TRIGONOMETRIC AND HYPERBOLIC IDENTITIES

Group (A):

TRIGONOMETRIC

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}$$
$$\sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}$$

HYPERBOLIC

$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{\cosh x}{\sinh x}$$
$$\operatorname{sech} x = \frac{1}{\cosh x}, \quad \operatorname{csch} x = \frac{1}{\sinh x}$$

Group (B)

$$\cos^2 x + \sin^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$\cot^2 x + 1 = \csc^2 x$$

$$\cosh^2 x - \sinh^2 x = 1$$
$$1 - \tanh^2 x = \operatorname{sech}^2 x$$
$$\coth^2 x - 1 = \operatorname{csch}^2 x$$

Group (C)

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x . \tan y}$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$
$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$
$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

Group (D)

$$\sin 2x = 2 \sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 2\cos^2 x - 1$$
$$= 1 - 2\sin^2 x$$

Hence
$$\boxed{\begin{aligned} \cos^2 x &= \tfrac{1}{2}(1 + \cos 2x) \\ \sin^2 x &= \tfrac{1}{2}(1 - \cos 2x) \end{aligned}}$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

$$\sinh 2x = 2 \sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$
$$= 2\cosh^2 x - 1$$
$$= 1 + 2\sinh^2 x$$

Hence
$$\boxed{\begin{aligned} \cosh^2 x &= \tfrac{1}{2}[\cosh 2x + 1] \\ \sinh^2 x &= \tfrac{1}{2}[\cosh 2x - 1] \end{aligned}}$$

$$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$$

Group (E)

$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$
$$\tan(-x) = -\tan x$$

$$\sinh(-x) = -\sinh x$$
$$\cosh(-x) = \cosh x$$
$$\tanh(-x) = -\tanh x$$

Group (F)

$$\sin x \cos y = \tfrac{1}{2}[\sin(x-y) + \sin(x+y)]$$
$$\sin x \sin y = \tfrac{1}{2}[\cos(x-y) - \cos(x+y)]$$
$$\cos x \cos y = \tfrac{1}{2}[\cos(x-y) + \cos(x+y)]$$

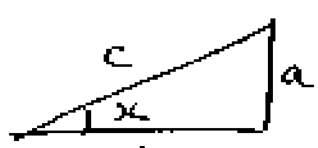
$$\sinh x \cosh y = \tfrac{1}{2}[\sinh(x+y) + \sinh(x-y)]$$
$$\sinh x \sinh y = \tfrac{1}{2}[\cosh(x+y) - \cosh(x-y)]$$
$$\cosh x \cosh y = \tfrac{1}{2}[\cosh(x+y) + \cosh(x-y)]$$

Special Values

$$\sin 0 = 0, \quad \cos 0 = 1, \quad \tan 0 = 0$$

$$\sinh 0 = 0, \quad \cosh 0 = 1, \quad \tanh 0 = 0$$

Definitions:



$$\sin x = \frac{\text{opposite}}{\text{Hyp.}}$$

$$\cos x = \frac{\text{adjacent}}{\text{Hyp.}}$$

$$\tan x = \frac{\text{opp.}}{\ldots}$$

a : opposite
b : adjacent
c : Hypotenuse

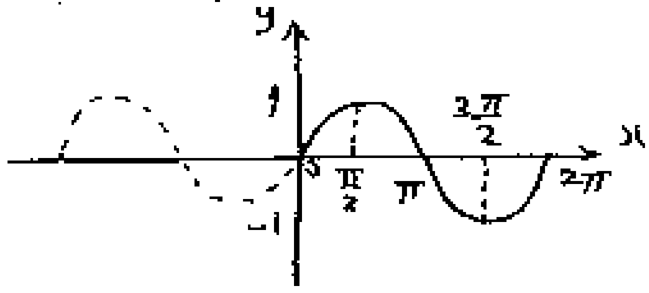
$$\sinh x = \frac{e^{x} - e^{-x}}{2} \text{ or } \tfrac{1}{2}[e^{x} - e^{-x}]$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2} \text{ or } \tfrac{1}{2}[e^{x} + e^{-x}]$$

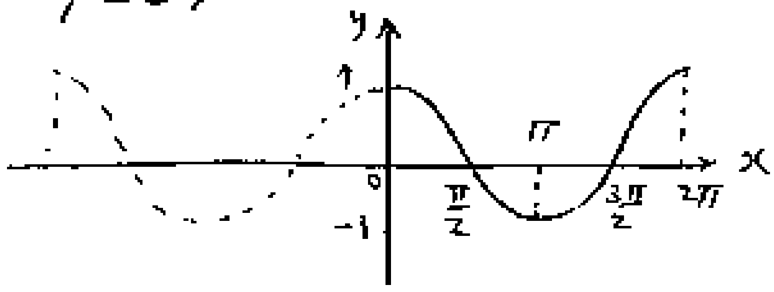
$$\tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

Standard Graphs

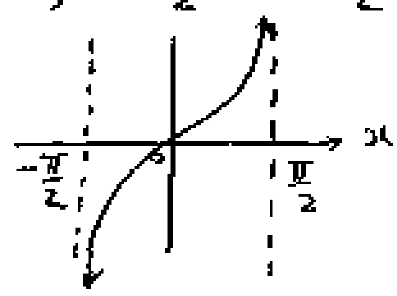
$y = \sin x, \quad 0 \le x \le 2\pi$



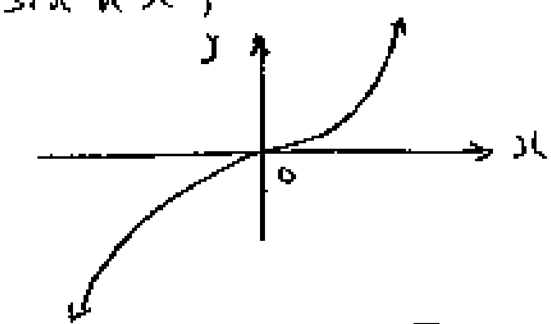
$y = \cos x \quad 0 \le x \le 2\pi$



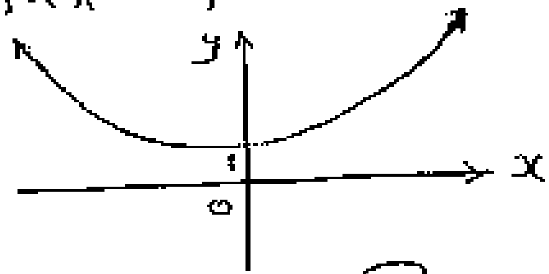
$y = \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$



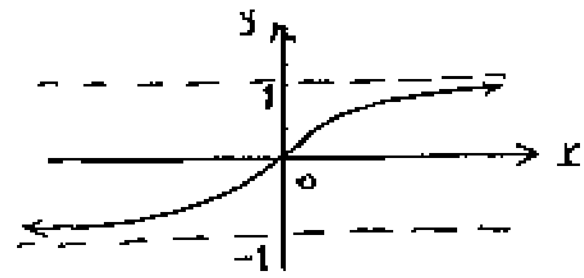
$y = \sinh x, \quad x \in \mathbb{R}$



$y = \cosh x, \quad x \in \mathbb{R}$

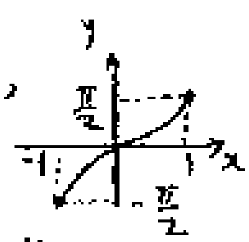


$y = \tanh x, \quad x \in \mathbb{R}$

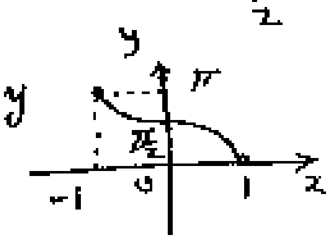


Inverse functions

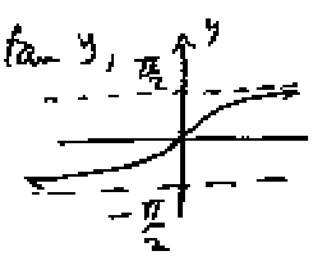
$y = \sin^{-1} x \iff x = \sin y$, $x \in [-1, 1]$, $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$



$y = \cos^{-1} x \iff x = \cos y$, $x \in [-1, 1]$, $y \in [0, \pi]$



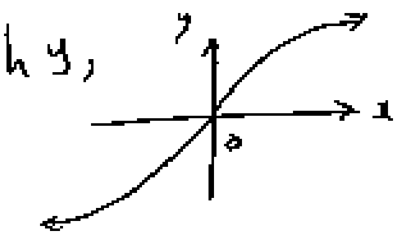
$y = \tan^{-1} x \iff x = \tan y$, $x \in \mathbb{R}$, $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$



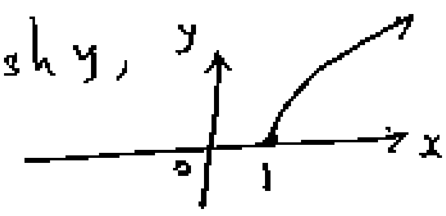
For $\csc^{-1} x$, $\sec^{-1} x$, and $\cot^{-1} x$ use:

$$\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right), \quad |x| \ge 1$$
$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right), \quad |x| \ge 1$$
$$\cot^{-1} x = \begin{cases} \tan^{-1} \frac{1}{x}, & x > 0 \\ \pi + \tan^{-1} \frac{1}{x}, & x < 0 \end{cases}$$

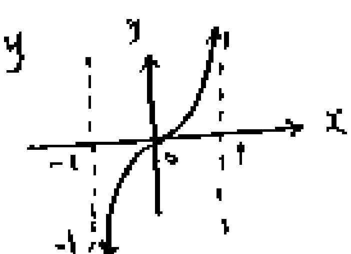
$y = \sinh^{-1} x \iff x = \sinh y$, $x \in \mathbb{R}$, $y \in \mathbb{R}$



$y = \cosh^{-1} x \iff x = \cosh y$, $x \in [1, \infty)$, $y \in [0, \infty)$



$y = \tanh^{-1} x \iff x = \tanh y$, $x \in (-1, 1)$, $y \in \mathbb{R}$

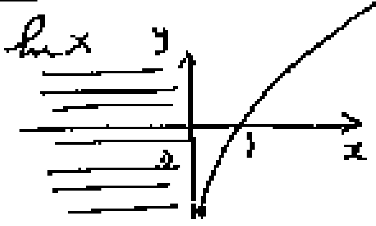


For $\operatorname{csch}^{-1} x$, $\operatorname{sech}^{-1} x$, and $\coth^{-1} x$ use:

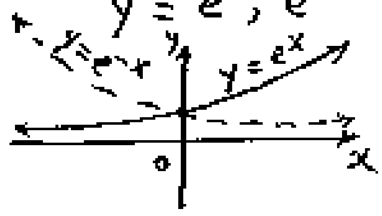
$$\operatorname{csch}^{-1} x = \sinh^{-1}\left(\frac{1}{x}\right), \quad x \neq 0$$
$$\operatorname{sech}^{-1} x = \cosh^{-1}\left(\frac{1}{x}\right), \quad x \in (0, 1]$$
$$\coth^{-1} x = \tanh^{-1}\left(\frac{1}{x}\right), \quad |x| > 1$$

Miscellaneous:

$y = \ln x$



$y = e^x, e^{-x}$



$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right), \quad x \in \mathbb{R}$$
$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right), \quad x \ge 1$$
$$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right), \quad -1 < x < 1$$

properties / special values

$\ln x + \ln y = \ln(xy)$,

$\ln x - \ln y = \ln\left(\frac{x}{y}\right)$, $\ln x^n = n \ln x$

$e^{\ln x} = x$, $\ln e^x = x$

$\ln e = 1$, $\ln 1 = 0$, $e^0 = 1$

$e^{+\infty} = +\infty$, $e^{-\infty} = 0$, $\ln(0^+) = -\infty$,

$\ln \infty = +\infty$, $\frac{1}{0^+} = +\infty$, $\frac{1}{0^-} = -\infty$

$\tan^{-1} \infty = \frac{\pi}{2}$, $\tan^{-1}(-\infty) = -\frac{\pi}{2}$, $\frac{1}{\pm\infty} = 0$.

Derivatives: SAMPLE

$(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(\tan x)' = \sec^2 x$

$(\sinh x)' = \cosh x$, $(\cosh x)' = \sinh x$, $(\tanh x)' = \operatorname{sech}^2 x$

$(\cot x)' = -\csc^2 x$, $(\sec x)' = \sec x \tan x$, ...

$(\coth x)' = -\operatorname{csch}^2 x$, $(\operatorname{sech} x)' = -\operatorname{sech} x \tanh x$, ...

$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$, $(\sinh^{-1} x)' = \frac{1}{\sqrt{1+x^2}}$

$(\tan^{-1} x)' = \frac{1}{1+x^2}$, $(\tanh^{-1} x)' = \frac{1}{1-x^2}$, $|x| < 1$

$(\sec^{-1} x)' = \frac{1}{|x|\sqrt{x^2-1}}$, $(\cosh^{-1} x)' = \frac{1}{\sqrt{x^2-1}}$