial y} = x^3 + \frac{2}{x^3+2y}$.

Higher order Derivatives: The derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ may be referred to as First order partial Derivatives of f. we define "four" 2nd. order Derivatives as follows:

(1) $\frac{\partial^2 z}{\partial x^2}$ = Der. of z w.r.to x twice!, may write z_{xx}

(2) $\frac{\partial^2 z}{\partial y^2}$ = ~ ~ ~ ~ ~ y ~ . , ~ ~ z_{yy}

(3) $\frac{\partial^2 z}{\partial y \partial x}$ = Der. of z w.r.to x, then w.r.to y, may write z_{xy}

(4) $\frac{\partial^2 z}{\partial x \partial y}$ = ~ ~ ~ ~ ~ y, ~ ~ ~ ~ x. may write z_{yx}

In general: z_{xy} and z_{yx} are not equal .. however under some Continuity Conditions they could be equal. This will be the case in this course!

Special Equations

(1) The equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is called "Laplace Equation".
A function "u" satisfying Laplace equation is called a "Harmonic function".

(2) The equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ is the so called "Wave Eq."

EX: show that $u(x,y) = e^{3x} \sin 3y$ is Harmonic.

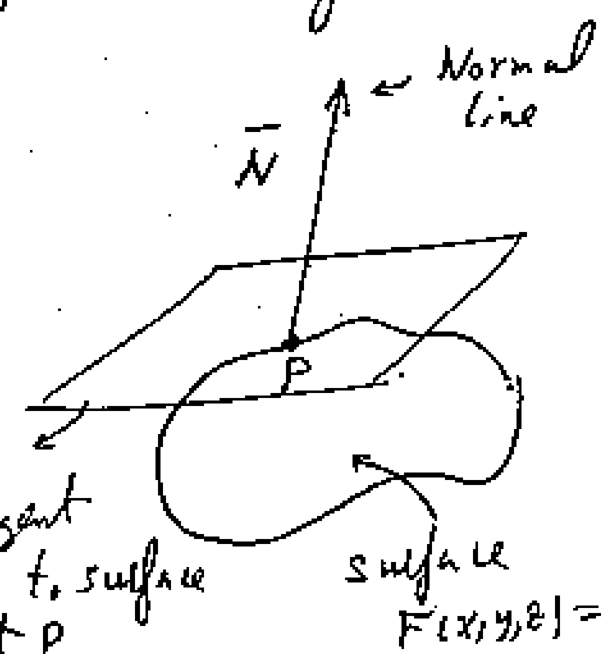
solution: Need to show: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$?

Indeed, $\frac{\partial u}{\partial x} = 3e^{3x}\sin 3y$, $\frac{\partial^2 u}{\partial x^2} = 9e^{3x}\sin 3y$

$\frac{\partial u}{\partial y} = 3e^{3x}\cos 3y$, $\frac{\partial^2 u}{\partial y^2} = -9e^{3x}\sin 3y$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 9e^{3x}\sin 3y - 9e^{3x}\sin 3y = 0$ .. verified!

Tangent plane / Normal line to a surface:



Let $F(x,y,z) = 0$ be an equation of a surface and $P(x_0,y_0,z_0)$ be a pt. on it.

A vector $\bar{N}$ normal to surface is given by

$$\bar{N} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)\Big|_P$$

Note: If $F(x,y,z)$ is a function of three independent variables x,y,z then $\frac{\partial F}{\partial x}$ = Der. of F w.r.to x holding both y, and z constants.
Similarly for $\frac{\partial F}{\partial y}$, and $\frac{\partial F}{\partial z}$.

Now: Eq. of Tangent plane: $\bar{r}\cdot\bar{N} = \bar{r}_0\cdot\bar{N}$, $\bar{r}_0 = $ pt. P

Eq. of normal line $\bar{r} = \bar{r}_0 + t\bar{N}$, $\bar{r}_0 = $ pt. P

EX: Find equations of tangent plane and normal line to the hyperboloid $4x^2 - 7y^2 + 3z^2 = 9$ at the point $(1,-1,2)$.

Solution: we need to find a vector $\bar{N}$ normal to surface at $P(1,-1,2)$

This vector is given by $\bar{N} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)\Big|_P$.

Note that $4x^2 - 7y^2 + 3z^2 - 9 = 0 \leftarrow$ Must be zero $\Rightarrow$

$$F = L.H.S = 4x^2 - 7y^2 + 3z^2 - 9$$

$$F_x = 8x, \quad F_y = -14y, \quad F_z = 6z$$

$$\bar{N} = (8x, -14y, 6z)\Big|_{(1,-1,2)} = (8, 14, 12)$$

$\therefore$ Eq. of tangent plane is $\quad \bar{r}\cdot\bar{N} = \bar{r}_0\cdot\bar{N}$

$$(x,y,z)\cdot(8,14,12) = (1,-1,2)\cdot(8,14,12)$$

$$\Rightarrow \quad 8x + 14y + 12z = 8 - 14 + 24 = 18$$

parametric equations of normal line: $\quad \bar{r} = \bar{r}_0 + t\bar{N}$

$$(x,y,z) = (1,-1,2) + t(8,14,12)$$

$$\Rightarrow \quad x = 1 + 8t, \quad y = -1 + 14t, \quad z = 2 + 12t \quad t \in \mathbb{R}$$

Implicit Differentiation:

Let $F(x,y,z) = 0$ be a given relation which defines say "z" implicitly as a function of x, and y; that is $z = z(x,y)$. Then we can determine $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ using the powerful formulae:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

provided $F_z \neq 0$

observe that if $F(x,y,z) = 0$ defines x as $x(y,z)$, then as above

$$\frac{\partial x}{\partial y} = -\frac{F_y}{F_x}, \quad \frac{\partial x}{\partial z} = -\frac{F_z}{F_x}, \quad F_x \neq 0.$$

Ex: If $x^3y + ze^{xyz} + \sin(3x-2y+9z) = 4$

defines y implicitly as a function of x, and z, find $\frac{\partial y}{\partial x}$.

Solution: Here $x^3y + ze^{xyz} + \sin(3x-2y+9z) - 4 = 0$

$$\therefore \quad F(x,y,z) = x^3y + ze^{xyz} + \sin(3x-2y+9z)$$

$$\therefore \quad \frac{\partial F}{\partial x} = 3x^2y + 2yz\,e^{xyz} + 3\cos(3x-2y+9z)$$

$$\frac{\partial F}{\partial y} = x^3 + 2xz\,e^{xyz} - 2\cos(3x-2y+9z)$$

$$\therefore \quad \frac{\partial y}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{3x^2y + 2yz\,e^{xyz} + 3\cos(3x-2y+9z)}{x^3 + 2xz\,e^{xyz} - 2\cos(3x-2y+9z)}$$

The chain rule for a function of several variables

There are several cases to be examined:

Case (1): If $z = f(t)$ and $t = t(x,y)$, that is t is a function of x, and y, then $\frac{\partial z}{\partial x} = f'(t)\frac{\partial t}{\partial x}$, $\frac{\partial z}{\partial y} = f'(t)\frac{\partial t}{\partial y}$.

Ex: If $z = f(3x+7y)$, find $\frac{\partial^2 z}{\partial y \partial x}$ if $f''(t) = \cos t$

Solution: $z = f(3x+7y)$ ... let $t = 3x+7y$ (hence $\frac{\partial t}{\partial x} = 3$, $\frac{\partial t}{\partial y} = 7$)

$$\therefore \quad z = f(t)$$

By chain rule: $\frac{\partial z}{\partial x} = f'(t)\frac{\partial t}{\partial x} = 3f'(t)$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = 3f''(t)\frac{\partial t}{\partial y} = (3)(7)f''(t)$$

$$\therefore \quad \frac{\partial^2 z}{\partial y \partial x} = 21 f''(t) = 21\cos t \leftarrow \text{recall}$$

$$= 21\cos(3x+7y)$$

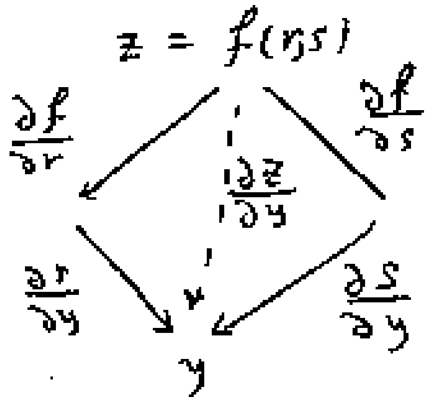
Case (2): If $W = f(t)$, and $t = t(x,y,z)$, then

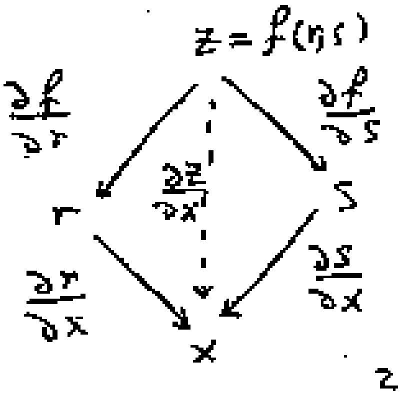
$$\frac{\partial W}{\partial x} = f'(t)\frac{\partial t}{\partial x}, \quad \frac{\partial W}{\partial y} = f'(t)\frac{\partial t}{\partial y}, \quad \frac{\partial W}{\partial z} = f'(t)\frac{\partial t}{\partial z}$$

Case (3): If $z = f(r,s)$ where $r = r(x,y)$, $S = S(x,y)$. Then

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial y}$$

see figures.





EX: If $z = f(3x-2y, 4x+y^2)$, find $\frac{\partial z}{\partial y}$ if $f(u,v) = u^2 v^{-3}$

solution $z = f(3x-2y, 4x+y^2)$
$= f(u,v)$ where $u = 3x-2y$, $v = 4x+y^2$

Now, $f(u,v) = u^2v^{-3} \Rightarrow \frac{\partial f}{\partial u} = 2uv^{-3}$, $\frac{\partial f}{\partial v} = -3u^2v^{-4}$

$u = 3x-2y \Rightarrow \frac{\partial u}{\partial y} = -2$

$v = 4x+y^2 \Rightarrow \frac{\partial v}{\partial y} = 2y$

By chain rule: $\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}$ ← sub. all quantities from above!

$= 2uv^{-3}(-2) + (-3u^2v^{-4})\cdot 2y$

$= -4uv^{-3} - 6u^2v^{-4}y$ ← but $\left.\begin{matrix} u = 3x-2y \\ v = 4x+y^2 \end{matrix}\right\}$

$= -4(3x-2y)(4x+y^2)^{-3} - 6(3x-2y)^2(4x+y^2)^{-4}y$

EX: If $z = u(2x+y, x-3y)$, find $\frac{\partial z}{\partial x}$ if u is such that $\frac{\partial u}{\partial x} = y$, $\frac{\partial u}{\partial y} = x$.

soln. $z = u(2x+y, x-3y) = u(r,s)$, where $r = 2x+y$, $S = x-3y$

By c-rule, $\frac{\partial z}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}$

Now: $r = 2x+y \Rightarrow \frac{\partial r}{\partial x} = 2$ ✓ $s = x-3y \Rightarrow \frac{\partial s}{\partial x} = 1$ ✓

$\frac{\partial u}{\partial x} = y \Rightarrow \frac{\partial u}{\partial r} = S$ ✓, $\frac{\partial u}{\partial y} = x \Rightarrow \frac{\partial u}{\partial s} = r$ ✓ (Note $\begin{matrix} x \to r \\ y \to s \end{matrix}$)

$\therefore \frac{\partial z}{\partial x} = s(2) + r(1) = 2S + r = 2(x-3y) + (2x+y)$
$= 4x - 5y$ #

Support Material

Some Review from AMAT 217

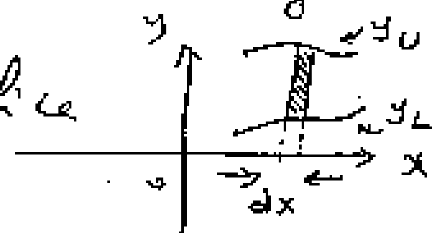
Assumptions: (1) Let $y = f(x)$, $y = g(x)$ be continuous for all $x \in [a, b]$. Assume that the graph of f lies entirely above the graph of g on $[a, b]$. We shall call $y = f(x)$ "the upper curve", and $y = g(x)$ "the lower curve" and denote by $y_U = f(x)$, $y_L = g(x)$ respectively.

(2) Let $x = p(y)$, $x = q(y)$ be continuous for all $y \in [c, d]$. Assume the graph of $q(y)$ lies to the left of the graph of p on $[c, d]$. We shall call $x = q(y)$ "the left curve" and $x = p(y)$ "the Right curve" and denote by $x_L = q(y)$, $x_R = p(y)$ respectively.

Slice Rule: If curves are expressed in the form $y = f(x)$, $y = g(x)$, we slice vertically. Hence thickness of slice is $\underline{dx}$

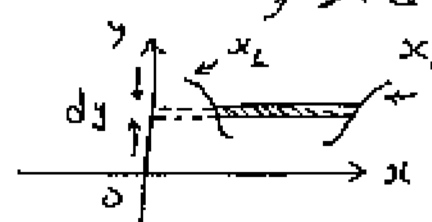
However if curves expressed in the form $x = p(y)$, $x = q(y)$ we slice Horizontally. Hence thickness of slice is $\underline{dy}$

vertical slice



slice height $= y_U - y_L$

Horizontal slice



slice height $= x_R - x_L$

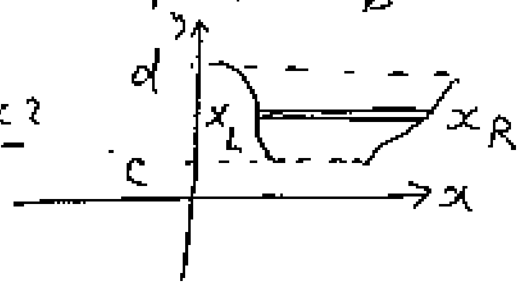
---

1 Area between curves: The area "A" is given by

Case 1

$$A = \int_a^b \text{height} \cdot \text{Thickness} = \int_a^b (y_U - y_L)\, dx$$

Case 2



$$A = \int_c^d \text{height} \cdot \text{Thickness} = \int_c^d (x_R - x_L)\, dy$$

[2] Volume of solid of revolution : Two Methods

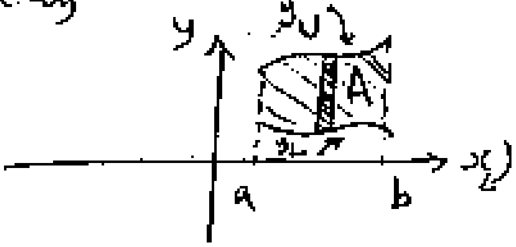
(a) <u>The Washer Method</u>: Use if slice is <u>perpendicular</u> to axis of rotation. In this case volume V is given by

$$V = \pi \int_{\alpha}^{\beta} (r_o^2 - r_i^2)\ \text{Thickness}$$

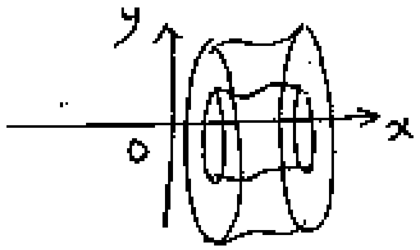
where r_o = outer radius = distance from outer curve to axis of rotation
r_i = Inner radius = ~ ~ inner ~ ~ ~ ~ ~

For instance if area "A" shown in figure below is rotated about x-axis, the volume V is given by

$$V_x = \pi \int_a^b (y_U^2 - y_L^2)\, dx$$



Area rotated



solid generated

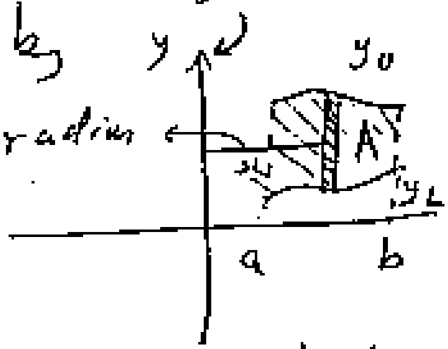
(b) <u>The cylinderical shells Method</u>: Use if slice is <u>parallel</u> to axis of rotation. The volume V is given by

$$V = 2\pi \int_{\alpha}^{\beta} \text{radius}\cdot\text{height}\cdot\text{Thickness}$$

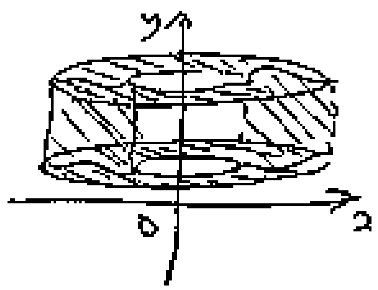
where radius = distance from centre of slice to axis of rotation

For instance if area "A" shown in figure below is rotated about y-axis, the volume V is given by

$$V_y = 2\pi \int_a^b x(y_U - y_L)\, dX$$



area rotated

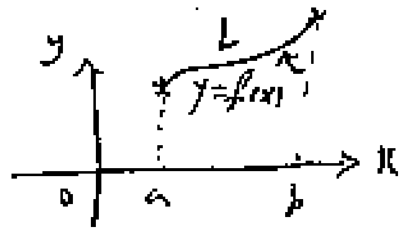


solid generated

[3] Arc length:

Let $y = f(x)$ and assume $f'(x)$ exists and is continuous on $[a,b]$. Then the length of the arc of $f(x)$ on $[a,b]$ is given by

$$L = \int_a^b \sqrt{1+(f'(x))^2}\,dx \quad \text{or} \quad \int_a^b \sqrt{1+y'^2}\,dx$$



__Note__: Similar formula holds if graph is expressed in the form $x = g(y)$, $y \in [c,d]$, namely

$$L = \int_c^d \sqrt{1+(g'(y))^2}\,dy$$

[4] Area of surface of revolution

under same conditions above, the area of the surface generated by revolving the arc of $y = f(x)$, $x \in [a,b]$ about x-axis is given by

$$S_x = 2\pi \int_a^b y\,ds, \quad y > 0, \qquad ds = \sqrt{1+\left(\tfrac{dy}{dx}\right)^2}\,dx \text{ or } \sqrt{1+\left(\tfrac{dx}{dy}\right)^2}\,dy$$

Similar formula hold for rotation about y-axis, namely $S_y = 2\pi \int x\,ds$, $x >$

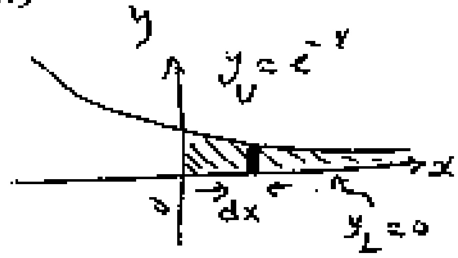
— — — — — —

__Examples__:

(1) Find the volume of the solid generated by revolving the region bounded by $y = e^{-x}$ and x-axis, $0 \le x < \infty$

(a) About x-axis

(b) About y-axis



__Solution__ (a) slice vertically ($\perp$ to axis of rotation)

$\therefore$ Use Washer method!

$$V_x = \pi \int_a^b (r_o^2 - r_i^2)\,dx = \pi \int_0^\infty [(e^{-x})^2 - (0)^2]\,dx$$

$$= \pi \int_0^\infty e^{-2x}\,dx \leftarrow \text{Improper!}$$

$r_o = e^{-x}$

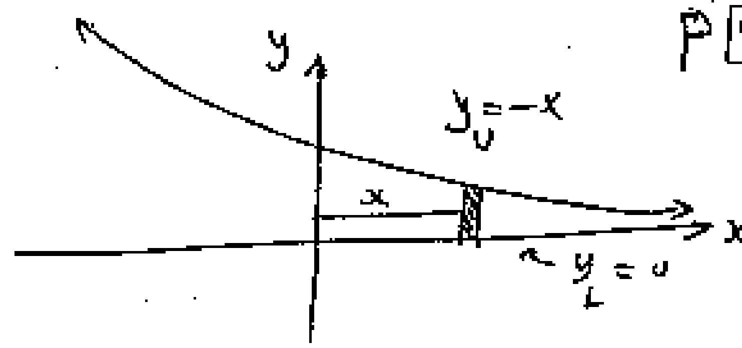
No inner radius ($r_i = 0$)

$$= \pi \lim_{R\to\infty} \int_0^R e^{-2x}\,dx = \frac{\pi}{-2} \lim_{R\to\infty} e^{-2x}\Big|_0^R$$

$$= -\frac{\pi}{2}\left[\lim_{R\to\infty}(e^{-2R} - e^0)\right] = -\frac{\pi}{2}[0-1] = \frac{\pi}{2}$$

— — — — —

(b) The slice is parallel to axis of rotation .. Use shell method!



$$V_y = 2\pi \int_a^b \text{radius} \cdot \text{height} \cdot \text{Thickness}$$

$$= 2\pi \int_0^\infty x e^{-x} dx \leftarrow \text{Improper}$$

$$= 2\pi \lim_{R\to\infty} \int_0^R x e^{-x} dx \leftarrow \text{by parts}$$

$$= 2\pi \lim_{R\to\infty} -e^{-x}(x+1) \Big|_0^R$$

$$= 2\pi \lim_{R\to\infty} \left[-e^{-R}(R+1) + 1\right] = 2\pi[0+1] = 2\pi$$

radius $= x$
height $y_U - y_L = e^{-x} - 0 = e^{-x}$
Thickness $= dx$

Ex: Find the arc length of $y = \frac{3}{2} x^{\frac{2}{3}}$ from $x = 3\sqrt{3}$ to $x = 16\sqrt{2}$

soln. arc length $L = \int_a^b ds$, $ds = \sqrt{1 + (f'(x))^2}\, dx$

here $y = f(x) = \frac{3}{2} x^{\frac{2}{3}} \Rightarrow f'(x) = x^{-\frac{1}{3}} \Rightarrow (f'(x))^2 = x^{-\frac{2}{3}} = \frac{1}{x^{\frac{2}{3}}}$

$$\therefore L = \int_{3\sqrt{3}}^{16\sqrt{2}} \sqrt{1 + \frac{1}{x^{\frac{2}{3}}}}\, dx = \int_{3\sqrt{3}}^{16\sqrt{2}} \sqrt{\frac{x^{\frac{2}{3}} + 1}{x^{\frac{2}{3}}}}\, dx = \int_{3\sqrt{3}}^{16\sqrt{2}} \frac{\sqrt{x^{\frac{2}{3}}+1}}{x^{\frac{1}{3}}}\, dx = \int x^{-\frac{1}{3}} \sqrt{x^{\frac{2}{3}}+1}\, dx$$

let $u = x^{\frac{2}{3}} + 1$, $du = \frac{2}{3} x^{-\frac{1}{3}} dx \Rightarrow x^{-\frac{1}{3}} dx = \frac{3}{2} du$

$$\therefore L = \frac{3}{2}\int \sqrt{u}\, du = \frac{3}{2} \cdot \frac{2}{3} u^{\frac{3}{2}} = u^{\frac{3}{2}} = \left(x^{\frac{2}{3}} + 1\right)^{\frac{3}{2}} \Big|_{3\sqrt{3}}^{16\sqrt{2}}$$

$$= \left[(\sqrt[3]{x^2}) + 1\right]^{\frac{3}{2}} \Big|_{3\sqrt{3}}^{16\sqrt{2}} = 9^{\frac{3}{2}} - 4^{\frac{3}{2}}$$

$$= (\sqrt{9})^3 - (\sqrt{4})^3 = 27 - 8 = 19$$

Ex: Find the arc length of $y = \frac{1}{2} x^2$ from $x = 0$ to $x = 1$

solution: Done before! .. Redo it!

Ans. $\frac{1}{2}\left[\sqrt{2} + \ln(\sqrt{2}+1)\right]$

AMAT 217

REVIEW PROBLEMS

(I) Find each of the following

(a) $\int e^{-4x+1}\,dx$

(b) $\int e^{19x}\,dx$

(c) $\int x\sqrt{1+x^2}\,dx$

(d) $\int x\cos(1+x^2)\,dx$

(e) $\frac{d}{dx}\sin^{-1}(7x)$

(f) $\frac{d}{dx}\tan^{-1}(2x)$

(g) $\int \frac{x}{1+x^2}\,dx$

(h) $\int_{-1}^{0} \frac{1}{1+x^2}\,dx$

(i) $\frac{d}{dx}\ln(\sin x)$

(j) $\frac{d}{dx}\ln(x^3-2x+1)$

(k) $\int_{1}^{e^2} \frac{1}{x}\,dx$

(l) $\int_{-2}^{-1} \frac{1}{x}\,dx$

(m) $\frac{d}{dx}\int_{0}^{2x} \sin(t^3)\,dt$

(n) $\frac{d}{dx}\int_{1}^{x^3} e^{-2t^2}\,dt$

(II) Find the area enclosed by $y=-x$, and $y=x^2$

(III) Find the area enclosed by $y=2-x^2$, and $y=x^2$

(IV) Find the length of the arc $y=\frac{2}{3}x^{\frac{3}{2}}$, $0\leq x\leq 8$

(V) Find the length of the arc $y=\cosh x$, $0\leq x\leq \ln 2$

(VI) Find the volume obtained by rotating the region enclosed by $y=-x$, $y=x$, $x=1$

about y.-axis.

(VII) Find the volume obtained by rotating the region enclosed by $y=x^2$, $y=x$

(a) about the line $y=-1$.

(b) about the line $x=2$.

AMAT 217

SOLUTIONS TO REVIEW PROBLEMS

(I) (a) $\int e^{-4x+1}\,dx = \dfrac{e^{-4x+1}}{-4} + C$ (b) $\int e^{19x}\,dx = \dfrac{e^{19x}}{19} + C$

(c) $\int x\sqrt{1+x^2}\,dx$. let $w = 1+x^2 \Rightarrow dw = 2x\,dx \Rightarrow \frac{1}{2}dw = x\,dx$

$\therefore I = \frac{1}{2}\int \sqrt{w}\,dw = \frac{1}{2}\cdot\dfrac{w^{3/2}}{3/2} + C = \frac{1}{3}(1+x^2)^{3/2} + C.$

(d) $\int x\cos(1+x^2)\,dx$ .. let $w = 1+x^2 \Rightarrow dw = 2x\,dx \Rightarrow \frac{1}{2}dw = x\,d$

$\therefore I = \frac{1}{2}\int \cos w\,dw = \frac{1}{2}\sin w + C = \frac{1}{2}\sin(1+x^2) + C$

(e) $\dfrac{d}{dx}\sin^{-1}(7x) = \dfrac{(7x)'}{\sqrt{1-(7x)^2}} = \dfrac{7}{\sqrt{1-49x^2}}$

(f) $\dfrac{d}{dx}\tan^{-1}(2x) = \dfrac{(2x)'}{1+(2x)^2} = \dfrac{2}{1+4x^2}$

(g) $\int \dfrac{x}{1+x^2}\,dx$. let $w = 1+x^2 \Rightarrow dw = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}dw$

$\therefore I = \frac{1}{2}\int \frac{1}{w}\,dw = \frac{1}{2}\ln|w| + C = \frac{1}{2}\ln(1+x^2) + C$

(h) $\int_{-1}^{0}\dfrac{1}{1+x^2}\,dx \;\underline{\underline{\text{Table}}}\; \tan^{-1}x\Big|_{-1}^{0} = \tan^{-1}0 - \tan^{-1}(-1)$

$= 0 + \tan^{-1}1 = 0 + \frac{\pi}{4} = \frac{\pi}{4}$

(i) $\dfrac{d}{dx}\ln(\sin x) = \dfrac{\cos x \;\leftarrow u'}{\sin x \;\leftarrow u} = \cot x$

(j) $\dfrac{d}{dx}\ln(x^3-2x+1) = \dfrac{3x^2-2}{x^3-2x+1}$

(k) $\int_1^{e^2}\frac{1}{x}\,dx \;\underline{\underline{\text{Table}}}\; \ln|x|\Big|_1^{e^2} = \ln e^2 - \ln 1$

$= 2\ln e - \ln 1$

$= 2(1) - 0 = 2$

(l) $\int_{-2}^{-1}\frac{1}{x}\,dx = \ln|x|\Big|_{-2}^{-1} = \ln|-1| - \ln|-2|$

$= \ln 1 - \ln 2 = 0 - \ln 2 = -\ln 2$

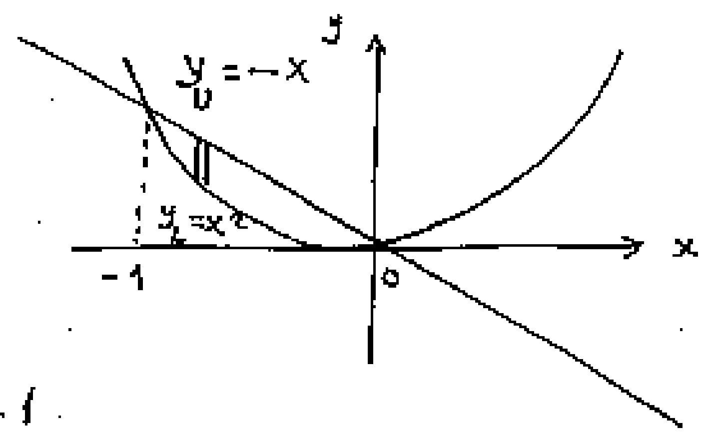
(m) $\dfrac{d}{dx}\int_0^{2x}\sin(t^3)\,dt = \sin(t^3)\Big|_{\text{at } t=2x}(2x)' = \sin(8x^3).\,2$

$= 2\sin(8x^3)$

(n) $\dfrac{d}{dx}\int_1^{x^3} e^{-2t^2}\,dt = e^{-2t^2}\Big|_{t=x^3}(x^3)'$

$= e^{-2(x^3)^2}\cdot 3x^2 = 3x^2e^{-2x^6}$

(II) First sketch!
Next: Find where graphs intersect?



$y = x^2$, $y = -x$

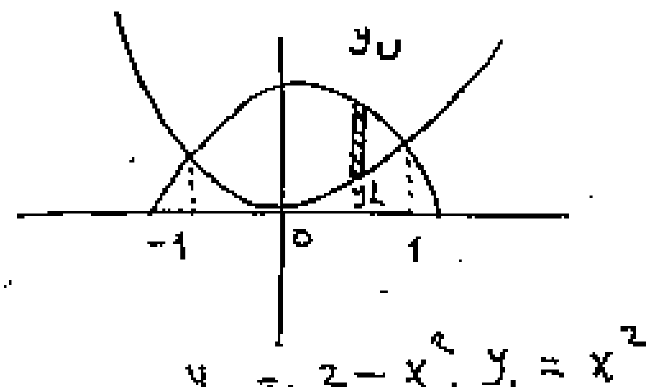
$\Rightarrow x^2 = -x \Rightarrow x^2 + x = 0$

$x(x+1) = 0 \Rightarrow x = 0, -1$

$\therefore$ area $A = \int_a^b (y_U - y_L)\, dx = \int_{-1}^{0} (-x - x^2)\, dx = -\frac{1}{2}x^2 - \frac{1}{3}x^3 \Big|_{-1}^{0}$

$= \frac{1}{6}$

(III) First sketch!
Next: where curves intersect?



Let us solve $y = 2 - x^2$, $y = x^2$

$\therefore 2 - x^2 = x^2$

$2 = 2x^2 \Rightarrow x = \pm 1$

area $A = \int_{-1}^{1} [(2 - x^2) - x^2]\, dx = 2\int_{-1}^{1} (1 - x^2)\, dx = 2\left[x - \frac{1}{3}x^3\right]_{-1}^{1} = 8/3$

(IV) $y = \frac{2}{3} x^{\frac{3}{2}}$, $0 \le x \le 8$

Arc length $L = \int_a^b \sqrt{1 + y'^2}\, dx$ .. here $y' = \frac{2}{3} \cdot \frac{3}{2} x^{\frac{1}{2}} = x^{\frac{1}{2}} = \sqrt{x}$

$= \int_0^8 \sqrt{1 + (\sqrt{x})^2}\, dx = \int_0^8 (1 + x)^{\frac{1}{2}} dx$

$= \frac{2}{3}(1 + x)^{\frac{3}{2}} \Big|_0^8 = \frac{2}{3}\left[9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right] = \frac{2}{3}\left[3^3 - 1\right]$

$= \frac{52}{3}$

(V) $y = \cosh x$, $0 \le x \le \ln 2$

$y' = \sinh x$, $1 + y'^2 = 1 + \sinh^2 x \xrightarrow{id.} \cosh^2 x$

$\therefore$ arc length $L = \int_a^b \sqrt{1 + y'^2}\, dx = \int_0^{\ln 2} \sqrt{\cosh^2 x}\, dx = \int_0^{\ln 2} \cosh x\, dx$

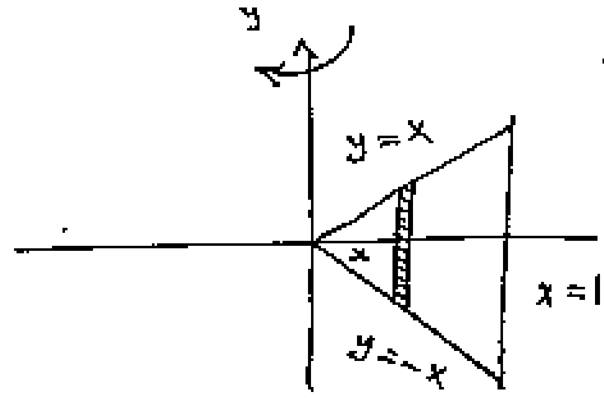
$= \sinh x \Big|_0^{\ln 2} = \sin(\ln 2) - \sinh 0 = \sinh(\ln 2)$

Note: $\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{e^{\ln 2} - e^{\ln 2^{-1}}}{2} = \frac{2 - 2^{-1}}{2} = \frac{2 - \frac{1}{2}}{2} = \frac{3}{4}$

$\therefore L = \frac{3}{4}$

(VI)

Since slice is // to axis of rotation, use cylinderical shells!



$$V_y = 2\pi \int_a^b \text{radius} \cdot \text{height} \cdot \text{Thickness}$$

$$= 2\pi \int_0^1 x(2x)\,dx = 4\pi \cdot \frac{x^3}{3}\Big|_0^1 = \frac{4\pi}{3}.$$

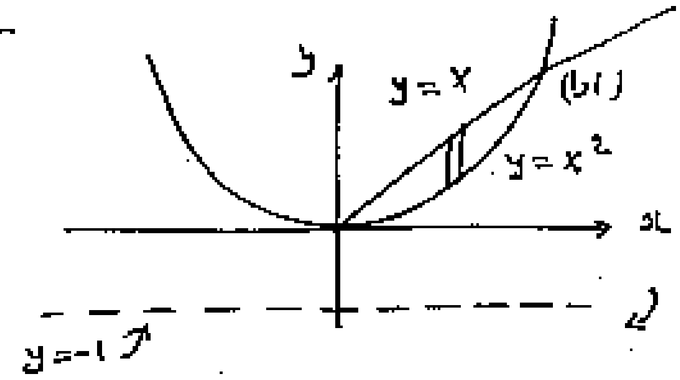
height $y_U - y_L = x-(-x) = 2x$

Thickness dx,

radius x

(VII) (a) slice is perpendicular to axis of rotation

$\therefore$ Use Washer method



$$V = \pi \int_a^b (r_o^2 - r_i^2) \cdot \text{Thickness}$$

$$= \pi \int_0^1 [(x+1)^2 - (x^2+1)^2]\,dx$$

$$= \pi \int_0^1 (x^2+2x+1) - (x^4+2x^2+1)\,dx$$

$$= \pi \int_0^1 (2x - x^2 - x^4)\,dx$$

$$= \pi \left[x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{5}x^5\right]_0^1 = \frac{7}{15}$$

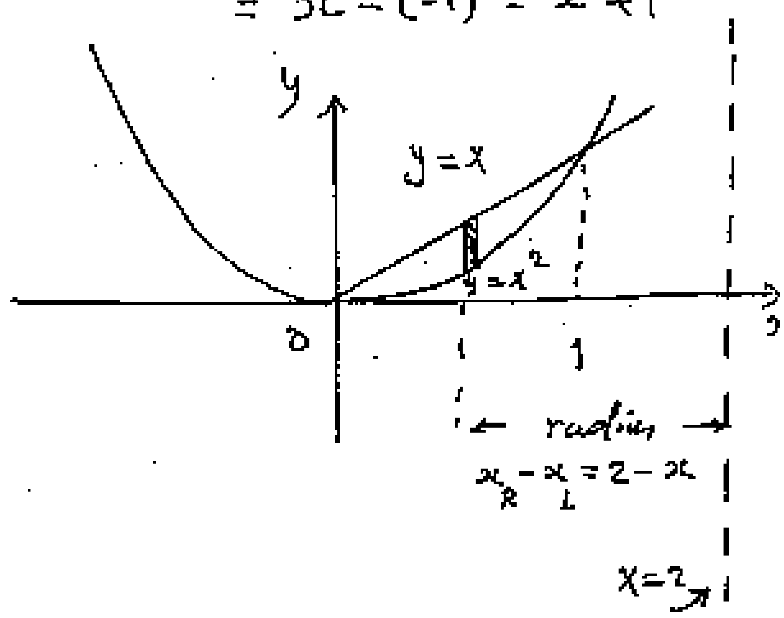
Thickness dx

r_o = outer radius = distance from upper curve to axis of rotation $= x-(-1) = x+1$

r_i = Inner radius = distance from lower curve to axis of rotation $= x^2-(-1) = x^2+1$

(b) slice is parallel to axis of rotation

$\therefore$ Use shells!



$$V = 2\pi \int_a^b \text{radius} \cdot \text{height} \cdot \text{Thickness}$$

$$= 2\pi \int_0^1 (2-x)(x-x^2)\,dx$$

$$= 2\pi \int_0^1 (2x - 3x^2 + x^3)\,dx$$

$$= 2\pi \left[x^2 - x^3 + \tfrac{1}{4}x^4\right]_0^1$$

$$= 2\pi\left[1 - 1 + \tfrac{1}{4}\right] = \frac{\pi}{2}$$

rad. $= 2-x$

height $y_U - y_L = x - x^2$

Thickness $= dx$

MID TERM TEST

WINTER 1999

(I) Calculate the following

(a) $\int x \sin(x+1)\,dx$

(b) $\int \sin^{-1} x\,dx$

(c) $\int \frac{1}{x^2-4}\,dx$

(d) $\int_0^1 \frac{2}{\sqrt{1+x^2}}\,dx$

(e) $\int \frac{1}{x(x+1)^2}\,dx$

(f) $\int \frac{1}{x^2+5x+6}\,dx$

(g) $\int \frac{x-1}{x+1}\,dx$

(h) $\int \frac{1}{x^2+8x+17}\,dx$

(i) The value of the Trapezoidal rule applied to $\int_2^5 x\,dx$ with $n = 10$

(j) $\int_0^{\frac{\pi}{2}} \frac{d}{dx}\left(\frac{\sin x}{x}\right) dx$

(II) Find $\iint_R y\,dA$ where R is the region enclosed by $y = 2x$, $y = -x$, and $y = 4$

(III) Find $\iint_R \frac{1}{1+y^2}\,dA$ where R is the region enclosed by $y = x$, $y = 1$, $x = 0$.

(IV) Use double integrals to find the volume bounded by the Cone $z = 9 - \sqrt{x^2+y^2}$ and the plane $z = 0$.

(V) Use double integrals to find $\bar{y}$ of the centroid of the region bounded by $x = 2$, $y = 0$, $y = \sqrt{x}$

Additional problems: on Vectors

(I) Find the equation of the plane passing through the points $(-1,1,-1)$, $(-3,4,-2)$, and $(2,-2,-1)$

(II) Find parametric equations for the line perpendicular to the plane $3x - y + 2z = 6$ that goes through the point $(1,-2,-1)$

(III) Find the distance from $(-3,-2,1)$ to the plane $2x - y - z = 1$

(IV) Find parametric equations for the line described by the planes $2x + 3y - z = 1$, and $3x - 2y + z = 2$

AMAT 219 SOLUTIONS TO MID TERM WINTER 99

(1)

(a) $\int x \sin(x+1)\,dx$ ← by parts once

$= -x\cos(x+1) + \int \cos(x+1)\,dx$

$= -x\cos(x+1) + \sin(x+1) + C$

$$\begin{array}{cc} x & \sin(x+1) \\ 1 & -\cos(x+1) \end{array}$$

(b) $\int \sin^{-1}x\,dx$ ← by parts once

$$\begin{array}{cc} \sin^{-1}x & 1 \\ \frac{1}{\sqrt{1-x^2}} & x \end{array}$$

$= x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx$ → Let $w = 1-x^2$, $-\frac{1}{2}dw = x\,dx$

$= x\sin^{-1}x + \frac{1}{2}\int \frac{1}{\sqrt{w}}\,dw = x\sin^{-1}x + \frac{1}{2}\cdot\frac{2}{1}w^{\frac{1}{2}} + C$

$= x\sin^{-1}x + \sqrt{1-x^2} + C$

(c) $\int \frac{1}{x^2-4}\,dx$ ← use PFD .. By inspection,

$\frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)} = \frac{1}{2-(-2)}\left[\frac{1}{x-2} - \frac{1}{x+2}\right]$

$\therefore I = \frac{1}{4}\int\left(\frac{1}{x-2} - \frac{1}{x+2}\right)dx = \frac{1}{4}\left[\ln|x-2| - \ln|x+2|\right] + C$

(d) $\int_0^1 \frac{2}{\sqrt{1+x^2}}\,dx = 2\int_0^1 \frac{dx}{\sqrt{1+x^2}} \overset{\text{Table}}{=} 2\sinh^{-1}x\Big|_0^1$

$= 2\left(\sinh^{-1}1 - \sinh^{-1}0\right) = 2\sinh^{-1}1$

or (optional) $= 2\ln\left(1+\sqrt{2}\right)$.

(Since $\sinh^{-1}x = \ln\left(x+\sqrt{1+x^2}\right)$).

(e) $\int \frac{1}{x(x+1)^2}\,dx$ ← Use PFD

$\frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

verify $A = 1$, $B = -1$, and $C = -1$

$\therefore I = \int\left(\frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2}\right)dx = \ln|x| - \ln|x+1| - \int \frac{dx}{(x+1)^2}$

warning: This is not $\ln|x+1|^2$!!

$= \ln|x| - \ln|x+1| - \int (x+1)^{-2}\,dx$

$= \ln|x| - \ln|x+1| - \frac{(x+1)^{-1}}{-1} + C$

$= \ln|x| - \ln|x+1| + \frac{1}{x+1} + C$

(f) $\int \frac{1}{x^2+5x+6}\,dx$ ← Since x^2+5x+6 is factorable,
use PFD!

$$\frac{1}{x^2+5x+6} = \frac{1}{(x+2)(x+3)} = \frac{1}{3-2}\left(\frac{1}{x+2} - \frac{1}{x+3}\right) = \frac{1}{x+2} - \frac{1}{x+3}$$

$$\therefore I = \int\left(\frac{1}{x+2} - \frac{1}{x+3}\right) dx = \ln|x+2| - \ln|x+3| + C$$

(g) $\int \frac{x-1}{x+1}\,dx$ .. easier, let $w = x+1 \Rightarrow dw = dx$, and $\boxed{x = w-1}$

$$\therefore I = \int \frac{(w-1)-1}{w}\,dw = \int \frac{w-2}{w}\,dw = \int\left(1-\frac{2}{w}\right)dw$$

$$= w - 2\ln|w| + C = (x+1) - 2\ln|x+1| + C.$$

(h) $\int \frac{1}{x^2+8x+17}\,dx$ .. Note that $x^2+8x+17$ is not factorable

∴ Complete the squares!

$$x^2+8x+17 = (x+4)^2+1$$

$$\therefore I = \int \frac{1}{(x+4)^2+1}\,dx \quad \text{.. Now let } x+4 = t \Rightarrow dx = dt$$

$$= \int \frac{1}{t^2+1}\,dt \overset{\text{Table}}{=} \frac{1}{1}\tan^{-1}\left(\frac{t}{1}\right) + C = \tan^{-1}(x+4) + C$$

(i) $I = \int_2^5 x\,dx$ .. Since integrand $f(x) = x$ is a polynomial of degree less than Two, the required T_{10} equals to exact value of I!

$$\therefore T_{10} = \int_2^5 x\,dx = \frac{1}{2}x^2\Big|_2^5 = \frac{1}{2}\left[5^2-2^2\right] = \frac{21}{2}$$

(j) $\int_0^{\frac{\pi}{2}} \frac{d}{dx}\left(\frac{\sin x}{x}\right)dx$ ← Improper at lower limit

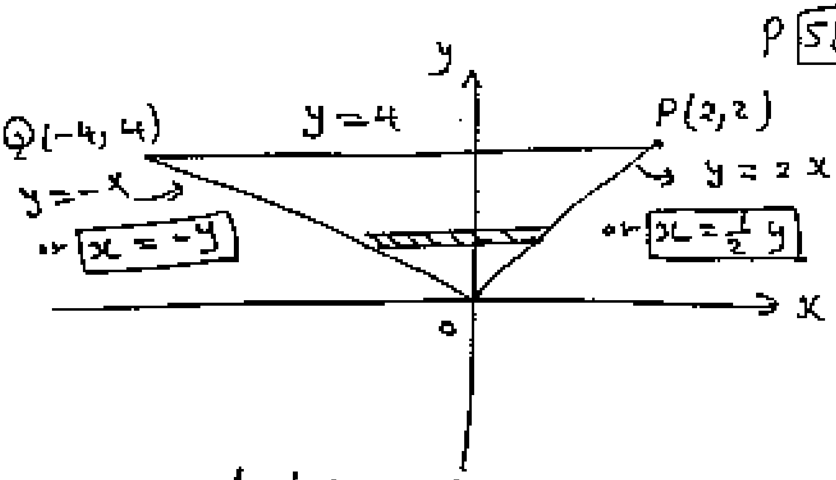
$$\therefore I = \lim_{R\to 0^+} \int_R^{\frac{\pi}{2}} \frac{d}{dx}\left(\frac{\sin x}{x}\right)dx \quad \text{.. Note } \int \text{ \& } \frac{d}{dx} \text{ cancels!}$$

$$= \lim_{R\to 0^+} \frac{\sin x}{x}\Big|_R^{\frac{\pi}{2}} = \lim_{R\to 0^+}\left[\frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} - \frac{\sin R}{R}\right]$$

$$= \frac{2}{\pi} - \underbrace{\lim_{R\to 0^+}\frac{\sin R}{R}}_{=1} \leftarrow \text{by L'H Rule}$$

$$= \frac{2}{\pi} - 1$$

(II) $\iint_R y\,dA$



clearly R is an x-simple region:

$$-y \leq x \leq \tfrac{1}{2}y$$
$$0 \leq y \leq 4$$

To find P: solve $y=2x,\ y=4$
$\therefore 2x=4 \Rightarrow x=2$
$\therefore P(2,4)$

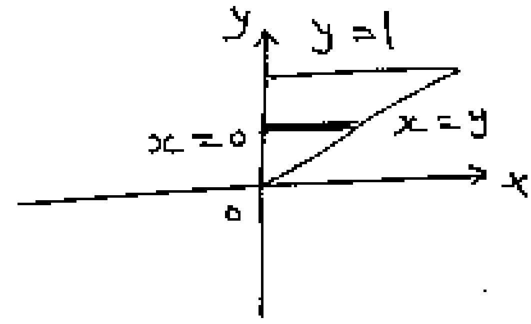
To find Q: solve $y=-x,\ y=4$
$\therefore -x=4 \Rightarrow x=-4$
$\therefore Q(-4,4)$

$$I = \int_0^4 \int_{-y}^{\frac{1}{2}y} y\,dx\,dy$$

$$= \int_0^4 y\,(x)\Big|_{-y}^{\frac{1}{2}y}dy = \int_0^4 y\left[\tfrac{1}{2}y+y\right]dy$$

$$= \frac{3}{2}\int_0^4 y^2\,dy = \frac{3}{2}\cdot\frac{1}{3}\,y^3\Big|_0^4 = \frac{3}{2}\cdot\frac{1}{3}\cdot 64 = 32$$

(III) $\iint_R \frac{1}{1+y^2}\,dA$



Note: If we integrate w.r.t. x 1st. the problem becomes much easier!
$\therefore$ let us treat R as an x-simple region!

R: $0 \leq x \leq y$
$0 \leq y \leq 1$

$$\therefore\ I = \int_0^1\int_0^y \frac{1}{1+y^2}\,dx\,dy = \int_0^1 \frac{1}{1+y^2}\left(\int_0^y dx\right)dy$$

$$= \int_0^1 \frac{1}{1+y^2}\,(x)\Big|_0^y\,dy = \int_0^1 \frac{y}{1+y^2}\,dy \longleftarrow$$

let $w=1+y^2$
$dw = 2y\,dy$
$\Rightarrow y\,dy = \frac{1}{2}dw$

$$= \frac{1}{2}\int \frac{1}{w}\,dw = \frac{1}{2}\ln|w|$$
$$= \frac{1}{2}\ln(1+y^2)\Big|_0^1$$
$$= \frac{1}{2}\left[\ln 2 - \ln 1\right] = \frac{1}{2}\ln 2$$

(IV) $V = \iint_R z\,dA$ .. hence $z = 9-\sqrt{x^2+y^2}$

and R is the region enclosed by the projection of $z=9-\sqrt{x^2+y^2}$ onto $z=0$ (the xy-plane), namely enclosed by $0=9-\sqrt{x^2+y^2}$ or $x^2+y^2=81$ (a circle).

Use polar coordinates: $x^2+y^2=r^2$, $dA = r\,dr\,d\theta$

$$\therefore\ z = 9-\sqrt{x^2+y^2} = 9-\sqrt{r^2} = 9-r$$

$$\therefore V = \int_0^{2\pi}\int_0^{9} (9-r)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{9} (9-r)\,r\,dr$$

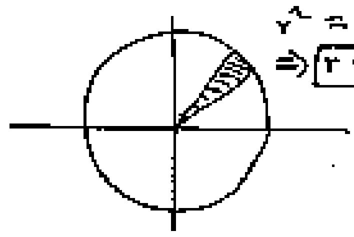
$$= 2\pi \int_0^{9} (9r - r^2)\,dr$$

$$= 2\pi \left[\frac{9}{2}r^2 - \frac{1}{3}r^3 \right]_0^9$$

$$= 2\pi \left[\frac{9}{2}(9)^2 - \frac{1}{3}9^3 \right] = 2\pi (9)^3 \left[\frac{1}{2} - \frac{1}{3} \right]$$

$$= 2\pi \cdot \frac{729}{6} = 243\pi$$

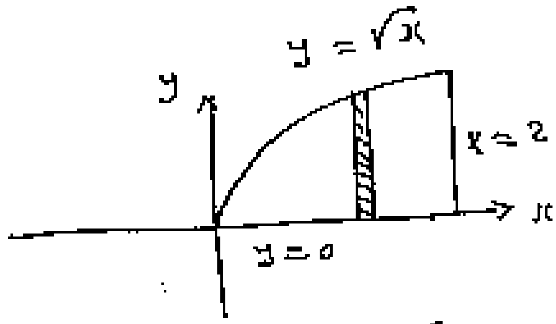
P $\boxed{59}$



$0 \leq r \leq 9$
$0 \leq \theta \leq 2\pi$

(V) For centroid $\delta = 1$

$\therefore dm = \delta\, dA = dA$



$0 \leq y \leq \sqrt{x}$
$0 \leq x \leq 2$

$$\therefore m = \iint_R dm = \iint_R dA$$

$$= \int_0^2 \left\{ \int_0^{\sqrt{x}} dy \right\} dx$$

$$= \int_0^2 y \Big|_0^{\sqrt{x}} dx = \int_0^2 (\sqrt{x} - 0)\,dx$$

$$= \frac{2}{3} x^{\frac{3}{2}} \Big|_0^2 = \frac{2}{3}\left[2^{\frac{3}{2}} - 0 \right] = \frac{2}{3}\left[2\sqrt{2} \right] = \frac{4\sqrt{2}}{3}.$$

Next, $M_{y=0} = \iint_R y\, dm = \iint_R y \cdot dA = \int_0^2 \int_0^{\sqrt{x}} y\,dy\,dx$

$$= \frac{1}{2}\int_0^2 y^2 \Big|_0^{\sqrt{x}} dx = \frac{1}{2}\int_0^2 \left[(\sqrt{x})^2 - 0^2 \right] dx$$

$$= \frac{1}{2}\int_0^2 x\,dx = \frac{1}{2} \cdot \frac{1}{2} x^2 \Big|_0^2 = 1$$

$$\therefore \bar{y} = \frac{M_{y=0}}{m} = \frac{1}{\frac{4\sqrt{2}}{3}} = \frac{3}{4\sqrt{2}}$$

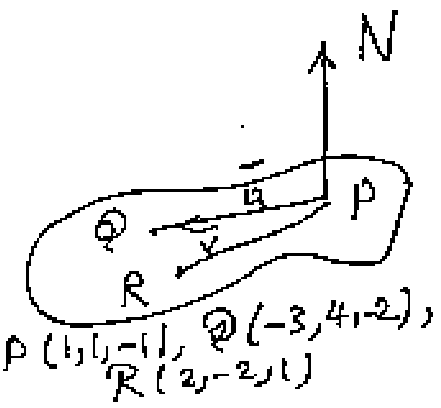
Additional problems:

(I) For equation of a plane, we need

(i) A point: choose say $P(-1,1,-1) \Rightarrow \bar{r}_0 = (-1,1,-1)$

(ii) A normal vector $\bar{N}$: This is $\bar{u} \times \bar{v}$

where $\bar{u} = (-4,3,-1)$, $\bar{v} = (1,-3,2)$



$$\begin{matrix} \bar{u} \\ \bar{v} \end{matrix} \begin{pmatrix} -4 & 3 & -1 \\ 1 & -3 & 2 \end{pmatrix}$$

$\therefore \quad \bar{N} = \bar{u} \times \bar{v} = (3, 7, 9)$

$\therefore$ equation of plane is given by

$$\bar{r} \cdot \bar{N} = \bar{r}_0 \cdot \bar{N}$$

$$(x, y, z) \cdot (3, 7, 9) = (1, 1, -1) \cdot (3, 7, 9)$$

$$\Rightarrow 3x + 7y + 9z = +3 + 7 - 9$$
$$= 1$$

(II) For eq. of line we need

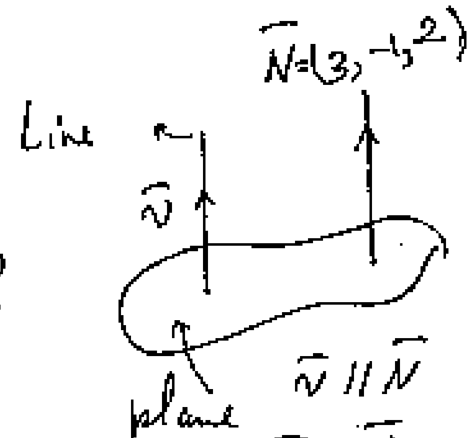
(i) <u>A point</u>: Given as $(1, -2, -1) \Rightarrow \bar{r}_0 = (1, -2, -1)$

(ii) <u>A direction vector</u> $\bar{v}$: Take $\bar{v} = \bar{N}$

$\therefore \bar{v} = (3, -1, 2)$

Eq. of line: $\bar{r} = \bar{r}_0 + t\,\bar{v}$

$$(x, y, z) = (1, -2, -1) + t(3, -1, 2), \quad t \in \mathbb{R}$$



$\therefore$ <u>Take</u>: $\bar{v} = \bar{N}$

(III) Distance $D = \dfrac{|2(-3) - (-2) - 1 - 1|}{\sqrt{2^2 + (-1)^2 + (-1)^2}}$

$$= \frac{6}{\sqrt{6}} \doteq \sqrt{6}$$

Note: 1st write $2x - y - z = 1$ as $2x - y - z - 1 = 0$

(IV) Just solve system
$$\begin{cases} 2x + 3y - z = 1, & \text{---} (*) \\ 3x - 2y + z = 2 & \text{---} (**) \end{cases}$$

Either using matrices or method of elimination (from high school!).

Adding, we get, $5x + y = 3$

$$\Rightarrow \boxed{y = 3 - 5x}$$

substitute $y = 3 - 5x$ into $(*)$ we get:

$$2x + 3(3 - 5x) - z = 1 \qquad \text{(easier to solve for } z \text{ !!)}$$

$$\therefore \boxed{z = 8 - 13x}$$

Now, if we let $x = t$, $t \in \mathbb{R}$.

$\therefore$ parametric equations are
$$\begin{aligned} x &= t \\ y &= 3 - 5t \\ z &= 8 - 13t \end{aligned} \qquad t \in \mathbb{R}$$

MID TERM TEST

WINTER 2000

(1) Calculate the following

(a) $\int x \ln x \, dx$ (b) $\int \sin^{-1} x \, dx$ (c) $\int \frac{1}{(1-x^2)^{\frac{3}{2}}} dx$

(d) $\int \frac{x^2}{x-1} dx$ (e) $\int \frac{1}{(x+1)(x+2)} dx$ (f) $\int \frac{dx}{(x-1)^2 x}$

(g) $\int \frac{1}{x^2+2x+2} dx$ (h) $\int \frac{1}{x^3-3x^2+2x} dx$

(i) The value of Simpson's rule applied to $\int_0^2 x^2 \, dx$ with $n = 8$

(j) $\int_{-1}^{1} \frac{1}{x^2} dx$

(2) Find $\iint_R x^2 \, dA$ where R is the region enclosed by $y = 1-x^2$, and $y = 0$

(3) Use double integrals to find the volume of the triangular region pyramid given by $x \geq 0$, $y \geq 0$, $z \geq 0$,
$$x + y + z \leq 2$$

(4) Use double integral to find the volume bounded by the paraboloids $z = 18 - x^2 - y^2$, and $z = x^2 + y^2$

(5) Use double integrals to find the y-coordinate of the centre of mass of the region bounded by $y = x$, $y = 0$, and $x = 1$ with density $\delta(x,y) = x^2$.

Additional questions: on vectors

(I) Find the point of intersection (if it exists) of the lines $\bar{r} = (-1,2,1) + s(2,-1,2)$, and $\bar{r} = (3,2,1) + t(2,1,-2)$

(II) Find $(\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d})$ if $\bar{a} = (1,0,-1)$, $\bar{b} = (0,1,0)$, $\bar{c} = (0,0,1)$, and $\bar{d} = (3,-4,1)$

AMAT 219
SOLUTIONS TO MIDTERM
WINTER 2000

(1) (a) $\int x \ln x \, dx$ ← by parts once

$$\begin{array}{ccc} \ln x & & x \\ & \nwarrow & \\ \frac{1}{x} & \xleftarrow{-\int} & \frac{x^2}{2} \end{array}$$

$$= \frac{x^2}{2}\ln x - \int \frac{1}{x}\cdot\frac{x^2}{2}\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx = \frac{x^2}{2}\ln x - \frac{1}{4}x^2 + C$$

(b) $\int \sin^{-1} x \, dx$ ← by parts once

$$\begin{array}{ccc} \sin^{-1} x & & 1 \\ & \nwarrow & \\ \frac{1}{\sqrt{1-x^2}} & \xleftarrow{-\int} & x \end{array}$$

$$= x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx$$ ← use $w = 1 - x^2$

$$= x\sin^{-1}x + \sqrt{1-x^2} + C$$

(c) $\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx$. let $1 - x^2 = a^2 - b^2x^2 \Rightarrow a = 1, b = 1$

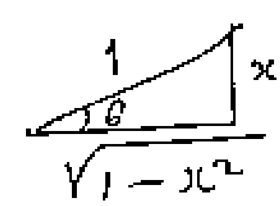
$\therefore$ Use $x = \frac{a}{b}\sin\theta \Rightarrow$ $\boxed{x = \sin\theta}$ $\boxed{dx = \cos\theta\, d\theta}$

ad $\boxed{1 - x^2 = 1 - \sin^2\theta = \cos^2\theta}$

$$I = \int \frac{1}{(\cos^2\theta)^{\frac{3}{2}}}\cos\theta\, d\theta = \int \frac{\cos\theta}{\cos^3\theta}\,d\theta = \int \sec^2\theta\, d\theta$$

$$= \tan\theta + C \;\underset{\Delta}{\overset{\text{From}}{=\!=}}\; \frac{x}{\sqrt{1-x^2}} + C.$$

$\sin\theta = \frac{x}{1}$



(d) $\int \frac{x^2}{x-1}\,dx$ ← Improper ⇒ use long division

$$\begin{array}{r|l} & x + 1 \\ \hline x - 1 & x^2 \\ & x^2 - x \\ \hline & +x \\ & x - 1 \\ \hline & +1 \end{array}$$

$$\therefore \int \frac{x^2}{x-1}\,dx = \int \left(x + 1 + \frac{1}{x-1}\right)dx$$

$$= \frac{x^2}{2} + x + \ln|x-1| + C$$

<u>Note</u>: Do same problem using $w = x - 1$ instead!

(e) $$\int \frac{1}{(x+1)(x+2)}\,dx \;\underset{\text{inspection}}{\overset{\text{By}}{=\!=}}\; \frac{1}{2-1}\int\left(\frac{1}{x+1} - \frac{1}{x+2}\right) = 1\int\left(\frac{1}{x+1} - \frac{1}{x+2}\right)dx$$

$$= \ln|x+1| - \ln|x+2| + C$$

(f) $\int \frac{dx}{(x-1)^2 x}$ ← use PFD

$$\frac{1}{(x-1)^2 x} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x}$$

$$\Rightarrow \quad 1 = A(x-1)x + Bx + C(x-1)^2$$

put $x=0$: $1 = C(0-1)^2 \Rightarrow \boxed{C=1}$

put $x=+1$: $1 = B \cdot 1 \Rightarrow \boxed{B=1}$

put $x=2$: $1 = 2A + 2B + C \Rightarrow 1 = 2A + 2(1) + 1 \Rightarrow A = -1$

$$\therefore I = \int \left\{ \frac{-1}{x-1} + \frac{1}{(x-1)^2} + \frac{1}{x} \right\} dx$$

$$= \int \left\{ -\frac{1}{x-1} + (x-1)^{-2} + \frac{1}{x} \right\} dx$$

$$= -\ln|x-1| - (x-1)^{-1} + \ln|x| + C$$

(g) $\int \frac{1}{x^2+2x+2}\,dx$ .. complete square!

$$x^2+2x+2 = (x+1)^2 + 1$$

$\therefore \int \frac{1}{(x+1)^2+1}\,dx$ -- let $t = x+1 \Rightarrow dt = dx$

$$= \int \frac{1}{t^2+1}\,dt \ \underset{\text{Table}}{=} \ \tan^{-1}\left(\frac{t}{1}\right) + C = \tan^{-1}(x+1) + C$$

(h) $\int \frac{1}{x^3-3x^2+2x}\,dx$ ← use PFD

1st. $x^3-3x^2+2x = x(x^2-3x+2) = x(x-1)(x-2)$

$$\therefore \frac{1}{x^3-3x^2+2x} = \frac{1}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}$$

verify that $A =$, $B =$ and $C =$!

$$\therefore I = \int \left(\frac{1}{2}\cdot\frac{1}{x} - \frac{1}{x-1} + \frac{1}{2}\cdot\frac{1}{x-2} \right) dx$$

$$= \frac{1}{2}\ln|x| - \ln|x-1| + \frac{1}{2}\ln|x-2| + C$$

(i) $I = \int_0^2 x^2\,dx$ .. Since $f(x) = x^2$ is a polynomial of degree less than 4, then $S_8 =$ Exact value of $I = \frac{1}{3}x^3\Big|_0^2 = \frac{8}{3}$.

(j) $\int_{-1}^{1} \frac{1}{x^2}\,dx$ .. warning .. this integral is improper since integrand $\frac{1}{x^2}$ does not exist at $x = 0 \in [-1,1]$.

Now $I = \int_{-1}^{1} \frac{1}{x^2}\,dx = 2\int_0^1 \frac{1}{x^2}\,dx$ (Since $\frac{1}{x^2}$ is Even function!)

$$I = 2\lim_{R\to 0^+}\int_R^1 x^{-2}\,dx = 2\lim_{R\to 0^+} -x^{-1}\Big|_R^1$$

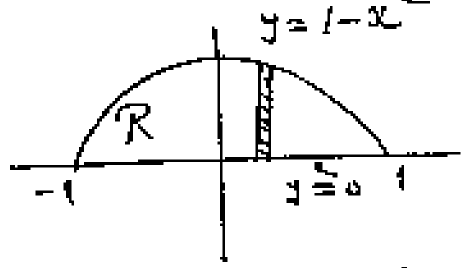
$$= -2\lim_{R\to 0^+}\left[\frac{1}{x}\right]_R^1 = -2\lim_{R\to 0^+}\left[1-\frac{1}{R}\right] = -2\left[1-\frac{1}{0^+}\right]$$

$$= -2[1-\infty] = +\infty$$

$\therefore$ Improper Integral diverge to $+\infty$.

---

(2) $\displaystyle\iint_R x^2\,dA$



$$= \int_{-1}^{1}\int_0^{1-x^2} x^2\,dy\,dx$$

R: $0 \le y \le 1-x^2$
$-1 \le x \le 1$

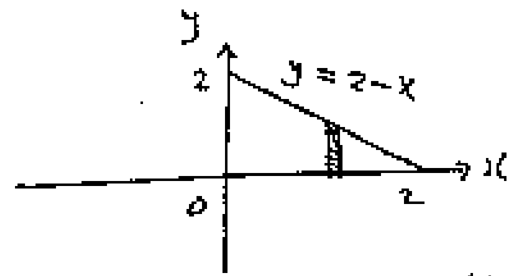
$$= \int_{-1}^{1} x^2 \cdot [y]_0^{1-x^2}\,dx$$

$$= \int_{-1}^{1} x^2(1-x^2)\,dx = \int_{-1}^{1}(x^2-x^4)\,dx = \frac{4}{15}$$

---

(3) Here $x+y+z=2 \Rightarrow z = 2-x-y$ .. Base R shown

volume $V = \displaystyle\iint_R z\,dA$



$$= \int_0^2\int_0^{2-x}(2-x-y)\,dy\,dx$$

$$= \int_0^2 \frac{(2-x-y)^2}{(-1)(2)}\Big|_0^{2-x}\,dx$$

R: region enclosed by
$x \ge 0,\ y \ge 0,\ 2-x-y=0$

$$= -\frac{1}{2}\int_0^2\left[0-(2-x)^2\right]dx$$

$0 \le y \le 2-x$
$0 \le x \le 2$

$$= \frac{1}{2}\,\frac{(2-x)^3}{(-1)\cdot 3}\Big|_0^2 = -\frac{1}{6}\left[0-2^3\right] = 4/3$$

---

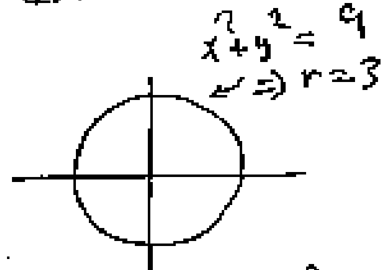
(4) Here $z_{upper} = 18 - x^2 - y^2$, $z_{lower} = x^2+y^2$

Base: The region enclosed by the curve obtained by eliminating "z" between z_u, and z_L : we get

$$x^2+y^2 = 18 - x^2 - y^2$$

or $2x^2+2y^2 = 18 \Rightarrow x^2+y^2 = 9$ (circular region)

$$V = \iint_R (z_U - z_L)\,dA = \iint_R\left[(18-x^2-y^2)-(x^2+y^2)\right]dA$$



Use polar coordinates!
$x^2+y^2 = r^2$, $dA = r\,dr\,d\theta$

R: $0 \le r \le 3$
$0 \le \theta \le 2\pi$

$$V = \int_0^{2\pi}\int_0^3 (18 - r^2 - r^2)\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^3 (18-2r^2) r\, dr$$

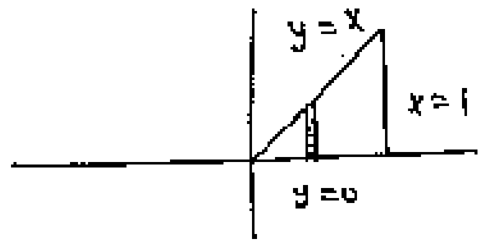
$$= 2\pi \cdot \int_0^3 (18r - 2r^3)\, dr = 2\pi \left[9r^2 - \tfrac{1}{2} r^4 \right]_0^3$$

$$= 2\pi \left[81 - \tfrac{81}{2} \right] = 2\pi (81) \left(1 - \tfrac{1}{2}\right)$$

$$= 81\pi$$

---

(5) $dm = \delta\, dA = x^2\, dA$



$\therefore\ m = \iint_R dm = \iint_R x^2\, dA$

$$= \int_0^1 \int_0^x x^2\, dy\, dx = \int_0^1 x^2 \cdot y \Big|_0^x dx$$

$R:\ 0 \le y \le x,\ 0 \le x \le 1$

$$= \int_0^1 x^2 (x - 0)\, dx = \tfrac{1}{4} x^4 \Big|_0^1 = \tfrac{1}{4}$$

$$M_{y=0} = \iint_R y\, dm = \iint_R y x^2\, dA = \int_0^1 x^2 \left(\int_0^x y\, dy\right) dx$$

$$= \int_0^1 x^2 \cdot \tfrac{1}{2}\, y^2 \Big|_0^x dx = \tfrac{1}{2}\int_0^1 x^2 (x^2 - 0^2)\, dx = \tfrac{1}{10}$$

$$\therefore\ \bar{y} = \frac{M_{y=0}}{m} = \frac{\frac{1}{10}}{\frac{1}{4}} = \frac{2}{5}$$

<u>Additional problems</u>

(II) very easy! Ans. $\bar{a} \times \bar{b} = (1, 0, 1)$, $\bar{c} \times \bar{d} = (4, 3, 0)$

$\therefore\ (\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d}) = (-3, 4, 3)$.

(I) we need to solve equations of lines

$\bar{r} = (-1, 2, 1) + s(2, -1, 2)$ --- (1)

$\bar{r} = (3, 2, 1) + t(2, 1, -2)$ --- (2)

equate R.H.S of (1), (2):

$$(-1, 2, 1) + s(2, -1, 2) = (3, 2, 1) + t(2, 1, -2)$$

<u>Comparing Components:</u>

$-1 + 2s = 3 + 2t$ or $2s - 2t = 4$

$2 - s = 2 + t$ or $-s - t = 0$

$1 + 2s = 1 - 2t$ or $2s + 2t = 0$

Consider 1st. Two equations: $2s - 2t = 4$, $-s - t = 0$

solve to get $t = -1$, $s = 1$

<u>Note</u>: Do $t = -1$, $s = 1$ satisfy 3rd. Eq. $2s + 2t = 0$?

Yes! indeed: $2(1) + 2(-1) = 0$ ✓

$\therefore$ lines intersect and pt. of intersection may be found by putting $s = 1$ in Eq. #(1) to get $\bar{r} = (-1, 2, 1) + 1(2, -1, 2) = (1, 1, 3)$.

FINAL EXAM
WINTER 1999

(I) Calculate the following

(a) $\int x \sin(2x)\,dx$

(b) $\int \ln(x+1)\,dx$

(c) $\int_0^2 \frac{x^2}{x+2}\,dx$

(d) $\int \frac{x}{x^2+6x+10}\,dx$

(e) $\int \frac{1}{x^2+x}\,dx$

(f) $\int \frac{x+11}{x^3-5x^2+6x}\,dx$

(g) $\int \sqrt{1-x^2}\,dx$

(h) $\iint_R \sqrt{x^2+y^2}\,dA$, R is the region in 1st. quadrant bounded by x-axis, y-axis, and $x^2+y^2=1$

(i) Find two orthogonal vectors from $\bar{a}=(2,1,3,1)$, $\bar{b}=(-3,-2,3,0)$, and $\bar{c}=(-3,5,0,1)$

(j) The equation of the plane containing the line $\bar{r}=(2t,t+1,3-t)$ and the point $P(0,3,5)$

(k) The distance from the point $(1,2,3)$ to the plane $x-y+2z=3$

(l) $(\bar{a}\times\bar{b})\cdot(2\bar{a}+3\bar{b})$ when $\bar{a}=(2,1,-1)$, $\bar{b}=(100,99,98)$

(m) $\frac{\partial z}{\partial x}$ where $z=\cos(x^2+y^3)$

(n) $\frac{\partial z}{\partial x}$ where $z=u(x^2-y^2,x^2+y^2)$, and $u(x,y)$ is such that $\frac{\partial u}{\partial x}=y$, $\frac{\partial u}{\partial y}=x$

(o) $\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}$ where $z(x,y)=f(x-y)$ for some single variable function f

(p) The line tangent to the curve $x=t^2+1$, $y=t^4+2t^2+1$, $t\geq 0$ at the point $P(2,4)$

(q) The plane tangent to the hyperboloid $x^2-3y^2+2z^2=7$ at $(1,2,3)$

(r) The length of the curve $\bar{r}=(1-t^3,1-t^2)$, $0\leq t\leq 2$

(s) The parametric equations of the line of intersection of the planes $x-3y+2z=1$, $2x+y-z=3$

(t) Find S_{10} for $\int_0^2 x^3 dx$.

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(2) Given $x^3y + xz^2 + \cos(xyz) = 0$, Find $\frac{\partial x}{\partial z}$.

(3) A particle follows the curve $\bar{r} = (t^2, t^{\frac{3}{2}} - t, 2t)$ starting at time $t = 0$ until $t = 9$. It then leaves the curve tangentially and continues uniformally at the same exit speed along tangent line. What is its position at $t = 14$?

(4) Find the centre of mass of the planar region $x^2 + y^2 \leq 4$, $y \geq 0$ where the density is given by $\delta = \sqrt{x^2 + y^2}$.

(5) Use double integrals to find volume of that part of the sphere $x^2 + y^2 + z^2 \leq 4$ inside cylinder $x^2 + y^2 \leq 1$

(6) Use double integrals to find volume under $z = 1 - x^2 - y^2$ and above the region $x \geq 0$, $y \geq 0$, $x + y \leq 1$

(7) show that y-coordinate of the centroid of the planar region enclosed by $y = x^2$, $y = 2 - x$ is $\bar{y} = \frac{8}{5}$.

(8) Compute $\iiint_R z \, dV$ where R is the region enclosed by $4x + 2y + z = 4$, $x = 0$, $y = 0$, $z = 0$

(9) Use spherical coordinates to find $\iiint_R z \, dV$ where R is the hemi-sphere $x^2 + y^2 + z^2 \leq a^2$, $z \geq 0$

Form #(4): Refer to solution in W 2000.

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(I) (a) $\int x \sin(2x)\,dx$ ← by parts once

$x \quad \sin 2x$

$1 \xleftarrow{-\int} -\frac{\cos 2x}{2}$

$= -\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x\,dx$

$= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C$

*** (b) $\int \ln(x+1)\,dx$ ← by parts once

$\ln(x+1) \quad 1$

$\frac{1}{x+1} \xleftarrow{-\int} x+1$

$= (x+1)\ln(x+1) - \int 1\,dx$

$= (x+1)\ln(x+1) - x + C$

Note: $\int 1\,dx$ = either x or x + any constant say $x+1$ in this case

(c) $\int_0^2 \frac{x^2}{x+2}\,dx$ ← use long division 1st.

$$\begin{array}{r|l} & x-2 \\ \hline x+2 & x^2 \\ & x^2+2x \\ \hline & -2x \\ & -2x-4 \\ \hline & +4 \end{array}$$

$= \int_0^2 \left(x-2+\frac{4}{x+2}\right)dx$

$= \frac{x^2}{2} - 2x + 4\ln|x+2| \Big|_0^2 = (2-4+4\ln 4) - (0-0+4\ln 2)$

$= -2+4\ln 4 - 4\ln 2$

or $= -2+4(\ln 4 - \ln 2)$

$= -2+4\ln\left(\frac{4}{2}\right) = -2+4\ln 2$

*(d) $\int \frac{x}{x^2+6x+10}\,dx$ ← Note $x^2+6x+10$ is not factorable

$\therefore$ Complete the squares 1st.

$x^2+6x+10 = (x+3)^2+1$

$\therefore I = \int \frac{x}{x^2+6x+10}\,dx = \int \frac{x}{(x+3)^2+1}\,dx$

let $x+3=t \Rightarrow dx = dt$ $\therefore$ also $x = t-3$

$\therefore I = \int \frac{t-3}{t^2+1}\,dt = \int \frac{t}{t^2+1}\,dt - 3\int \frac{1}{t^2+1}\,dt$

$= \frac{1}{2}\ln|t^2+1| - 3\tan^{-1} t$

$= \frac{1}{2}\ln|(x+3)^2+1| - 3\tan^{-1}(x+3) + C$

(e) $\int \frac{1}{x^2+x}\,dx \leftarrow$ PFD

By inspection $I = \int \left(\frac{1}{x} - \frac{1}{x+1}\right)dx = \ln|x| - \ln|x+1| + C$

(f) $\int \frac{x+11}{x^3-5x^2+6x}\,dx \leftarrow$ PFD

$$\frac{x+11}{x^3-5x^2+6x} = \frac{x+11}{x(x^2-5x+6)} = \frac{x+11}{x(x-2)(x-3)}$$

$$\therefore \frac{x+11}{x(x-2)(x-3)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-3}$$

$$\Rightarrow x+11 = A(x-2)(x-3) + Bx(x-3) + C(x)(x-2)$$

put $x=0$: $11 = 6A$, $A = \frac{11}{6}$

put $x=2$: $13 = -2B$, $B = -\frac{13}{2}$

put $x=3$: $14 = 3C$, $C = \frac{14}{3}$

$$\therefore I = \int\left(\frac{11}{6}\cdot\frac{1}{x} - \frac{13}{2}\cdot\frac{1}{x-2} + \frac{14}{3}\cdot\frac{1}{x-3}\right)dx$$

$$= \frac{11}{6}\ln|x| - \frac{13}{2}\ln|x-2| + \frac{14}{3}\ln|x-3| + C.$$

(g) $\int \sqrt{1-x^2}\,dx$ $\qquad 1-x^2 = a^2 - b^2x^2 \Rightarrow a = b = 1$

$\therefore$ use $x = \frac{a}{b}\sin\theta \Rightarrow \boxed{x = \sin\theta}$, $\boxed{dx = \cos\theta\,d\theta}$

and $\boxed{1-x^2 = 1-\sin^2\theta = \cos^2\theta}$

$$I = \int \sqrt{\cos^2\theta}\cdot\cos\theta\,d\theta = \int \cos^2\theta\,d\theta$$ $\leftarrow$ use double angle identity

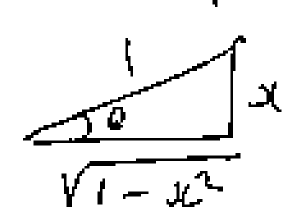
$$= \frac{1}{2}\int(1+\cos 2\theta)\,d\theta = \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$$

$$= \frac{1}{2}\left[\theta + \frac{1}{\cancel{2}}\cdot\cancel{2}\sin\theta\cos\theta\right] + C$$

$$= \frac{1}{2}\left[\theta + \sin\theta\cos\theta\right] + C$$

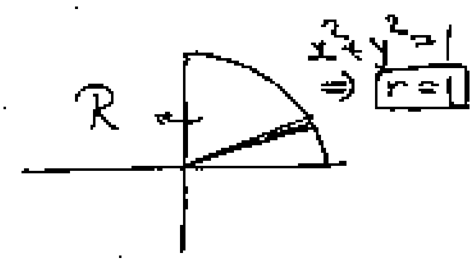
$\underset{A}{\underline{\text{From}}}$ $\frac{1}{2}\left[\sin^{-1}x + \frac{x}{1}\cdot\frac{\sqrt{1-x^2}}{1}\right] + C$

$\sin\theta = \frac{x}{1}$



Note: $\sin\theta = x$
$\Rightarrow \theta = \sin^{-1}x$

(h) $\iint_R \sqrt{x^2+y^2}\,dA$



use polar coordinates: $x^2+y^2=r^2$, $dA = r\,dr\,d\theta$

$$I = \int_0^{\frac{\pi}{2}}\int_0^1 \sqrt{r^2}\cdot r\,dr\,d\theta = \frac{\pi}{6}$$

R: $0 \leq r \leq 1$
$0 \leq \theta \leq \frac{\pi}{2}$

(i) $\bar{a} = (2,1,3,1)$, $\bar{b} = (-3,-2,3,0)$, $\bar{c} = (-3,5,0,1)$

Recall: Two vectors are orthogonal if their dot product is zero.

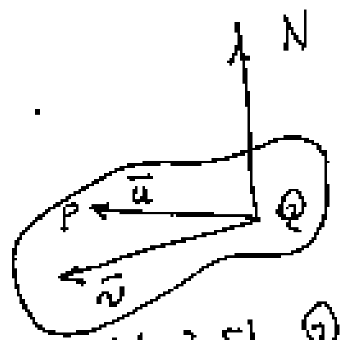
Indeed, $\bar{a}\cdot\bar{c} = (2,1,3,1)\cdot(-3,5,0,1) = -6+5+0+1 = 0$

$\therefore\ \bar{a} \perp \bar{c}$

(j) From eq. of line

$\bar{r} = (2t, t+1, 3-t)$
$= (0,1,3) + t(2,1,-1)$

$\therefore$ It passes through $Q(0,1,3)$ and has direction $\bar{v} = (2,1,-1)$.



$P(0,3,5)$, $Q(0,1,3)$

let $\bar{u} = \overline{PQ} = (0,1,3)-(0,3,5) = (0,-2,-2)$

$\therefore$ A vector normal to plane is $\bar{N} = \bar{u}\times\bar{v}$
$= (4,-4,4)$

$$\begin{array}{c} \bar{u} \\ \bar{v} \end{array}\begin{pmatrix} 0 & -2 & -2 \\ 2 & 1 & -1 \end{pmatrix}$$

$\therefore$ Eq. of plane is $\bar{r}\cdot\bar{N} = \bar{r}_0\cdot\bar{N}$, $\bar{r}_0 = P = (0,3,5)$

$(x,y,z)\cdot(4,-4,4) = (0,3,5)\cdot(4,-4,4)$

$\Rightarrow 4x - 4y + 4z = 0 - 12 + 20$

$\Rightarrow x - y + z = 2$

(K) Distance $D = \dfrac{|1-2+2(3)-3|}{\sqrt{(1)^2+(-1)^2+(2)^2}}$

$= \dfrac{2}{\sqrt{6}}$

Note: write plane as $x - y + 2z - 3 = \underline{\underline{0}}$

(l) 1st. $\bar{a}\times\bar{b} = (197, -296, 98)$,

$$\begin{array}{c} \bar{a} \\ \bar{b} \end{array}\begin{pmatrix} 2 & 1 & -1 \\ 100 & 99 & 98 \end{pmatrix}$$

$2\bar{a}+3\bar{b} = 2(2,1,-1)+3(100,99,98)$
$= (304, 299, 292)$

$\therefore\ (\bar{a}\times\bar{b})\cdot(2\bar{a}+3\bar{b}) = (197,-296,98)\cdot(304,299,292)$
$= 0$

(m) $z = \cos(x^2+y^2)$, $\dfrac{\partial z}{\partial x} = -\sin(x^2+y^2)\cdot 2x = -2x\sin(x^2+y^2)$

(n) $z = u(x^2-y^2, x^2+y^2)$. let $x^2-y^2 = r$, $x^2+y^2 = s$

$\therefore z = u(r,s)$

By chain rule $\dfrac{\partial z}{\partial x} = \dfrac{\partial u}{\partial r}\cdot\dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial s}\cdot\dfrac{\partial s}{\partial x}$ (*)

But $r = x^2-y^2 \Rightarrow \boxed{\dfrac{\partial r}{\partial x} = 2x}$, $s = x^2+y^2 \Rightarrow \boxed{\dfrac{\partial s}{\partial x} = 2x}$

and $\dfrac{\partial u}{\partial x} = y$, $\dfrac{\partial u}{\partial y} = x \Rightarrow \dfrac{\partial u}{\partial r} = s$, $\dfrac{\partial u}{\partial s} = r$ (we have just replaced x by r and y by s.

It follows that

$$\frac{\partial u}{\partial r} = s = x^2+y^2, \quad \frac{\partial u}{\partial s} = r = x^2-y^2$$

hence (*) $\Rightarrow$ $\frac{\partial z}{\partial x} = (x^2+y^2)\cdot 2x + (x^2-y^2)\cdot 2x$

$$= 4x^3$$

(o) let $x-y=t \Rightarrow \frac{\partial t}{\partial x} = 1, \ \frac{\partial t}{\partial y} = -1$ and that

$\therefore \ z = f(t)$

$$\frac{\partial z}{\partial x} = f'(t)\frac{\partial t}{\partial x} = f'(t)\cdot 1, \quad \frac{\partial^2 z}{\partial x^2} = f''(t)\frac{\partial t}{\partial x} = f''(t)\cdot 1$$

Similarly, $\frac{\partial z}{\partial y} = f'(t)\frac{\partial t}{\partial y} = -f'(t)$,

$$\frac{\partial^2 z}{\partial y^2} = -f''(t)\frac{\partial t}{\partial y} = -f''(t)\cdot(-1) = f''(t)$$

$$\therefore \ \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = f''(t) - f''(t) = 0$$

(p) Need: (1) A point: Given as $P(2,4)$

(2) The slope: This is $\frac{dy}{dx} = \left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=t_0}$

here $x = t^2+1, \quad y = t^4+2t^2+1$

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 4t^3+4t$$

Now, at $x=2$, $t^2+1=2 \Rightarrow t^2=1 \Rightarrow t = \text{only } +1$ since $t \geqslant 0$

$$\therefore \text{slope } m = \left.\frac{4t^3+4t}{2t}\right|_{t=1} = 4$$

Eq. is $y-4 = 4(x-2)$ or $y = 4x-4$

(q) Here $\bar{r_0} = (1,2,3)$

let $F(x,y,z) = x^2-3y^2+2z^2-7 = 0$

A vector normal to plane is given by

$$\bar{N} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)_{P(1,2,3)} = (2x,-6y,4z)\Big|_{(1,2,3)}$$

$$= (2,-12,12)$$

Eq. is $\bar{r}\cdot\bar{N} = \bar{r_0}\cdot\bar{N}$

$$(x,y,z)\cdot(2,-12,12) = (1,2,3)\cdot(2,-12,12)$$

$$2x-12y+12z = 2-24+36 \Rightarrow x-6y+6z = 7$$

(r) Arc length

$$L = \int_a^b \sqrt{x'^2 + y'^2}\, dt$$

here $x = 1 - t^3$, $y = 1 - t^2$, $0 \leq t \leq 2$

$$\therefore L = \int_0^2 \sqrt{(-3t^2)^2 + (-2t)^2}\, dt = \int_0^2 \sqrt{9t^4 + 4t^2}\, dt$$

$$= \int_0^2 t\sqrt{9t^2 + 4}\, dt$$

let $w = 9t^2 + 4 \Rightarrow dw = 18t\, dt \Rightarrow t\, dt = \frac{1}{18} dw$

$$L = \frac{1}{18}\int \sqrt{w}\, dw = \frac{1}{18} \cdot \frac{2}{3} w^{\frac{3}{2}}$$

$$= \frac{1}{27} (9t^2+4)^{\frac{3}{2}} \Big|_0^2 = \frac{1}{27}\left[(40)^{\frac{3}{2}} - 4^{\frac{3}{2}}\right]$$

Note: $4^{\frac{3}{2}} = (2^2)^{\frac{3}{2}} = 2^3 = 8$, $(40)^{\frac{3}{2}} = 40\sqrt{40} = 80\sqrt{10}$

$$\therefore L = \frac{1}{27}\left[80\sqrt{10} - 8\right] = \frac{8}{27}\left[10\sqrt{10} - 1\right]$$

(s) solve system $x - 3y + 2z = 1$, $2x + y - z = 3$

Augmented matrix $\left[\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 2 & 1 & -1 & 3 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1}$

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 0 & 7 & -5 & 1 \end{array}\right] \xrightarrow{R_2 \to \frac{R_2}{7}} \left[\begin{array}{cccc} 1 & -3 & 2 & 1 \\ 0 & 1 & -\frac{5}{7} & \frac{1}{7} \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + 3R_2} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{7} & \frac{10}{7} \\ 0 & 1 & -\frac{5}{7} & \frac{1}{7} \end{array}\right]$$

$\therefore z = t$, $t \in \mathbb{R}$ and $x - \frac{1}{7}z = \frac{10}{7} \Rightarrow x = \frac{10}{7} + \frac{1}{7}t$

$y - \frac{5}{7}z = \frac{1}{7} \Rightarrow y = \frac{1}{7} + \frac{5}{7}t$

$\therefore$ parametric eq.s are $x = \frac{10}{7} + \frac{1}{7}t$, $y = \frac{1}{7} + \frac{5}{7}t$, $z = t$, $t \in \mathbb{R}$

(t) Since $f(x) = x^3$ is a polynomial of degree less than four,

S_{10} = Exact value of I

$$= \int_0^2 x^3\, dx = \frac{1}{4} x^4 \Big|_0^2 = \frac{1}{4} \cdot 2^4 = \frac{1}{4} \cdot 16 = 4$$

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(2) Let $F(x,y,z) = x^3y + xz^2 + \cos(xyz) = 0$

$$\therefore \frac{\partial F}{\partial x} = 3x^2y + z^2 - yz\sin(xyz),$$

$$\frac{\partial F}{\partial z} = 0 + 2xz - xy\sin(xyz)$$

$$\therefore \frac{\partial x}{\partial z} = -\frac{F_z}{F_x} = -\frac{2xz - xy\sin(xyz)}{3x^2y + z^2 - yz\sin(xyz)}$$

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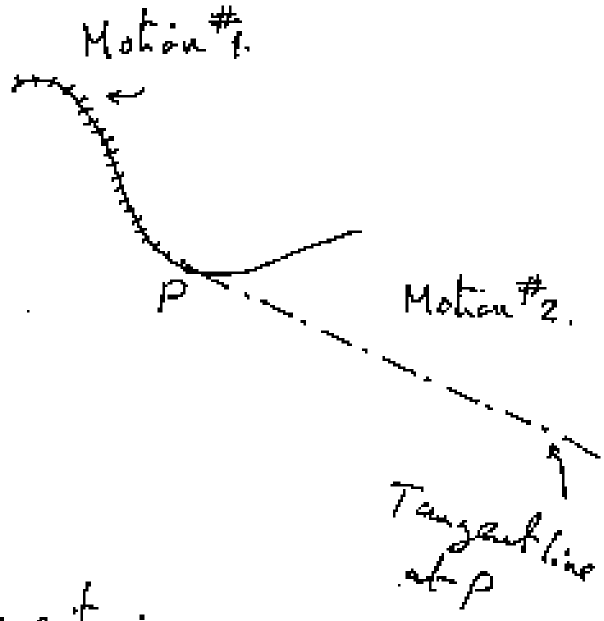
(3) We just need the position $\bar{r}$ and the velocity $\bar{v}$ at the end of 1st. motion $(t=9)$



$$\bar{r} = (t^2, t^{\frac{3}{2}} - t, 2t),$$

$$\bar{v} = \frac{d\bar{r}}{dt} = (2t, \tfrac{3}{2}t^{\frac{1}{2}} - 1, 2)$$

at $t=9$, $\bar{r}(9) = (9^2, 9^{\frac{3}{2}} - 9, 2(9))$

$= (81, 18, 18) =$ position

and velocity $\bar{v}(9) = (2(9), \tfrac{3}{2}(9)^{\frac{1}{2}} - 1, 2)$

$= (18, \tfrac{7}{2}, 2)$

For 2nd. motion : Need eq. of Tangent line :

$$\bar{r} = \bar{r}(9) + t\,\bar{v}(9)$$

$$= (81, 18, 18) + t(18, \tfrac{7}{2}, 2)$$

at $t = 14$ from the start, $t = 14 - 9 = 5$ from 2nd. motion

$$\therefore \bar{r} = (81, 18, 18) + 5(18, \tfrac{7}{2}, 2) = (171, \tfrac{71}{2}, 28)$$

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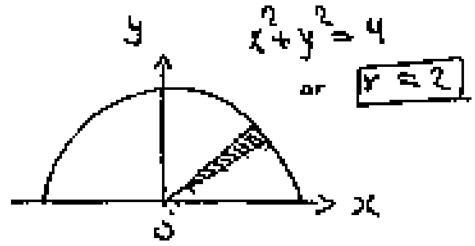
(4) $dm = \delta\, dA = \sqrt{x^2+y^2}\, dA$



using polar coordinates,

$$dm = \sqrt{r^2}.\, r\,dr\,d\theta = r^2\,dr\,d\theta$$

R: $0 \le r \le 2$
$0 \le \theta \le \pi$

$$\therefore m = \iint_R dm = \iint_R r^2\,dr\,d\theta = \int_0^{\pi}\int_0^2 r^2\,dr\,d\theta$$

$$= \pi \cdot \tfrac{1}{3} r^3 \Big|_0^2 = \frac{8\pi}{3}$$

From symmetry, it is obvious that centre of mass will lie on y-axis.. hence $\bar{x} = 0$. To find $\bar{y}$, we need to Compute $M_{y=0} = \iint_R y\,dm = \int_0^{\pi}\int_0^2 (r\sin\theta).\, r^2\,dr\,d\theta$

$$M_{y=0} = \int_0^{\pi} \sin\theta \, d\theta \, . \int_0^2 r^3 \, dr$$

$$= -\cos\theta \Big|_0^{\pi} . \ \frac{1}{4} r^4 \Big|_0^2 = (2)(4) = 8$$

$$\therefore \ \bar{y} = \frac{M_{y=0}}{m} = \frac{8}{\frac{8\pi}{3}} = \frac{3}{\pi}$$

$\therefore$ Centre of mass is $(\bar{x}, \bar{y}) = (0, \frac{3}{\pi})$.

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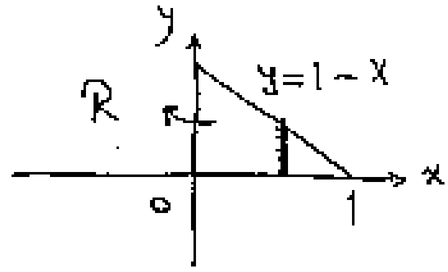
*** (6) $V = \iint_R z \, dA$



where $z = 1 - x^2 - y^2$, and R is as shown in figure.

$R: \ 0 \leq y \leq 1-x$
$\quad 0 \leq x \leq 1$

$$V = \int_0^1 \int_0^{1-x} (1 - x^2 - y^2) \, dy \, dx$$

$$= \int_0^1 y - yx^2 - \frac{1}{3} y^3 \Big|_0^{1-x} dx = \int_0^1 \left[(1-x) - x^2(1-x) - \frac{1}{3}(1-x)^3\right] dx$$

$$= x - \frac{x^2}{2} - \frac{1}{3}x^3 + \frac{1}{4}x^4 - \frac{1}{3} \cdot \frac{(1-x)^4}{(-1)(4)} \Big|_0^1$$

$$= 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{12} = \frac{1}{3}$$

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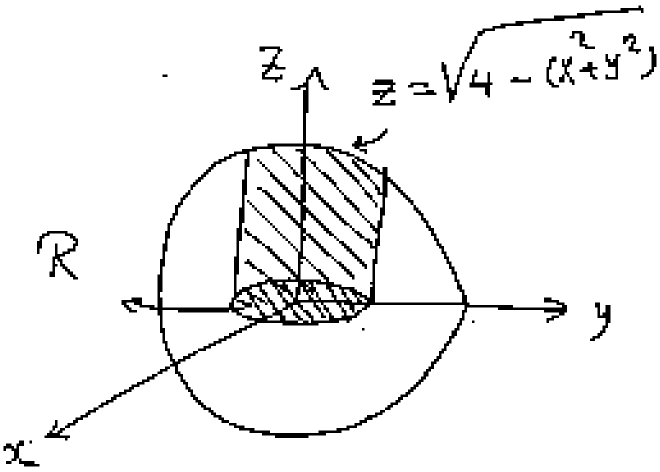
(5)



Volume $V = 2 \iint_R z \, dA$

here $x^2 + y^2 + z^2 = 4$

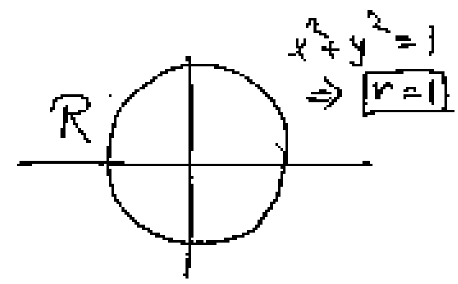
$$z = +\sqrt{4 - (x^2 + y^2)}$$

and R is the region bounded by

$$x^2 + y^2 = 1$$

volume = 2 × volume of upper part (shaded).

Use polar Coordinates: $z = \sqrt{4 - r^2}$, $dA = r\,dr\,d\theta$



$$\therefore V = 2 \int_0^{2\pi} \int_0^1 \sqrt{4 - r^2} \, . \, r \, dr \, d\theta$$

$0 \leq r \leq 1$
$0 \leq \theta \leq 2\pi$

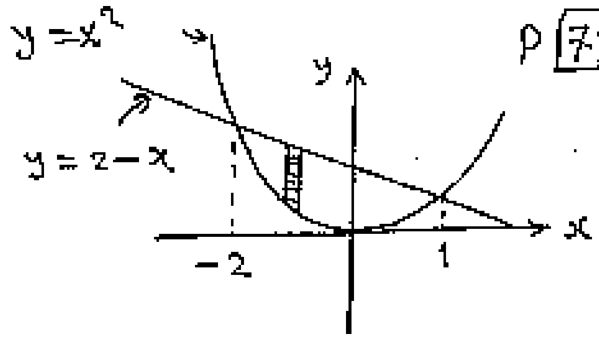
$$= 2 \int_0^{2\pi} d\theta \, . \int_0^1 r\sqrt{4 - r^2} \, dr$$

let $w = 4 - r^2$!

$$= 2(2\pi) \, . \, -\frac{1}{3}(4 - r^2)^{\frac{3}{2}} \Big|_0^1 = \frac{4\pi}{3}\left[8 - 3\sqrt{3}\right]$$

— — — — — —

(7) 1st. : Find where graphs intersect?



$$y = x^2, \quad y = 2 - x$$
$$\therefore x^2 = 2 - x$$
$$x^2 + x - 2 = 0$$
$$(x+2)(x-1) = 0$$
$$x = -2, 1$$

$$R: \quad x^2 \leq y \leq 2 - x, \quad -2 \leq x \leq 1$$

For the centroid $\delta = 1 \Rightarrow dm = \delta\, dA = dA$

$$\therefore m = \iint_R dm = \iint_R dA = \int_{-2}^{1} \int_{x^2}^{2-x} dy\, dx = \int_{-2}^{1} y\Big|_{x^2}^{2-x} dx$$
$$= \int_{-2}^{1} (2 - x - x^2)\, dx = 2x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 \Big|_{-2}^{1} = 9/2$$

Next,

$$M_{y=0} = \iint_A y\, dm = \iint_R y\, dA$$
$$= \int_{-2}^{1} \int_{x^2}^{2-x} y\, dy\, dx = \frac{1}{2}\int_{-2}^{1} y^2\Big|_{x^2}^{2-x} dx$$
$$= \frac{1}{2}\int_{-2}^{1} \left[(2-x)^2 - x^4\right] dx$$
$$= \frac{1}{2}\left[\frac{(2-x)^3}{(-1)(3)} - \frac{x^5}{5}\right]_{-2}^{1} =$$
$$= -\frac{1}{2}\left[\frac{(2-x)^3}{3} + \frac{x^5}{5}\right]_{-2}^{1} = \frac{72}{10} = \frac{36}{5}$$

$$\therefore \bar{y} = \frac{M_{y=0}}{m} = \frac{36/5}{9/2} = \frac{8}{5}$$

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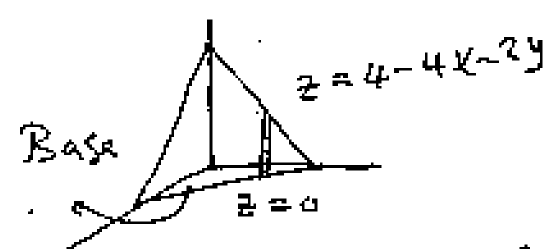
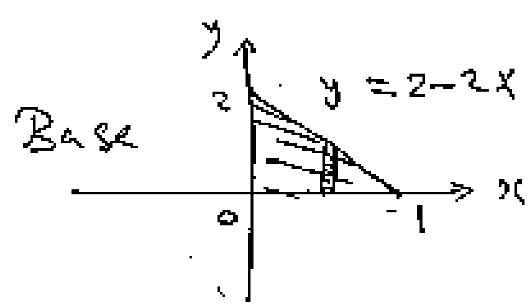
(8) To find Base, put $z = 0$ to get,



$$4x + 2y = 4$$
$$\Rightarrow y = 2 - 2x$$

$$0 \leq z \leq 4 - 4x - 2y$$



$$0 \leq y \leq 2 - 2x$$
$$0 \leq x \leq 2$$

$$I = \iiint_R z\, dV = \int_0^1 \int_0^{2-2x} \int_0^{4-4x-2y} z\, dz\, dy\, dx$$
$$= \frac{1}{2}\int_0^1 \int_0^{2-2x} z^2\Big|_0^{4-4x-2y} dy\, dx$$
$$= \frac{1}{2}\int_0^1 \int_0^{2-2x} (4 - 4x - 2y)^2\, dy\, dx$$
$$= \frac{1}{2}\int_0^1 \frac{(4-4x-2y)^3}{(-2)(3)}\Big|_0^{2-2x} dx = \frac{1}{12}\int_0^1 (4-4x)^3\, dx$$
$$= \frac{1}{12}\, \frac{(4-4x)^4}{(-4)(4)}\Big|_0^1 = \frac{1}{12}\cdot\frac{4^4}{4^2} = \frac{16}{12} = \frac{4}{3}$$

FINAL EXAM

WINTER 2000

(1) Calculate each of the following

(a) $\int x e^{x+1} dx$ (b) $\int \tan^{-1} x \, dx$ (c) $\int \frac{x-1}{x+1} dx$

(d) $\int \frac{1}{x^2+2x+3} dx$ (e) $\int \frac{1}{x^2-1} dx$ (f) $\int \frac{1}{x^2(x+1)} dx$

(g) $\int \frac{1}{(4+x^2)^{\frac{3}{2}}} dx$ (h) $\iint\limits_R dA$, when R is the triangle with vertices $(0,0)$, $(0,1)$, and $(1,1)$

(i) The equation of the plane containing the three points $(1,1,2)$, $(2,3,3)$, and $(1,-1,0)$

(j) The equation describing the line of intersection of the plane $x+y+z=1$, and $2x+3y+z=2$

(k) The distance from the point $(1,2,3)$ to the plane $2x-y+3z=4$.

(l) $(\bar{a}\times\bar{b})\times(\bar{a}\times\bar{c})$ where $\bar{a}=(1,0,0)$, $\bar{b}=(0,2,4)$, and $\bar{c}=(0,0,1)$

(m) $\frac{\partial}{\partial y}\left(y e^{x^2+y^2}\right)$

(n) $\frac{\partial z}{\partial x}$ where $z=u(2x-y,\, 3x+y)$ and $u(r,s)=r^{10}s^{10}$

(o) $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}$ where $z(x,y)=f(x+2y)$, where $f(t)$ is such that $f''(t)=e^{t^2}$

(p) The length of the curve $\bar{r}(t)=t^3\bar{i}+(t^2+1)\bar{j}$, $0\le t\le 1$

(q) The plane tangent to the ellipsoid $2x^2+3y^2+4z^2=9$ at the point $(1,-1,1)$.

(r) For $z(x,y)=x^3-3xy^2$, find the points (x,y) in the plane where $\frac{\partial z}{\partial x}=0$, and $\frac{\partial z}{\partial y}=0$

(s) $\iint\limits_R y\,dA$ where R is the region $x^2+y^2\le 1$, $y\ge 0$

(2) Given that $x^2y+xz+\sin(xyz)=0$, determine $\frac{\partial y}{\partial z}$.

(3) The position vector of a particle at time t is given by $\bar{r}=t^2\bar{i}+(t^{\frac{3}{2}}-t)\bar{j}+2t\bar{k}$. Find the <u>speed</u> of the particle at time $t=4$

(4) Find the centre of mass of the planar region bounded by $y = x$, $y = -x$, and $y = 1$ with density $\delta(x,y) = y$.

(5) Use double integrals to find the volume of the region in the first octant ($x, y, z \geq 0$) bounded by $y = 1 - x^2$, and $z = 2 - x$

(6) Use double integrals to find the volume enclosed by $z = 2 - \sqrt{x^2 + y^2}$, and $z = 0$

(7) Find the centroid of the planar region $x^2 + y^2 \leq a^2$, $x \geq 0$, $y \geq 0$

(8) Compute $\iiint_R z \, dV$ where R is the region in first octant bounded by $2x + y + z = 2$

(9) Use spherical coordinates to find the centroid of the hemi-sphere $x^2 + y^2 + z^2 \leq R^2$, $z \geq 0$

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AMAT 219
SOLUTIONS TO FINAL EXAM
WINTER 2000

(1) (a) $\int x e^{x+1} dx$ ← by parts once

$u = x$, $dV = e^{x+1} dx$, $du = 1 dx$, $V = e^{x+1}$

$= x e^{x+1} - \int e^{x+1} dx = x e^{x+1} - e^{x+1} + C$

(b) $\int \tan^{-1} x \, dx$ ← by parts once

$u = \tan^{-1} x$, $dV = 1 dx$, $du = \frac{dx}{1+x^2}$, $V = x$

$= x \tan^{-1} x - \int \frac{x}{1+x^2} dx$ ← let $w = 1+x^2$

$= x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C$

(c) $\int \frac{x-1}{x+1} dx$ ← use either PFD or better let $w = x+1$ $\therefore dw = dx$

Note: $\boxed{x = w-1}$

$= \int \frac{w-1}{w} dw = \int (1 - \frac{1}{w}) dw$

$= w - \ln|w| + C = x+1 - \ln|x+1| + C$

(d) $\int \frac{1}{x^2+2x+3} dx$. Since x^2+2x+3 is not factorable, just complete the squares to get $x^2+2x+3 = (x+1)^2 + 2$

$\therefore \int \frac{1}{(x+1)^2+2} dx$ -- let $t = x+1 \Rightarrow dt = dx$

$= \int \frac{1}{t^2+(\sqrt{2})^2} dt \underset{\text{Table}}{=} \frac{1}{\sqrt{2}} \tan^{-1}(\frac{t}{\sqrt{2}}) + C = \frac{1}{\sqrt{2}} \tan^{-1}(\frac{x+1}{\sqrt{2}}) + C$

(e) $\int \frac{1}{x^2-1} dx$ ← use PFD (By inspection shown in class!)

$= \frac{1}{2} \int (\frac{1}{x-1} - \frac{1}{x+1}) dx = \frac{1}{2} [\ln|x-1| - \ln|x+1|] + C$

(f) $\int \frac{1}{x^2(x+1)} dx$ ← use PFD

$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} \Rightarrow \boxed{1 = Ax(x+1) + B(x+1) + Cx^2}$

put $x = 0$: $1 = B$, next: put $x = -1$, $1 = C$

Finally if we put $x = 1$, $1 = 2A + 2B + C \Rightarrow 1 = 2A + 2(1) + 1$

$\Rightarrow A = -1$

$\therefore \frac{1}{x^2(x+1)} = -\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}$

$\therefore I = \int (-\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}) dx = \int (-\frac{1}{x} + x^{-2} + \frac{1}{x+1}) dx$

$= -\ln|x| - x^{-1} + \ln|x+1| + C$

(g) $\int \frac{1}{(4+x^2)^{\frac{3}{2}}}\,dx$ .. Integrand has the quantity

$$a^2 + b^2x^2 = 4 + x^2$$

with $a = 2$, $b = 1$ .. hence use the substitution

$$x = \frac{a}{b}\tan\theta = 2\tan\theta$$

$$\therefore\ 4 + x^2 = 4 + 4\tan^2\theta = 4(1+\tan^2\theta) = 4\sec^2\theta$$

and $dx = 2\sec^2\theta\, d\theta$

$$\therefore\ I = \int \frac{1}{(4\sec^2\theta)^{\frac{3}{2}}} \cdot 2\sec^2\theta\, d\theta = \frac{2}{8}\int \frac{\sec^2\theta}{\sec^3\theta}\, d\theta$$

$$= \frac{1}{4}\int \frac{1}{\sec\theta}\, d\theta = \frac{1}{4}\int \cos\theta\, d\theta$$

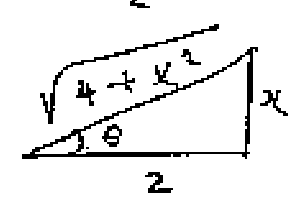
$$= \frac{1}{4}\sin\theta + C$$

$$\underset{\Delta}{\underline{\text{From}}}\quad \frac{1}{4}\cdot\frac{x}{\sqrt{4+x^2}} + C$$

Associated Δ

$x = 2\tan\theta$

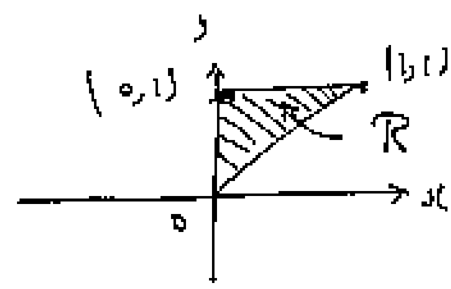
$\tan\theta = \frac{x}{2}$



(h) $\iint_R dA$ = area enclosed by region R

= area of Δ

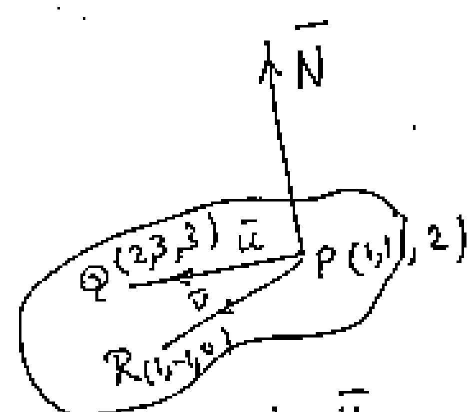
$= \frac{1}{2}(1)(1) = \frac{1}{2}$



(i) For an equation of a plane we need

(a) A point .. choose say $(1,1,2) \Rightarrow \bar{r}_0 = (1,1,2)$

(b) A normal vector : This is $\underline{\underline{\bar{N} = \bar{u}\times\bar{v}}}$



$\vec{PQ} = (1,2,1) = \bar{u}$

$\vec{PR} = (0,-2,-2) = \bar{v}$

$\bar{u}\times\bar{v} = (-2, 2, -2)$

Eq. of plane is given by

$$\bar{r}\cdot\bar{N} = \bar{r}_0\cdot\bar{N}\ ,\quad \bar{r} = (x,y,z)$$

$$(x,y,z)\cdot(-2,2,-2) = (1,1,2)\cdot(-2,2,-2)$$

$$-2x + 2y - 2z = -2 + 2 - 4$$

or $\quad x - y + z = 2$

(j) Just solve the system $x + y + z = 1$, $2x + 3y + z = 2$.

Augmented matrix $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 2 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 \end{array}\right]$

$\xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 0 \end{array}\right]$

$\therefore\ z = t,\ t\in\mathbb{R}$ and hence $\quad x + 2z = 1 \Rightarrow x = 1 - 2t$

$\qquad\qquad\qquad\qquad\qquad\qquad y - z = 0,\ y = t$

$\therefore$ parametric equations of line are given by

$$\begin{aligned} x &= 1 - 2t \\ y &= t \\ z &= t \end{aligned} \qquad t\in\mathbb{R}$$

(k) 1st. write eq. of plane where its R.H.S is zero.

$$2x - y + 3z - 4 = 0$$

$\therefore$ distance $D = \dfrac{|2(1) - (2) + 3(3) - 4|}{\sqrt{2^2 + (-1)^2 + (3)^2}} = \dfrac{5}{\sqrt{14}}$ units.

(l) $(\bar{a} \times \bar{b}) \times (\bar{a} \times \bar{c})$

$= (0, -4, 2) \times (0, -1, 0)$

$= \left(\begin{vmatrix} -4 & 2 \\ -1 & 0 \end{vmatrix}, -\begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 0 & -4 \\ 0 & -1 \end{vmatrix} \right)$

$= (2, 0, 0)$

$$\begin{array}{c|ccc|} & \oplus & \ominus & \oplus \\ \bar{a} & 1 & 0 & 0 \\ \bar{b} & 0 & 2 & 4 \end{array}$$

$\bar{a}\times\bar{b} = (0, -4, 2)$

$$\begin{array}{c|ccc|} \bar{a} & 1 & 0 & 0 \\ \bar{c} & 0 & 0 & 1 \end{array}$$

$\bar{a}\times\bar{c} = (0, -1, 0)$

(m) $\frac{\partial}{\partial y}\left(y e^{x^2+y^2}\right) \leftarrow$ "x" is held Constant!

$= $ (By product rule) $y \cdot e^{x^2+y^2}(2y) + 1 \cdot e^{x^2+y^2}$

$= e^{x^2+y^2}(2y^2+1)$.

(n) $U(r,s) = r^{10} s^{10}$ .. Now, replace r by $2x-y$, and s by $3x+y$

we get $U(2x-y, 3x+y) = (2x-y)^{10} \cdot (3x+y)^{10} \stackrel{\text{or}}{=} [(2x-y)(3x+y)]^{10}$

that is $Z = (6x^2 - xy - y^2)^{10}$

$\therefore \dfrac{\partial z}{\partial x} = 10(6x^2 - xy - y^2)^9 \cdot (12x - y)$

(o) $z = f(x+2y)$ .. let $x + 2y = t \Rightarrow \boxed{\dfrac{\partial t}{\partial x} = 1, \dfrac{\partial t}{\partial y} = 2}$

$\therefore z = f(t)$

By chain rule, $\dfrac{\partial z}{\partial x} = f'(t)\dfrac{\partial t}{\partial x} = f'(t) \cdot 1 = f'(t)$

$\dfrac{\partial^2 z}{\partial x^2} = f''(t)\dfrac{\partial t}{\partial x} = f''(t) \cdot 1 = f''(t)$

Next, $\dfrac{\partial z}{\partial y} = f'(t)\dfrac{\partial t}{\partial y} = 2f'(t)$, $\dfrac{\partial^2 z}{\partial y^2} = 2f''(t)\dfrac{\partial t}{\partial y} = 4f''(t)$

$\therefore \dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = f''(t) + 4f''(t) = 5f''(t)$

But $f''(t) = e^{t^2} = e^{(x+2y)^2}$

$\therefore \dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 5e^{(x+2y)^2}$

(p) Arc length of $\bar{r} = (x, y) = (t^3, t^2+1)$, $0 \le t \le 1$

is given by $L = \int_0^1 \sqrt{x'^2 + y'^2}\, dt = \int_0^1 \sqrt{(3t^2)^2 + (2t)^2}\, dt$

$= \int_0^1 \sqrt{9t^4 + 4t^2}\, dt = \int_0^1 t\sqrt{9t^2+4}\, dt$

let $w = 9t^2+4$ .. hence $dw = 18t\,dt \Rightarrow t\,dt = \frac{1}{18}\,dw$

New limits: when $t=0$, $w=4$, and when $t=1$, $w=9+4=13$

$$\therefore\ L = \frac{1}{18}\int_4^{13} \sqrt{w}\,dw = \frac{1}{18}\cdot\frac{2}{3}\ w^{\frac{3}{2}}\Big|_4^{13} = \frac{1}{27}\Big[\,w\sqrt{w}\,\Big]_4^{13}$$

$$= \frac{1}{27}\left[13\sqrt{13} - 8\right].$$

(q) we need:

(i) A point: Given as $P(1,-1,1) \Rightarrow \bar{r}_0 = (1,-1,1)$

(ii) A normal vector $\bar{N}$: This is $\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)\Big|_P$

where $F(x,y,z) = 2x^2+3y^2+4z^2-9 = 0$

$$\therefore\ \bar{N} = (4x, 6y, 8z)\Big|_{P(1,-1,1)} = \big(4(1), 6(-1), 8(1)\big) = (4,-6,8)$$

eq. of tangent plane is $\bar{r}\cdot\bar{N} = \bar{r}_0\cdot\bar{N}$

$$(x,y,z)\cdot(4,-6,8) = (1,-1,1)\cdot(4,-6,8)$$

$$\Rightarrow 4x - 6y + 8z = 4+6+8$$

or $2x - 3y + 4z = 9$.

(r) $z = x^3 - 3xy^2$

$\frac{\partial z}{\partial x} = 3x^2 - 3y^2$, $\quad \frac{\partial z}{\partial y} = -6xy$

want to solve the pair: $\frac{\partial z}{\partial x} = 0 \Rightarrow 3x^2 - 3y^2 = 0$ --- (1)

$\frac{\partial z}{\partial y} = 0 \Rightarrow -6xy = 0$ --- (2)

From (2): $-6xy = 0 \Rightarrow xy = 0 \Rightarrow$ Either $\boxed{x=0}$ or $\boxed{y=0}$

Case 1: If $x=0$, Eq. (1) becomes $3y^2 = 0$
$\Rightarrow y = 0$

$\therefore$ 1st. point is $(0,0)$

Case 2: If $y=0$, Eq. (1) becomes $3x^2 = 0$
$\Rightarrow x = 0$

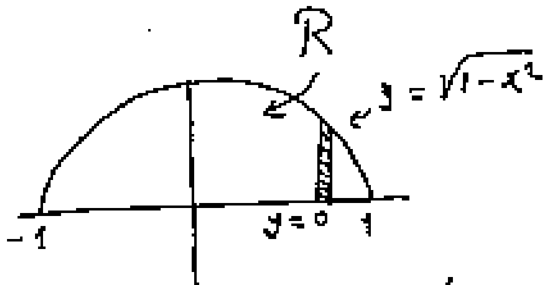
$\therefore$ The point is $(0,0)$ as well!

(s) $$\iint_R y\,dA = \int_{-1}^{1}\int_0^{\sqrt{1-x^2}} y\,dy\,dx$$

$$= \int_{-1}^{1} \frac{1}{2}y^2\Big|_0^{\sqrt{1-x^2}}\,dx$$

$$= \frac{1}{2}\int_{-1}^{1}(1-x^2)\,dx = \frac{1}{2}\left[x - \frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{2}{3}$$



region R is y-Simple

$0 \leq y \leq \sqrt{1-x^2}$

$-1 \leq x \leq 1$

---

(2) Let $F(x,y,z) = x^2y + xz + \sin(xyz) = 0$

$$\therefore \frac{\partial F}{\partial y} = x^2 + xz\cos(xyz), \quad \frac{\partial F}{\partial z} = x + xy\cos(xyz)$$

$$\therefore \frac{\partial y}{\partial z} = -\frac{\frac{\partial F}{\partial z}}{\frac{\partial F}{\partial y}} = -\frac{x + xy\cos(xyz)}{x^2 + xz\cos(xyz)}$$

---

(3) $\bar{r} = t^2\bar{i} + (t^{\frac{3}{2}} - t)\bar{j} + 2t\bar{k} \overset{\text{or}}{=} (t^2, t^{\frac{3}{2}} - t, 2t)$

velocity $\bar{v} = \frac{d\bar{r}}{dt} = (2t, \frac{3}{2}t^{\frac{1}{2}} - 1, 2)$

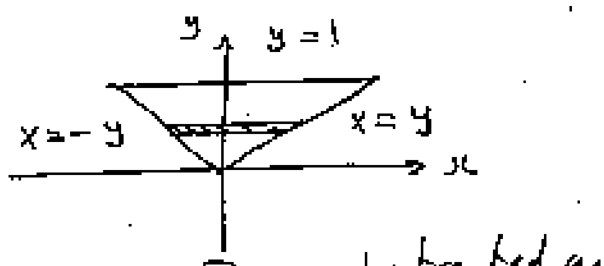
at $t = 4$, $\bar{v} = (2(4), \frac{3}{2}(4)^{\frac{1}{2}} - 1, 2) = (8, 2, 2) = 2(4,1,1)$

$\therefore$ speed $\|\bar{v}\| = \|2(4,1,1)\| = 2\sqrt{4^2+1^2+1^2} = 2\sqrt{18}$
$= 6\sqrt{2}$.

---

(4) The centre of mass $(\bar{x}, \bar{y})$ is given by $\bar{x} = \frac{M_{x=0}}{m}$, $\bar{y} = \frac{M_{y=0}}{m}$

where $m = \iint_R dm$, $dm = \delta\, dA = y\, dA$



region R may be treated as x-simple
$-y \leq x \leq y$, $0 \leq y \leq 1$

$$\therefore m = \iint_R y\, dA = \int_0^1 y\left(\int_{-y}^{y} dx\right) dy$$
$$= 2\int_0^1 y^2 dy = \frac{2}{3}$$

$$M_{y=0} = \iint_R y\, dm = \iint_R y \cdot y\, dA = \int_0^1 y^2\left(\int_{-y}^{y} dx\right) dy = 2\int_0^1 y^3 dy = \frac{1}{2}$$

$$\therefore \bar{y} = \frac{M_{y=0}}{m} = \frac{\frac{1}{2}}{\frac{2}{3}} = \frac{3}{4}.$$

Similarly you may show $\bar{x} = 0$ (or use symmetry !!)

$\therefore$ centre of mass $(\bar{x}, \bar{y}) = (0, \frac{3}{4})$

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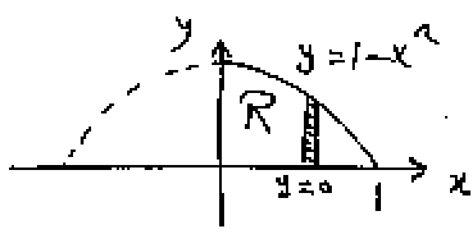
(5) Volume $\iint_R z\, dA = \int_0^1 \int_0^{1-x^2} (2-x)\, dy\, dx$



Base R:
$0 \leq y \leq 1 - x^2$
$0 \leq x \leq 1$

$$= \int_0^1 (2-x)\left[y\right]_0^{1-x^2} dx$$

$$= \int_0^1 (2-x)(1-x^2)\, dx$$

$$= \int_0^1 (2 - 2x^2 - x + x^3)\, dx$$

$$= 2x - \frac{2}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{4}x^4 \Big|_0^1$$

$$= 2 - \frac{2}{3} - \frac{1}{2} + \frac{1}{4} = \frac{13}{12}$$

---

(6) volume $V = \iint_R (z_U - z_L)\, dA$

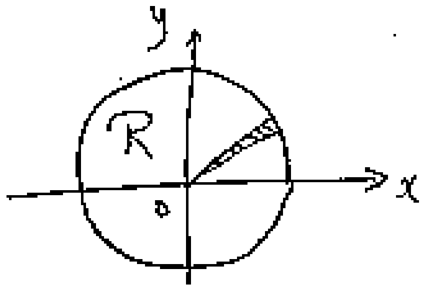
Here z_U = upper surface $= 2 - \sqrt{x^2+y^2}$, z_L = lower surface $= 0$

and R is the projection of $z = 2 - \sqrt{x^2+y^2}$ onto $z = 0$.

putting $z = 0$ in 1st. surface we get $0 = 2 - \sqrt{x^2+y^2}$

or $x^2+y^2 = 4$

$\therefore$ R is the region enclosed by the circle $x^2+y^2=4$



$\therefore V = \iint_R [2 - \sqrt{x^2+y^2} - 0]\, dA$

we shall use polar coordinates!

$x^2+y^2 = r^2$, $dA = r\,dr\,d\theta$

$x^2+y^2 = 4$
$\Rightarrow r^2 = 4$
$\Rightarrow r = 2$
$\therefore$ $0 \le r \le 2$
$0 \le \theta \le 2\pi$

$$V = \int_0^{2\pi}\int_0^2 (2-r).r\,dr\,d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\int_0^2 (2r - r^2)\,dr = 2\pi \cdot \frac{4}{3} = \frac{8\pi}{3}$$

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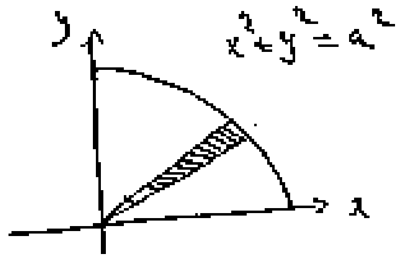
(7) For Centroid $\delta = 1$



$\therefore$ $m = \iint_R dm = \iint_R 1\, dA$ = area of R

$= \frac{1}{4}$(area of circle)

$= \frac{1}{4}(\pi a^2)$.

R: $0 \le r \le a$
$0 \le \theta \le \frac{\pi}{2}$

Note: From symmetry we see that $\bar{x} = \bar{y}$.

let us compute only $\bar{x}$!

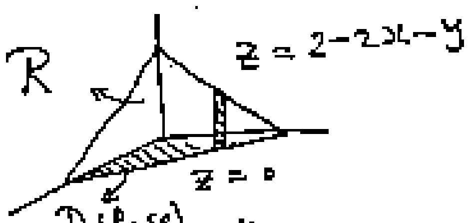
$M_{x=0} = \iint_R x\,dm = \iint_R x.1\,dA$ ← use polar coordinates

$$= \int_0^{\frac{\pi}{2}}\int_0^a r\cos\theta . r\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\cos\theta\,d\theta . \int_0^a r^2\,dr$$

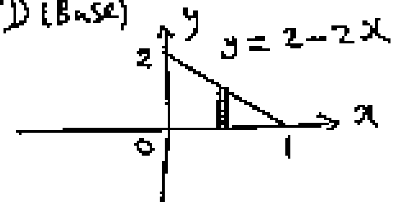
$$= \sin\theta\Big|_0^{\frac{\pi}{2}} \cdot \frac{1}{3}r^3\Big|_0^a = 1 \cdot \frac{a^3}{3}$$

$$\therefore \bar{x} = \frac{M_{x=0}}{m} = \frac{\frac{a^3}{3}}{\frac{1}{4}\pi a^2} = \frac{4}{3\pi}$$

$\Rightarrow$ Centroid $(\bar{x},\bar{y}) = \left(\frac{4}{3\pi}, \frac{4}{3\pi}\right)$



(8) $$\iiint_R z\,dV = \int_0^1\int_0^{2-2x}\int_0^{2-2x-y} z\,dz\,dy\,dx$$



$$= \frac{1}{2}\int_0^1\int_0^{2-2x} z^2\Big|_0^{2-2x-y}\,dy\,dx$$

D: $0 \le y \le 2-2x$
$0 \le x \le 1$

$$I = \frac{1}{2}\int_0^1\int_0^{2-2x}(2-2x-y)^2\,dy\,dx$$

Do not Expand!

$$I = \frac{1}{2}\cdot\int_0^1 \frac{(2-2x-y)^3}{(-1)(3)}\Big|_0^{2-2x} dx$$

$$= -\frac{1}{6}\int_0^1 \left[0 - (2-2x)^3\right] dx = \frac{1}{6}\,\frac{(2-2x)^4}{(-2)(4)}\Big|_0^1 = -\frac{1}{48}\left[0 - 16\right] = \frac{1}{3}$$

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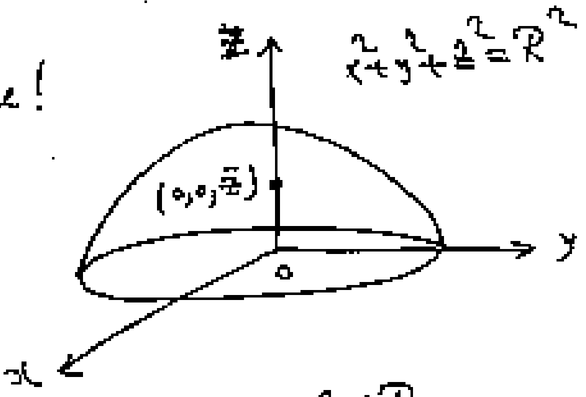
(9) From symmetry we know that the centroid must lie on the z-axis .. hence $\bar{x} = 0$, $\bar{y} = 0$!



Let us compute $\bar{z}$: Note: $\delta = 1$ in this case!

$$m = \iiint_R dm\ ,\quad dm = \delta\, dV = 1\, dV$$

$$\therefore\ m = \iiint_R dV = \text{volume of hemisphere} = \frac{1}{2}\left(\frac{4}{3}\pi R^3\right) = \frac{2}{3}\pi R^3.$$

$$R: \begin{array}{l} 0 \le \rho \le R \\ 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi/2 \end{array}$$

Next, $$M_{z=0} = \iiint_R z\, dm = \iiint_R z\, dV$$

we shall use spherical coordinates: $z = \rho\cos\varphi$, $dV = \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$

$$\therefore M_{z=0} = \int_0^{2\pi}\int_0^{\pi/2}\int_0^R (\rho\cos\varphi)\cdot \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^{\pi/2} \cos\varphi \sin\varphi\, d\varphi \cdot \int_0^R \rho^3 d\rho$$

$$= 2\pi \cdot \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2\varphi\, d\varphi \cdot \frac{R^4}{4}$$

Note: we have used identity $2\cos\varphi\sin\varphi = \sin 2\varphi \Rightarrow \cos\varphi\sin\varphi = \frac{1}{2}\sin 2\varphi$!

$$\therefore M_{z=0} = \cancel{2}\pi\cdot\frac{1}{\cancel{2}}\cdot \frac{-\cos 2\varphi}{2}\Big|_0^{\frac{\pi}{2}}\ \frac{R^4}{4}$$

$$= -\frac{\pi R^4}{8}\left[\cos\pi - \cos 0\right] = -\frac{\pi R^4}{8}\left[-1-1\right] = \frac{\pi R^4}{4}$$

$$\therefore\ \bar{z} = \frac{M_{z=0}}{m} = \frac{\frac{\pi R^4}{4}}{\frac{2}{3}\pi R^3} = \frac{3R}{8}$$

$$\therefore \text{ Centroid is } \left(0, 0, \frac{3R}{8}\right)$$
