Support Material

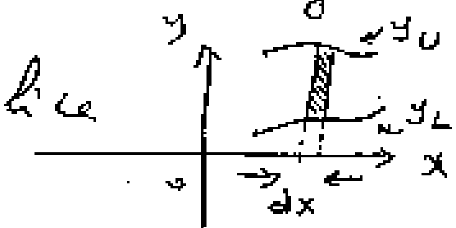
Some Review from AMAT 217

Assumptions: (1) let $y=f(x)$, $y=g(x)$ be continuous for all $x \in [a,b]$. Assume that the graph of f lies entirely above the graph of g on $[a,b]$. we shall call $y=f(x)$ "The upper curve", and $y=g(x)$ "The lower curve" and denote by $y_U = f(x)$, $y_L = g(x)$ respectively.

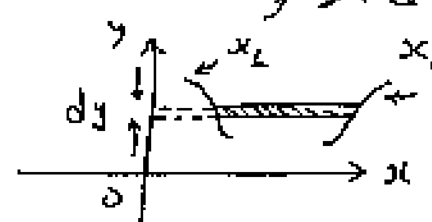
(2) Let $x=p(y)$, $x=q(y)$ be continuous for all $y \in [c,d]$. Assume the graph of $q(y)$ lies to the left of the graph of p on $[c,d]$. we shall call $x=q(y)$ "the left curve" and $x=p(y)$ "The Right curve" and denote by $x_L = q(y)$, $x_R = p(y)$ respectively.

Slice Rule: If curves are expressed in the form $y=f(x)$, $y=g(x)$, we slice Vertically. Hence thickness of slice is $\underline{dx}$.
However if curves expressed in the form $x=p(y)$, $x=q(y)$ we slice Horizontally. Hence thickness of slice is $\underline{dy}$

vertical slice



slice height = $y_U - y_L$

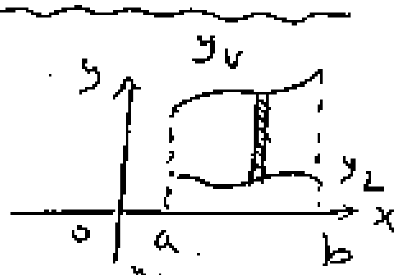


Horizontal slice

slice height = $x_R - x_L$

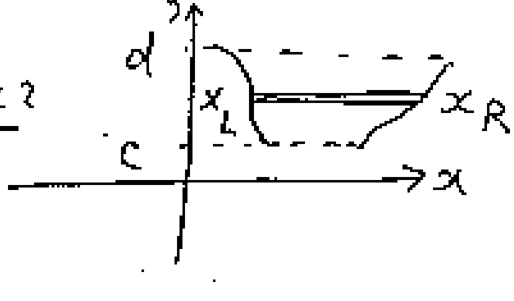
$\boxed{1}$ **Area between curves:** The area "A" is given by

Case 1



$$A = \int_a^b \text{height}.\text{Thickness} = \int_a^b (y_U - y_L)\, dx$$

Case 2



$$A = \int \text{height}.\text{Thickness} = \int_c^d (x_R - x_L)\, dy$$

2 Volume of solid of revolution : Two Methods

(a) <u>The Washer Method</u> : Use if slice is <u>perpendicular</u> to axis of rotation. In this case volume V is given by

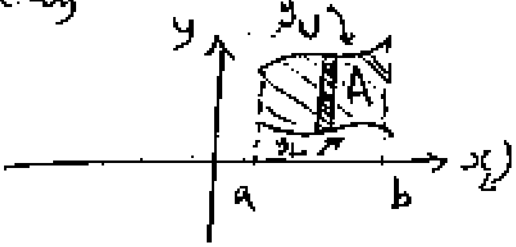
$$V = \pi \int_{\alpha}^{\beta} (r_o^2 - r_i^2)\ \text{Thickness}$$

where r_o = outer radius = distance from outer curve to axis of rotation

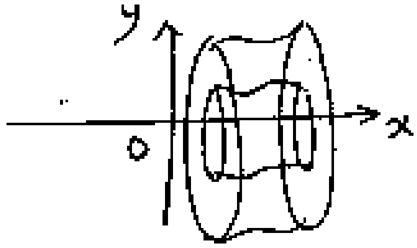
r_i = Inner radius = ~ ~ inner ~ ~ ~ ~ ~

For instance if area "A" shown in figure below is rotated about x-axis, the volume V is given by

$$V_x = \pi \int_a^b (y_U^2 - y_L^2)\, dx$$



Area rotated



solid generated

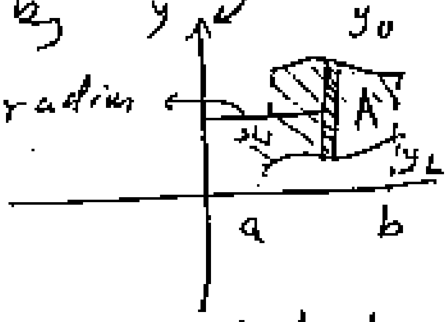
(b) <u>The Cylinderical Shells Method</u> : Use if slice is <u>parallel</u> to axis of rotation. The volume V is given by

$$V = 2\pi \int_{\alpha}^{\beta} \text{radius}\cdot\text{height}\cdot\text{Thickness}$$

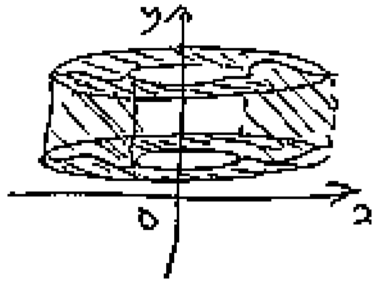
where radius = distance from centre of slice to axis of rotation

For instance if area "A" shown in figure below is rotated about y-axis, the volume V is given by

$$V_y = 2\pi \int_a^b x(y_U - y_L)\, dx$$



area rotated

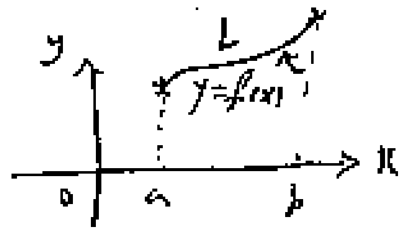


solid generated

$\boxed{3}$ Arc length:

Let $y = f(x)$ and assume $f'(x)$ exists and is continuous on $[a,b]$.
Then the length of the arc of $f(x)$ on $[a,b]$ is given by

$$L = \int_a^b \sqrt{1+(f'(x))^2}\,dx \quad \text{or} \quad \int_a^b \sqrt{1+y'^2}\,dx$$



<u>Note</u>: Similar formula holds if graph is expressed in the form $x = g(y)$, $y \in [c,d]$, namely

$$L = \int_c^d \sqrt{1+(g'(y))^2}\,dy$$

$\boxed{4}$ Area of surface of revolution

under same conditions above, the area of the surface generated by revolving the arc of $y = f(x)$, $x \in [a,b]$ about x-axis is given by

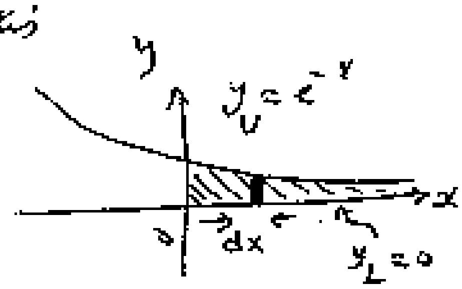
$$S_x = 2\pi \int_a^b y\,ds, \quad y > 0, \qquad ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \text{ or } \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

Similar formula hold for rotation about y-axis, namely $S_y = 2\pi \int x\,ds$, $x>$

<u>Examples</u>:

(1) Find the volume of the solid generated by revolving the region bounded by $y = e^{-x}$ and x-axis, $0 \leq x < \infty$

(a) About x-axis (b) About y-axis



<u>Solution</u> (a) slice vertically ($\perp$ to axis of rotation)

∴ Use Washer method!

$$V_x = \pi \int_a^b (r_o^2 - r_i^2)\,dx = \pi \int_0^\infty [(e^{-x})^2 - (0)^2]\,dx$$

$$= \pi \int_0^\infty e^{-2x}\,dx \leftarrow \text{Improper!}$$

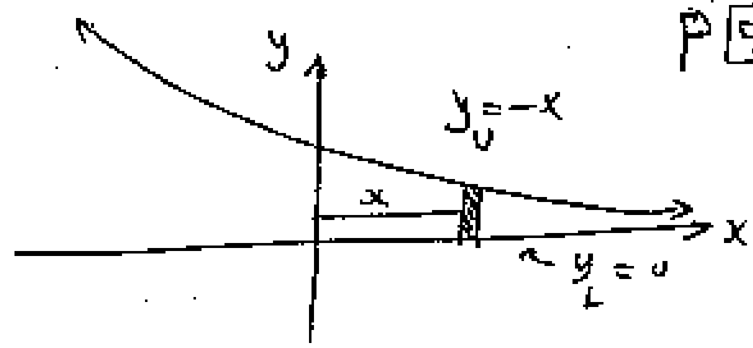
$r_o = e^{-x}$

No inner radius ($r_i = 0$)

$$= \pi \lim_{R\to\infty} \int_0^R e^{-2x}\,dx = \frac{\pi}{-2} \lim_{R\to\infty} e^{-2x}\Big|_0^R$$

$$= -\frac{\pi}{2}\left[\lim_{R\to\infty} (e^{-2R} - e^0)\right] = -\frac{\pi}{2}[0-1] = \frac{\pi}{2}$$

(b) The slice is parallel to axis of rotation .. Use shell method!



radius $= x$
height $y_U - y_L = e^{-x} - 0 = e^{-x}$
Thickness $= dx$

$$V_y = 2\pi \int_a^b \text{radius} \cdot \text{height} \cdot \text{Thickness}$$

$$= 2\pi \int_0^\infty x e^{-x} dx \leftarrow \text{Improper}$$

$$= 2\pi \lim_{R\to\infty} \int_0^R x e^{-x} dx \leftarrow \text{by parts}$$

$$= 2\pi \lim_{R\to\infty} -e^{-x}(x+1)\Big|_0^R$$

$$= 2\pi \lim_{R\to\infty} \left[-e^{-R}(R+1) + 1\right] = 2\pi[0+1] = 2\pi$$

<u>EX</u>: Find the arc length of $y = \frac{3}{2} x^{\frac{2}{3}}$ from $x = 3\sqrt{3}$ to $x = 16\sqrt{2}$

soln. arc length $L = \int_a^b ds$, $ds = \sqrt{1 + (f'(x))^2}\, dx$

here $y = f(x) = \frac{3}{2} x^{\frac{2}{3}} \Rightarrow f'(x) = x^{-\frac{1}{3}} \Rightarrow (f'(x))^2 = x^{-\frac{2}{3}} = \frac{1}{x^{2/3}}$

$$\therefore L = \int_{3\sqrt{3}}^{16\sqrt{2}} \sqrt{1 + \frac{1}{x^{\frac{2}{3}}}}\, dx = \int_{3\sqrt{3}}^{16\sqrt{2}} \sqrt{\frac{x^{\frac{2}{3}} + 1}{x^{\frac{2}{3}}}}\, dx = \int_{3\sqrt{3}}^{16\sqrt{2}} \frac{\sqrt{x^{\frac{2}{3}}+1}}{x^{\frac{1}{3}}}\, dx$$
$$= \int x^{-\frac{1}{3}} \sqrt{x^{\frac{2}{3}}+1}\, dx$$

let $u = x^{\frac{2}{3}} + 1$, $du = \frac{2}{3} x^{-\frac{1}{3}} dx \Rightarrow x^{-\frac{1}{3}} dx = \frac{3}{2} du$

$$\therefore L = \frac{3}{2} \int \sqrt{u}\, du = \frac{3}{2} \cdot \frac{2}{3} u^{\frac{3}{2}} = u^{\frac{3}{2}} = (x^{\frac{2}{3}} + 1)^{\frac{3}{2}}\Big|_{3\sqrt{3}}^{16\sqrt{2}}$$

$$= \left[(\sqrt[3]{x^2}) + 1\right]^{\frac{3}{2}}\Big|_{3\sqrt{3}}^{16\sqrt{2}} = 9^{\frac{3}{2}} - 4^{\frac{3}{2}}$$
$$= (\sqrt{9})^3 - (\sqrt{4})^3 = 27 - 8 = 19$$

EX: Find the arc length of $y = \frac{1}{2} x^2$ from $x = 0$ to $x = 1$

<u>solution</u>: Done before! .. Redo it!

Ans. $\frac{1}{2}\left[\sqrt{2} + \ln(\sqrt{2}+1)\right]$

AMAT 217

REVIEW PROBLEMS

(I) Find each of the following

(a) $\int e^{-4x+1}\,dx$

(b) $\int e^{19x}\,dx$

(c) $\int x\sqrt{1+x^2}\,dx$

(d) $\int x\cos(1+x^2)\,dx$

(e) $\frac{d}{dx}\sin^{-1}(7x)$

(f) $\frac{d}{dx}\tan^{-1}(2x)$

(g) $\int \frac{x}{1+x^2}\,dx$

(h) $\int_{-1}^{0} \frac{1}{1+x^2}\,dx$

(i) $\frac{d}{dx}\ln(\sin x)$

(j) $\frac{d}{dx}\ln(x^3-2x+1)$

(k) $\int_{1}^{e^2} \frac{1}{x}\,dx$

(l) $\int_{-2}^{-1} \frac{1}{x}\,dx$

(m) $\frac{d}{dx}\int_{0}^{2x} \sin(t^3)\,dt$

(n) $\frac{d}{dx}\int_{1}^{x^3} e^{-2t^2}\,dt$

(II) Find the area enclosed by $y=-x$, and $y=x^2$

(III) Find the area enclosed by $y=2-x^2$, and $y=x^2$

(IV) Find the length of the arc $y=\frac{2}{3}x^{\frac{3}{2}}$, $0\le x\le 8$

(V) Find the length of the arc $y=\cosh x$, $0\le x\le \ln 2$

(VI) Find the volume obtained by rotating the region enclosed by $y=-x$, $y=x$, $x=1$

about y-axis.

(VII) Find the volume obtained by rotating the region enclosed by $y=x^2$, $y=x$

(a) about the line $y=-1$.

(b) about the line $x=2$.

AMAT 217
SOLUTIONS TO REVIEW PROBLEMS

(I) (a) $\int e^{-4x+1}\,dx = \dfrac{e^{-4x+1}}{-4} + C$ (b) $\int e^{19x}\,dx = \dfrac{e^{19x}}{19} + C$

(c) $\int x\sqrt{1+x^2}\,dx$. let $w = 1+x^2 \Rightarrow dw = 2x\,dx \Rightarrow \frac{1}{2}dw = x\,dx$

$\therefore I = \frac{1}{2}\int \sqrt{w}\,dw = \frac{1}{2}\cdot \dfrac{w^{3/2}}{3/2} + C = \frac{1}{3}(1+x^2)^{3/2} + C.$

(d) $\int x\cos(1+x^2)\,dx$.. let $w = 1+x^2 \Rightarrow dw = 2x\,dx \Rightarrow \frac{1}{2}dw = x\,dx$

$\therefore I = \frac{1}{2}\int \cos w\,dw = \frac{1}{2}\sin w + C = \frac{1}{2}\sin(1+x^2) + C$

(e) $\dfrac{d}{dx}\sin^{-1}(7x) = \dfrac{(7x)'}{\sqrt{1-(7x)^2}} = \dfrac{7}{\sqrt{1-49x^2}}$

(f) $\dfrac{d}{dx}\tan^{-1}(2x) = \dfrac{(2x)'}{1+(2x)^2} = \dfrac{2}{1+4x^2}$

(g) $\int \dfrac{x}{1+x^2}\,dx$. let $w = 1+x^2 \Rightarrow dw = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}dw$

$\therefore I = \frac{1}{2}\int \frac{1}{w}\,dw = \frac{1}{2}\ln|w| + C = \frac{1}{2}\ln(1+x^2) + C$

(h) $\int_{-1}^{0}\dfrac{1}{1+x^2}\,dx \overset{\text{Table}}{=} \tan^{-1}x\Big|_{-1}^{0} = \tan^{-1}0 - \tan^{-1}(-1)$
$= 0 + \tan^{-1}1 = 0 + \frac{\pi}{4} = \frac{\pi}{4}$

(i) $\dfrac{d}{dx}\ln(\sin x) = \dfrac{\cos x \leftarrow u'}{\sin x \leftarrow u} = \cot x$

(j) $\dfrac{d}{dx}\ln(x^3-2x+1) = \dfrac{3x^2-2}{x^3-2x+1}$

(k) $\int_{1}^{e^2}\frac{1}{x}\,dx \overset{\text{Table}}{=} \ln|x|\Big|_{1}^{e^2} = \ln e^2 - \ln 1$
$= 2\ln e - \ln 1$
$= 2(1) - 0 = 2$

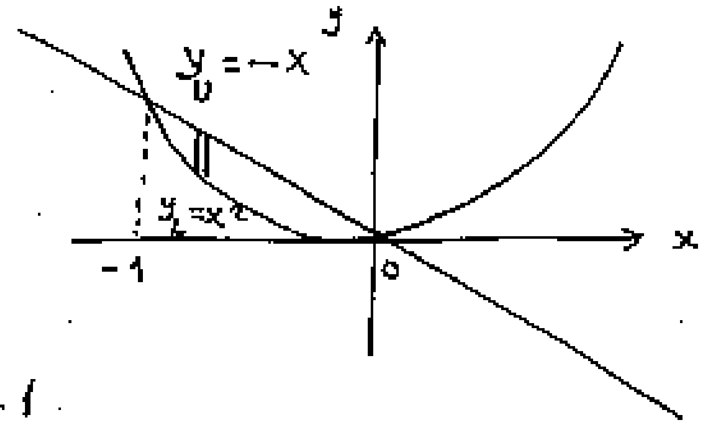
(l) $\int_{-2}^{-1}\frac{1}{x}\,dx = \ln|x|\Big|_{-2}^{-1} = \ln|-1| - \ln|-2|$
$= \ln 1 - \ln 2 = 0 - \ln 2 = -\ln 2$

(m) $\dfrac{d}{dx}\int_{0}^{2x}\sin(t^3)\,dt = \sin(t^3)\Big|_{\text{at } t=2x}(2x)' = \sin(8x^3).\,2$
$= 2\sin(8x^3)$

(n) $\dfrac{d}{dx}\int_{1}^{x^3} e^{-2t^2}\,dt = e^{-2t^2}\Big|_{t=x^3}(x^3)'$
$= e^{-2(x^3)^2}\cdot 3x^2 = 3x^2e^{-2x^6}$

(II) First sketch!

Next: Find where graphs intersect?



$y = x^2$, $y = -x$

$\Rightarrow x^2 = -x \Rightarrow x^2 + x = 0$

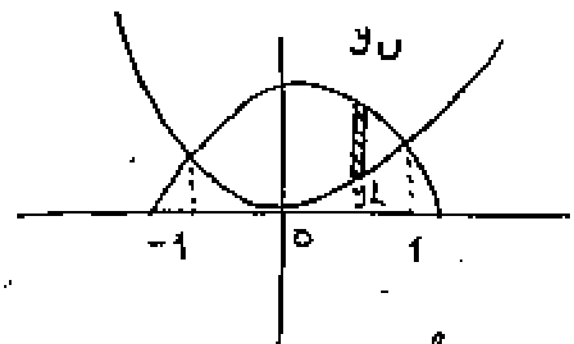
$x(x+1) = 0 \Rightarrow x = 0, -1$

$\therefore$ area $A = \int_a^b (y_U - y_L)\, dx = \int_{-1}^{0} (-x - x^2)\, dx = -\frac{1}{2}x^2 - \frac{1}{3}x^3 \Big|_{-1}^{0}$

$= \frac{1}{6}$

(III) First sketch!

Next: where curves intersect?



$y_U = 2 - x^2$, $y_L = x^2$

let us solve $y = 2 - x^2$, $y = x^2$

$\therefore 2 - x^2 = x^2$

$2 = 2x^2 \Rightarrow x = \pm 1$

area $A = \int_{-1}^{1} [(2 - x^2) - x^2]\, dx = 2\int_{-1}^{1} (1 - x^2)\, dx = 2\left[x - \frac{1}{3}x^3\right]_{-1}^{1} = 8/3$

(IV) $y = \frac{2}{3} x^{\frac{3}{2}}$, $0 \le x \le 8$

Arc length $L = \int_a^b \sqrt{1 + y'^2}\, dx$ .. here $y' = \frac{2}{3} \cdot \frac{3}{2} x^{\frac{1}{2}} = x^{\frac{1}{2}} = \sqrt{x}$

$= \int_0^8 \sqrt{1 + (\sqrt{x})^2}\, dx = \int_0^8 (1 + x)^{\frac{1}{2}} dx$

$= \frac{2}{3}(1 + x)^{\frac{3}{2}} \Big|_0^8 = \frac{2}{3}\left[9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right] = \frac{2}{3}\left[3^3 - 1\right]$

$= \frac{52}{3}$

(V) $y = \cosh x$, $0 \le x \le \ln 2$

$y' = \sinh x$, $1 + y'^2 = 1 + \sinh^2 x \xrightarrow{\text{id.}} \cosh^2 x$

$\therefore$ arc length $L = \int_a^b \sqrt{1 + y'^2}\, dx = \int_0^{\ln 2} \sqrt{\cosh^2 x}\, dx = \int_0^{\ln 2} \cosh x\, dx$

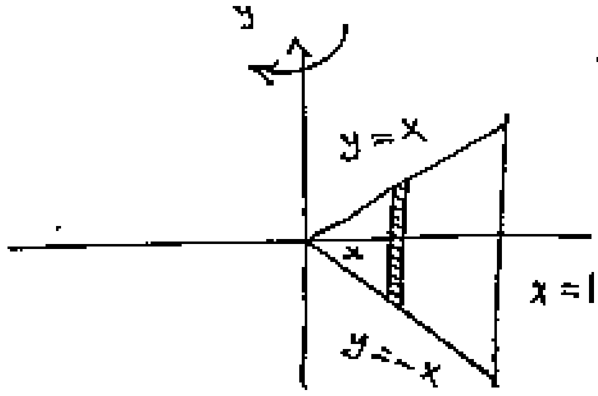
$= \sinh x \Big|_0^{\ln 2} = \sin(\ln 2) - \sinh 0 = \sinh(\ln 2)$

Note: $\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{e^{\ln 2} - e^{\ln 2^{-1}}}{2} = \frac{2 - 2^{-1}}{2} = \frac{2 - \frac{1}{2}}{2} = \frac{3}{4}$

$\therefore L = \frac{3}{4}$

(VI)

Since slice is // to axis of rotation, Use cylinderical shells!



$$V_y = 2\pi\int_a^b \text{radius} \cdot \text{height} \cdot \text{Thickness}$$

$$= 2\pi \int_0^1 x(2x)\,dx = 4\pi \cdot \frac{x^3}{3}\Big|_0^1$$

$$= \frac{4\pi}{3}.$$

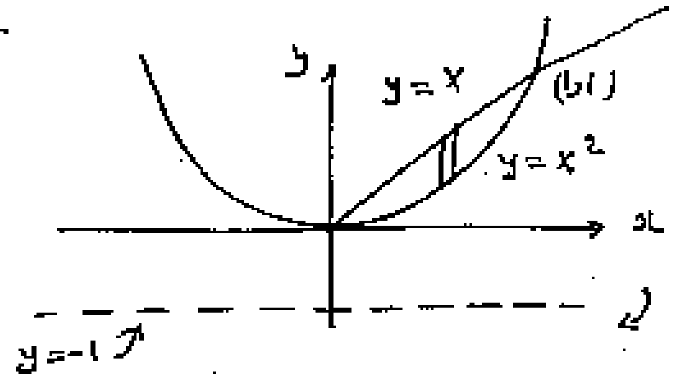
height $y_U - y_L = x-(-x) = 2x$

Thickness dx,

radius x

(VII) (a) slice is Perpendicular to axis of rotation

∴ Use Washer method



$$V = \pi \int_a^b (r_o^2 - r_i^2)\cdot \text{Thickness}$$

$$= \pi \int_0^1 [(x+1)^2 - (x^2+1)^2]\,dx$$

$$= \pi \int_0^1 (x^2+2x+1) - (x^4+2x^2+1)\,dx$$

$$= \pi \int_0^1 (2x - x^2 - x^4)\,dx$$

$$= \pi\left[x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{5}x^5\right]_0^1 = \frac{7}{15}$$

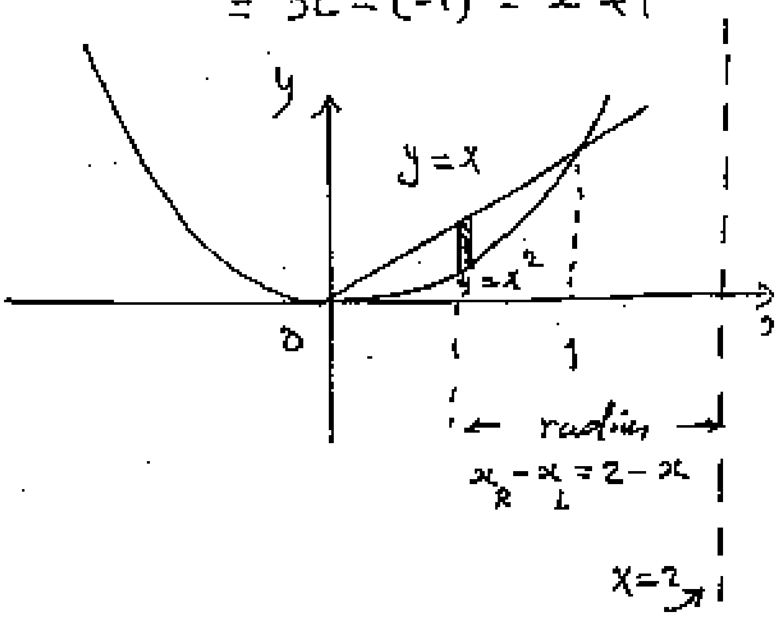
Thickness dx

r_o = outer radius = distance from upper Curve to axis of rotation $= x-(-1) = x+1$

r_i = Inner radius = distance from lower curve to axis of rotation $= x^2-(-1) = x^2+1$

(b) slice is parallel to axis of rotation

∴ Use shells!



$$V = 2\pi \int_a^b \text{radius} \cdot \text{height} \cdot \text{Thickness}$$

$$= 2\pi \int_0^1 (2-x)(x-x^2)\,dx$$

$$= 2\pi \int_0^1 (2x - 3x^2 + x^3)\,dx$$

$$= 2\pi\left[x^2 - x^3 + \tfrac{1}{4}x^4\right]_0^1$$

$$= 2\pi\left[1 - 1 + \tfrac{1}{4}\right] = \frac{\pi}{2}$$

rad. $= 2 - x$

height $y_U - y_L = x - x^2$

Thickness $= dx$
