Coalitional Congestion Games

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Working with Uri and Rann
Was for me a lot of fun
Playing along trough many days
Manipulating many games
Directed, guided with the tools
Which helped my research being cool
Sergey.

Contents

Abstract

In this thesis we study *Coalitional Congestion Games* (CCG), a family of non-cooperative games which is a natural extension of the well studied family of Congestion Games. Coalitional Congestion Games are similar to congestion games, replacing individual agents with coalitions of agents as the players of the game. Thus, a Congestion game and a coalition structure over the agents induce a Coalitional Congestion Game. A strategy for a coalition is the any combination of the strategies available to all of its members in the underlying Congestion Game. The utility of a coalition is the sum of its members' utilities.

CCG model similar situations as do Congestion Games, namely communication networks, transportation, flows, load balancing, routing and more. However, the new model better suits situations where the decision making is granted to a set of agents. For example, a transportation game played among a few fleet managers instead of among the drivers or routing games played among internet domains instead of individual computers.

The main research questions we address are derived from the literature on congestion games and can be roughly partitioned into two sets:

- Congestion games often have compelling properties, such as the existence of pure NE. We investigate under which circumstances do these properties carry over to CCG.
- What are the social welfare implications of working with coalitions, as opposed to endowing individual agents on the one hand, or the grand coalition on the other hand. In a way, this extends the literature on the Price of Anarchy the Price of Collusion.

Lists of Symbols and Acronyms

List of Symbols

BR(s)	Set of best reply strategies to the strategy profile s
α_r	The highest natural number satisfying $P_r(m) > P_N$
$c(\cdot)$	Congestion Function
$DR(s)$	The set of agents who select in s two different resources.
q, s, t	Arbitrary names for Strategy Profiles
i, j, k	Arbitrary names for agents and sub agents
$MA(s)$	Set of resources that an agent select them twice in s
MC(C)	Size of Maximal element of the Partition C
$NE(\cdot)$	The set of Pure NE Strategy Profiles in the game.
P_N	Highest cost of resource in NE of the (underlying) SCG
SA	Set of singleton agents in the CCG.
r, x	Arbitrary names for Resources
u, v, w	Arbitrary names for Congestion Vectors
$UR(s)$	The set of agents who in s select a unique resource.

List of Acronyms

CCG	Coalitional Congestion Games
CNC	Coalitional Non-Cooperative Game
FIP	Finite Improvement Path
NE	(Pure) Nash Equilibrium / Equilibria
PA	Price of Anarchy
PC	Price of Collusion
SCG	Simple Congestion Games
TC	Total Cost
TCR	Total Cost Ratio

Chapter 1

Introduction

In this thesis we study a family of noncooperative games called *Coalitional Congestion Games*, which are a natural derivative of a well studied family of games - Congestion Games. A Congestion Game models a situation where agents must choose among (subsets of) resources. Each resource is characterized by some cost function that is dependant on the accumulated usage of that resource, and is independent of the identity of the users. An agent's utility in such games is the total cost of the resources it has chosen.

In this paper we extend the model of Congestion Games into Coalitional Congestion Games (CCG), where the player in the congestion game is no longer the single agent but rather some subset of these agents (which we refer to as a coalition). Therefore, a CCG is given by congestion game coupled up with a set of a coalitions, which is an exogenously given partition over the set of agents. The utility of each player/coalition is the sum of the utility of its agents.

Congestion games have been used to model situations related to transportation, communication networks, assignment problems, economics and other fields, as we will show below. In all of these situations the decision maker (namely, the player) is the same as the agent itself. However, there are often cases where the decision and the utility is not related to the agent itself, but to some player who controls more than a single agent. Consider for example a standard traffic problem, modelled as a set of arcs in the network, where a driver chooses a path between two nodes. The collection of driver's choices induces congestion over the network can be viewed as a congestion game. In contrast, consider the same network and set of drivers, where each driver is an employee of one of a few shipping firms, and the decision which path to choose is handled by the shipping firm.

This is no longer a Congestion Game but rather a Coalitional Congestion Game.

Another motivating example to study coalitional congestion games is the model of computer networks, like the internet. Consider the situation where domain on web need to send a number of packets over internet. Each packet needs to be sent in one of the possible routes, but once again, it is the domain, not the packet that decides which link will be chosen. We can also look on the case when a router that receives a packet can alter the packet's route. In many cases there is also a coalition setting on the routers, e.g. routers from the same country or same building can coordinate their actions. The CCG model can be used for both of the described situations.

coalitional Congestion Games model is also used in problems of load balance, for example scheduling jobs on a computer with a number of processors. Each process needs a certain amount of time on one of the processors, when the delay depends on the processor quality and the load on in. If each process has a single job we are in the setting of a Congestion Game. However, often processes have a number of threads that run in parallel. When each process needs to choose a processor for its threads we are in the CCG setting.

In this work we focus on CCG where the underlying game is a Simple Congestion Game (SCG). A Simple Congestion Game is a Congestion Game where each agent selects exactly one resource from the resource set. We will also assume that resource use has a non negative cost, which is strictly increasing in congestion.

One preliminary question is whether, formally, a CCG is actually a congestion game. In the thesis we will demonstrate that the example below is not a congestion game.

Consider an example of two truck companies (1 and 2), with different sizes of truck fleets. Each truck needs to be sent to one of the two availible warehouses (A and B). The cost for a truck depends on the warehouse it goes to and how many trucks go to the same warehouse as it did. Note that in our setting the players who allocate the trucks are the shipping firms and not the individual drivers. Therefore, we need to use the CCG model.

Example 1 *Assume the costs over each resource, per truck, are given by the following table:*

Resource	1	2	3
A	1	3	5
B	1	2	4

In addition, there are 3 trucks, two in the first firm and one in the second, which induces the partition $C = [1, 2; 3]$. Therefore, the noncooperative game we should analyze is the following:

G^C costs	A	B
A,A	5+5,5	3+3,1
A,B	3+1,3	1+2,2
B,B	2+2,1	4+4,4

In the profile where the first company chooses to send one truck to A and one to B, and the second company sends the truck to A, the second company has the cost of 3 and the first company has a cost of 4 composed of the cost of 3 for the truck in A and 1 for the truck in B.

The most notable property of congestion games is that they have a pure NE. This has been shown by Rosenthal in [18]. In fact, Monderer and Shapley [13] showed that Congestion Games hold a stronger property - Exact Potential (they actually show an equivalence between Congestion Games and Potential Games). An intermediate property between existence of pure NE and existence of a Potential is the property known as 'Finite Improvement Path' (FIP) Property, where any profitable deviation sequence is of finite length.

As we will show, in general all these properties above do not carry over to CCG. However, we will provide some necessary and sufficient conditions for them, extending the results of Fotakis et al. [6] and Hayrapetyan et al. [7]. For example, if the coalitions are small then the CCG posses a pure NE. Also, Fotakis et al. [6] showed that if the resource cost functions are linear the game posses an Exact Potential and we show that under minimal conditions on the partition those are the only cases.

The other set of questions considers the implications of the coalitions on the social welfare, modelled as the sum of all agents' utilities. We pursue the line of thought initiated by Koutsoupias and Papadimitriou [10], who introduce the notion of the 'Price of Anarchy', and study the ratio between the total welfare in situations where the players are the coalitions compared with the situation where the players are the single agents. We bound social welfare ratio of Pure NE strategy profiles in the various settings. For example, we show that when we compare a CCG and its underlying SCG this ratio cannot exceed the number of agents in both directions. We will also provide tight bounds for Price of Anarchy for SCG, extending the results of Awerbuch et al. [2] and Chruistodolow and

Koutsiapis [3] to the most general setting.

1.1 Extensions and Future Research

We consider some extensions to the basic model as follows:

Throughout this paper we define the cost for the coalition as the sum of coalition members. However, in literature there are also other definitions for coalition cost. For example, Fotakis et al. [6] defined the coalition cost as the maximal cost of a coalition member. They show that in this model a Pure NE always exists, and we analyze the necessary and sufficient conditions for Pure NE in this model. An important future research question is to analyze the impact of various coalition cost definitions on the CCG.

Another assumption is that resource usage has a negative utility rather than positive. We check and analyze the social welfare ratios in the positive utility case and show that in this model those ratios are trivially bounded in the general case. An interesting question is to provide conditions for less trivial bounds.

Some additional future research which we point at is as follows:

This work leads to many other interesting questions. One of the basic ones is about dynamic coalition formation. In our model we assume that the coalitions are static and formed and unrelated to the played game. An important question is to check what coalitions can be formed in the game and how coalition members will distribute the cost and utility among them, as done in cooperative games.

Another question is identifying the cases when Pure NE strategy exist in CCG, and identifying their congestion vectors. We provide sufficient conditions to existence of Pure NE, but we do not show a necessary condition. Similarly we show sets of vectors that contain all Pure NE congestion vectors, but we do not identify their set.

In this paper we assume coalition members play a SCG, and provide only a preliminary result regarding other games. An interesting extension would be to look on different games played by the coalition members. This can be general congestion games, extended congestion games or a different set of games.

On the bounds of Social Welfare there is a gap between the tight bound on Price of Collusion provided in Hayrapetyan [7] and the non-tight bound we provide. Moreover, we provide the most general bound and if we provide some limitation on the SCG maybe these bounds can be improved.

1.2 Literature

In our paper we discuss formation of static Coalitions in Congestion Games. This previously was done in Fotakis et al. [6] and Hayrapetyan et al. [7]. The coalitions are formed by a exogenously given partition and each coalition member plays a Simple Congestion Game. Members of the same coalition can cooperate and the cost is per coalition. In coalition formation there are two basic approaches that define the coalitions cost we call *Cost Approach*, when a coalition pays the sum of coalition members costs, and *Time Approach*, when coalition pays the maximal cost among it's members. There is a great difference among the two approaches, as proven by Fotakis et al. in [6]: a time approach CCG has the FIP property (Theorem 1) whereas cost approach CCG may not (lemma 3). In the cost approach CCG model Hayrapetyan et al. [7] (Theorem 3.9) showed that if the cost functions are weakly convex the cost approach CCG has a Pure NE. Fotakis et al. showed in [6] (Theorem 6), that if a game has linear costs it has an Exact Potential. We provide further analysis for the cost approach CCG model further and provide additional results.

Coalition formation in games is also a question that have been researched deeply in Cooperative Games Theory. However, in Cooperative Game the utility of a coalition is not affected by outsiders, as noted by Montero [15]. Unfortunately, this is not the case if agents coordinate in a Non-Cooperative Game. There is no generally accepted model in Games Theory which describes the process of coalition formation and many solution concepts where agents act together, such as Strong NE, as described in Aumann [1], or Coalition Proof NE, as described in Moreno and Wooders [16]. Similarly to Hayrapetyan et al. [7] and Fotakis et al. [6], we analyze the game formed when the coalition structure is provided, focusing on the question "what is the game if the following coalitions are formed?" rather than "which coalitions will be formed?".

We would like to point out some previous research done in Congestion Games in literature. To our knowledge, the first time the model of congestion games was used in Wardrop [21], to model traffic routing. In this model each agent walks on a graph. The roads are represented by the resources and the strategies for an agent are the simple paths in the network between two nodes on the graph. Selecting road has a cost, which is the delay experienced by travellers on this road. This delay is a function of the road and the number of agents selecting this specific road and is independent on the identity of agents. Rosenthal in [18] showed that such game posses a Pure NE, implicitly using the existence of Exact Potential, formally defined by Monderer and Shapley in [13] who introduced the notation of congestion games. In Literature there are also references to a subset of Congestion Games - Simple Congestion Games (SCG) where each agent selects a single resource. Among other papers this model is dealt in [8], [9] and [20].

Congestion Games have many interesting properties. For example, those games have *Exact Potential*, as shown in Monderer and Shapley [13]. Moreover, any game that posses an Exact Potential is isomorphic to Congestion Games. A different proof to this is given in Voorneveld et al. [20]. In those two papers it is shown that existence of an Exact Potential leads to the existence of a Pure NE. In Voorneveld et al. [20] the equivalence between Strong NE, NE and Exact Potential Maximizer strategies is proven. Monderer and Shapley [13] define the *Finite Improvement Path (FIP) Property.* They show that Exact Potential leads to the FIP property which leads to existence of a Pure NE.

We are not the first to propose an extensions to the classical Congestion Games model. Among the many articles doing so are [11], [12] and [14]. An important extension which we would like to point out is the splittable flow model as described, for example, by Yang et al. [22]. In this model an agent may choose, as a pure strategy, to split his congestion between different resource subsets, as follows: assume that an agent needs to flow a unit of liquid on a given network. The agent can split this unit to fragments of various sizes, and as a pure strategy, he can use different routes for each fragment. In the classical (un-splittable) Congestion Game model, as presented for example, by Wardrop [21], pure strategies must use a single route for the entire unit. Our CCG model falls between the splittable and un-splittable model, in the following way: a coalition of k agents may split its flow, but only to fractions of size $1/k$. This can be done by giving each coalition member a single, but possibly a different, route. Therefore, CCG model can describe also a "limited" splittable flow, when a limited number of fragment sizes is allowed.

Another popular topic which is connected to Congestion Games is Price of Anarchy, introduced by Koutsoupias and Papadimitriou in [10]. This is a method to quantify the increase in cost due to selfishness of agents. Among others who deal with this topic are [4] and [19]. The most general SCG model, with n agents and m resources is analyzed in Awerbuch et al. [2] and Christodolow and Koutsiapis [3]. They show that in linear costs model the price of anarchy is between 2.5 and 2.618, whereas in d degree polynomial it is $d^{\theta(d)}$. Hayrapetyan et al. [7] define a similar concept, the Price of Collusion, which quantifies the increase in cost due to formations of coalitions. They provide bounds to it in various CCG settings, which we extend here.

1.3 Road Map

The chapters are arranged in the following order: First we define the model. After that we provide the results regarding Simple Congestion Games. Then, we analyze Pure NE, Exact Potential and FIP in CCG. It is followed by the chapter dealing with small coalition CCG. Then we turn to deal with the Total Costs and the Total Cost Ratio. Lastly, we provide some preliminary results in extended models.

Chapter 2

Model

2.1 Coalitional Games Model

Let $G = \{N, S, U\}$ be a non-cooperative game in strategic form. Let $C = \{C_1, \ldots, C_{n^c}\}$ be a Partition of N into n^c nonempty sets. Hence: $\cup_{k=1}^{n^c} C_k = N$ and $C_k \cap C_l = \emptyset \ \ \forall k \neq l \in [1, \ldots n^c]$.

The game G and the partition C form a Coalitional Non-Cooperative (CNC) Game $G^C = \{N^C, S^c, U^c\}$ defined as follows:

- N^C is the set of agents which are the elements of C.
- The strategy space is $S^c = \{S_k^c\}_{k \in C}$ where $S_k^c = \times_{i \in C_k} S_i$.

Note that $\times S_k = \times_{k \in C} \times_{i \in C_k} S_{k,i}$ is isomorphic to $S = \times_{i=1}^{n} S_i$, since we only changed the order of the coordinates. Thus, we can look on s^c as a vector in S.

- The utility function is defined as follows: $\forall s^c \in S^c \quad U_k^c(s^c) = \sum_{i \in C_k} U_i(s^c)$ and $U^c = \{U_k^c\}_{k \in C}$.

Let us define an auxiliary function over the partition: $MC(C) = max_{k \in C}|C_k|$ - describes the size of the maximal element in N^C.

For $L' \subseteq L$ let l denote it's component. For a vector $x \in \times_{l \in L} X_l$ and $L' \subseteq L$ we use the notation $x_{L'}$ to denote the projection of x onto $\times_{l \in L'} X_l$ and $x_{-L'}$ onto $\times_{l \notin L'} X_l$.

A Pure Nash Equilibrium (NE) of the game G^C is a strategy profile $s \in S^C$ such that $\forall k \in N^C$:

$$U_k(s) \geq U_k(s_{-k}, t_k) \;\; \forall t_k \in S_k^c \tag{2.1}$$

We use the term of agent to refer to agents in the game $G = \{N, S, U\}$ and to refer to agents in $G^C = \{N^C, S^C, U^C\}$. Furthermore we use the term *sub agent* to refer to an agent of G in the context of G^C. In G^C agent k will be called a *singleton agent* if $|C_k| = 1$ and *compound agent* if $|C_k| > 1$. We will refer to G as the *Underlying Game*.

For a game G we denote $NE(G)$ the set of Pure NE strategy profiles in G.

2.2 Coalitional Congestion Games

A congestion game is a game $G = \{N, R, \Sigma, P\}$ where N is the finite set of at least 2 agents, R is the finite set of at least 2 resources, $P = \{P_r\}_{r \in R}$ are the resource costs functions, where $P_r : [1, \dots, n] \to \mathbb{R}$ and $\Sigma = \times_{i \in N} \Sigma_i$, where $\Sigma_i \subseteq 2^R$, is the strategy space. Agent i selects $s_i \in \Sigma_i$ and pays $\sum_{r \in s_i} P_r(c(s)_r)$, where $c(s)_r$ is the number of agents who select r in s - $\sum_{j \in N} \mathbb{I}_{\{r \in s_j\}}(s_j)$. In utility terms, the utility of agent i is $U_i(s) = -\sum_{r \in s_i} P_r(c(s)_r)$.

$G = \{N, R, \Sigma, P\}$ is a Simple Congestion Game (SCG) if $\Sigma_i = R \;\; \forall i \in N$.

We will assume that the P_r functions are non-negative and strictly increasing ($P_r(1)$ can be zero). We also define $\Delta_r(j)$ as the increment in cost when the j^{th} agent joins the resource r: $\Delta_r(j) = P_r(j) - P_r(j-1), \; j \in [2, \dots n]$ where we define $\Delta_r(1) = 0$. We will say that in a strategy profile s the resource r has a cost p, if $P_r(c(s)_r) = p$.

Let G be a Congestion Game and C a partition of N as described above. Using C we can form a Coalitional Congestion Game (CCG) G^C as described above. Agent k's strategy in G^C is a vector in $\times_{i \in C_k} \Sigma_i$.

In a SCG strategy profile s we will denote $c(s)_r$ as the number of agents selecting r in the profile s (the congestion of r in s). Similarly for CCG strategy profile s^c we will denote $c(s^c)_r$ as the number of sub agents selecting r in the profile s^c

Fix a SCG (or CCG) G with R resources and n (sub-)agents. A *congestion vector* is an element of $\mathbb{N}^R$ whose elements sum up to n. A SCG (CCG) G and a strategy profile s

induce a congestion vector $c(s)$: $\{c(s)_r\}_{r \in R}$.

Strategy profile s of a Coalitional Congestion Games G^C induces a private congestion vector for each of the agents in G^C. Such vector for agent k will be c_k: $c_k(s_k)_r = |\{i \in C_k : s_{k,i} = r\}|$ which is an element if $\mathbb{N}^R$ whose elements sum up to $|C_k|$.

Let X be a subset of the strategy profiles space. We denote $c(X)$ as the corresponding set of congestion vectors: $\forall X \subseteq S \;\; c(X) = \{c(s) \; s.t : s \in X\}$

Remark 2.1 *Note that for any congestion vector v there is a strategy profile s of the SCG/CCG such that $c(s) = v$.*

Remark 2.2 *Let s_k and s'_k be two strategy profiles of agent k in a CCG. If $c_k(s_k) = c_k(s'_k)$ then $U_l(t_{-k}, s_k) = U_l(t_{-k}, s'_k) \;\; \forall l \in N^C$ and $\forall t \in S^c$.*

There can be many restrictions on the strategies agents may choose. The most natural one is to restrict sub agents of the same agent to select different resources, as follows:

Let G be a SCG. and C be a partition. Using G and C define a Restricted Coalitional Congestion Game $\overline{G^C}$, as a CCG, where sub agents of the same agent are restricted to select different strategies:
$\overline{G^C} = \{N^c, \overline{S^c}, U^c\}$, where N^c and U^c are as before and $\overline{S^c} = \{\overline{S^c_k}\}_{k \in C}$ where $\overline{S^c_k} = \{\times_{i \in C_k} S_{k,i} : s_{k,i} \neq s_{k,j} \;\; \forall i, j \in C_k\}$.

Remark 2.3 *Let G be a SCG and C a partition. The Restricted Coalitional Congestion Game $\overline{G}$ with underlying game G and a partition C is a Congestion game $\{C, R, \Sigma, P\}$ where $\Sigma_k = \{X \subset R : |X| = |C_k|\}$.*

A Coalitional Congestion Game is an extension to the classical model of Congestion Games. In regular congestion games each agent selects a vector in $\{0,1\}^R$ when in CCG he selects a vector in $R^{|C_k|}$, when agent can select a resource more than once. Therefore, a CCG is not a Congestion Game over the original set of resources R.

Remark 2.4 *Throughout this paper we will focus on CCG that have a SCG underlying game.*

2.3 Congestion Distance

Definition 1 *For two congestion vectors u, v we define the congestion distance between u and v as $d(u, v)$ as follows:*

$$d(u,v) = \frac{\sum_{r \in R} |v_r - u_r|}{2} \tag{2.2}$$

Remark 2.5 *Similarly we define distance for agent's private congestion vectors.*

Lemma 2.1 *d satisfies the triangle inequality ($d(u,v) + d(v,w) \geq d(u,w)$)*

Proof:

Let u, v, w be congestion vectors.

$d(u,v) + d(v,w) = \frac{\sum_{r \in R} |v_r - u_r|}{2} + \frac{\sum_{r \in R} |w_r - v_r|}{2} = \frac{\sum_{r \in R} |v_r - u_r| + |w_r - v_r|}{2} \geq \frac{\sum_{r \in R} |v_r - u_r + w_r - v_r|}{2} = \frac{\sum_{r \in R} |w_r - u_r|}{2} = d(u,w)$

□

Corollary 1 *d is a metric, since clearly that d is non-negative, symmetric and $d(u,v) = 0$ iff $u = v$. .*

Lemma 2.2 *Let G be a SCG or a CCG. Let u, v be congestion vectors of G. $d(u,v) \leq n$ and exist congestion vectors $u, v : d(u,v) = n$.*

Proof:

Note that:

$$d(u,v) = \frac{\sum_{r \in R} |v_r - u_r|}{2} \leq \frac{\sum_{r \in R} (|v_r| + |u_r|)}{2} = \frac{\sum_{r \in R} |v_r| + \sum_{r \in R} |u_r|}{2} = \frac{n+n}{2} = n$$

Therefore, $d(u,v) \leq n$

For the second part of the lemma take the following two vectors: $v_1 = n, v_2 = \ldots = v_R = 0, u_2 = n, u_1 = u_3 = \ldots = u_R = 0$. □

Remark 2.6 *Let G be a SCG or a CCG. For any two congestion vectors u, v of G $d(u,v)$ is an integer. To see this note that $\sum_{r:u_r>v_r} u_r - v_r = \sum_{r:v_r>u_r} v_r - u_r$. Therefore, $d(u,v) = \frac{\sum_{r:u_r>v_r} u_r - v_r + \sum_{r:v_r>u_r} v_r - u_r}{2} = \sum_{r:u_r>v_r} u_r - v_r$, which is an integer.*

Definition 2 *Two congestion vectors u, v will be called* adjacent *if $d(u,v) = 1$.*

Remark 2.7 *Let G be a SCG and G^C be a CCG with underlying game G and the partition C. Let s, s' be two strategy profiles in G^C. If $\exists! k$ and $\exists! i \in C_k$ s.t. : $c(s_l) = c(s'_l)$ $\forall l \neq k, s_{kj} = s'_{kj}$ $\forall j \in C_k \backslash \{i\}$ and $s_{ki} \neq s'_{ki}$ then $d(c(s), c(s')) = 1$. This is since the congestion is different in only two resources and by 1 in both of them.*

Definition 3 *A* congestion path *between u and v is a finite sequence of congestion vectors $(w_0, \ldots, w_l)$ where $w_0 = u, w_l = v$ and $d(w_i, w_{i+1}) = 1$ $\forall i \in [0, \ldots, l-1]$. The* length *of such path is l.*

Remark 2.8 *Let G be a SCG or a CCG. Let u, v be two congestion vectors of G and $w_o, \ldots, w_l$ a congestion path between u and v. The triangle inequality implies that $d(u,v) \leq \sum_{i=1}^{l} d(w_i - w_{i-1}) = l$. Therefore, the length of any congestion path between u and v is at least $d(u,v)$.*

Lemma 2.3 *Let there be a SCG (CCG) G with n (sub-)agents. Between any two congestion vectors u, v in G exists a congestion path with length $d(u,v)$.*

Proof:

Proof in induction on $d(u,v)$. If $d(u,v) = 0$ - it is the same vector and the path ($v = w_0 = u$) is a shortest congestion path, with $l = 0$. Similarly for the case when $l = 1$ the proof is trivial. Assume that the lemma is true for vectors u, v with $d(u,v) \leq m$ $(m \geq 1)$. Let u, v be two congestion vectors such that $d(u,v) = m + 1$.

Since $u \neq v$ and the sum of coordinates remains the same $\exists r, x : u_r < v_r$ *and* $u_x > v_x$. Let us define the following vector w as follows:

$$\left\{ \begin{array}{c} w_z = v_z \ \ \forall z \neq x, r \\ w_x = v_x + 1 \\ w_r = v_r - 1 \end{array} \right\}$$

Clearly w is a congestion vector and $d(w, v) = 1$. Also note that:

$$\begin{aligned} m+1 = d(u,v) = \frac{\sum_{z\in R}|v_z - u_z|}{2} = \frac{\sum_{z\neq x,r}|w_r - u_r| + u_x - v_x + v_r - u_r}{2} = \\ = \frac{\sum_{z\neq x,r}|w_z - u_z| + u_x - (w_x - 1) + (w_r + 1) - u_r}{2} = \\ = \frac{\sum_{z\in R}|w_z - u_z| + 2}{2} = d(u,w) + 1 \end{aligned} \tag{2.3}$$

Therefore, w is an adjacent vector to v, such that $d(u, w) = m$. By the induction hypothesis here is a congestion path from u to w of length m and a congestion path from w to v with length 1. Combining them we will get a congestion path of length $m + 1$ from u to v. □

Definition 4 *A congestion path between u and v is a is a* direct congestion path *if it's length is $d(u, v)$.*

A set of congestion vectors will be called connected *if between any two congestion vectors in the set there is a congestion path of elements in the set.*

A set of congestion vectors will be called convex *if between any two congestion vectors in the set there is a direct congestion path of elements in the set.*

Chapter 3

Simple Congestion Games NE

Before diving into the world of Coalitional Congestion Games, we need to equip ourselves with work tools that we will use, from the world of the SCG. Those tools will assist us in the following chapters. Therefore, our first step is to analyze SCG. The main result in this chapter shows that any SCG Pure NE strategy profile is a greedy profile. Among other results in this chapter we show that the set of Pure NE congestion vectors is a convex set and provide a bound for the highest resource cost in any strategy profile.

We use the definition of greedy behavior provided in Fotakis et al. [5]:

Let us consider a dynamic setting with the agents arriving to the game. The agents play only once and irrevocably choose their strategy upon arrival. Each new agent chooses one of the best reply strategies given the choices of the agents currently in the game.

We need some preliminary steps to identify congestion in Pure NE strategy profiles, as follows:

Lemma 3.1 *Let G be a SCG. Then* $\forall s, t \in NE(G)\;\; max_{r\in R} P_r(c(s)_r) = max_{r\in R} P_r(c(t)_r)$

Proof:

For a strategy profile s let us denote $max_{r\in R} P_r(c(s)_r) = P_N(s)$

Let $e, e' \in NE(G)$ where $P_N(e) < P_N(e')$.

Let us denote one of the resources with cost $P_N(e')$ in e' as x and denote an agent who chooses x in e' as i. We know that:

$$P_x(c(e_x)) \leq P_N(e) < P_N(e') = P_x(c(e'_x)) \tag{3.1}$$

Due to monotonicity of the cost functions we get that $c(e)_x < c(e')_x$. The total number of agents is the same in e and e', thus there is a resource r satisfying $c(e)_r > c(e')_r$. The highest cost in e was $P_N(e)$, thus $P_r(c(e)_r) \leq P_N(e)$. Combined with Equation 3.1 we get:

$$P_r(c(e'_r+1)) \leq P_r(c(e_r)) \leq P_N(e) < P_N(e') = P_x(c(e'_x)) \tag{3.2}$$

Therefore, agent i who chose x in e' can deviate to r and pay $P_r(c(e'_r)+1)$, which is less than $P_x(c(e')_x) = P_N(e')$. This contradicts the fact that e' is a Pure NE. □

Definition 5 *We will denote the highest cost for a resource in a Pure NE strategy profile of the SCG G as P_N: $P_N = P_N(s)$ when $s \in NE(G)$. For each resource r we define $\alpha_r : P_r(\alpha_r+1) \geq P_N$ and $P_r(\alpha_r) < P_N$.*

Lemma 3.2 *Let G be a SCG. Then $\exists K : |\{r : P_r(s) = P_N\}| = K \;\; \forall s \in NE(G)$*

Proof:

Let us denote the set of resources that $P_r(\alpha_r+1) \neq P_N$ (actually it is $> P_N$) as X.

From Lemma 3.1 we know that in all Pure NE of G all agents pay no more than P_N. Thus no resource is selected by more than α_r+1 agents. In addition, since exists agents that pay P_N, in order to prevent deviations all resources are selected by at least α_r agents. Therefore we can say:

$$e_r = \alpha_r \;\; \forall e \in NE(G) \;\; \forall r \in X \; \forall e \in NE(G) \tag{3.3}$$

$$\alpha_r \leq e_r \leq \alpha_r + 1 \;\; \forall e \in NE(G) \;\; \forall r \notin X \tag{3.4}$$

Let $e \in NE(G)$. Let us denote the number of resources selected by α_r+1 agents in e as K. Combined with Equations 3.3 and 3.4 we get:

$$\sum_{r\in R} \alpha_r = \sum_{r:e_r=\alpha_r} e_r + \sum_{r:e_r=\alpha_r+1} e_r - 1 = (\sum_{r\in R} e_r) - K = n - K \tag{3.5}$$

Let $e' \in NE(G)$. From equations 3.3, 3.4 and 3.5 we can say that:

$$\sum_{r\in R} c(e')_r = n = \sum_{r\in R} \alpha_r + K \tag{3.6}$$

$$\alpha_r \leq c(e')_r \leq \alpha_r + 1 \tag{3.7}$$

In e' no resource is selected by less than α_r agents. Thus in e' there are exactly K resources selected by $\alpha_r + 1$ and agents selecting them receive P_N, due to Lemma 3.1, when all other resources are selected by α_r agents. □

Definition 6 *In a SCG G, with P_N as defined in Definition 5, we define the number of resources with cost P_N in a Pure NE strategy profile of G as K.*

Lemma 3.3 *Let G be a SCG. Let P_N, K and α_r be as defined in Definitions 5 and 6. In any strategy profile s where there are resources with the cost of P_N and all other resources selected by α_r agents, is a Pure NE.*

Proof:

Note that due to Lemma 3.1 all agents in the strategy profile s pay at most P_N. Resources with cost lower than P_N are selected by α_r agents in s. Thus, if agent i deviates from s_i and selects $r \neq s_i$ we can say that $c(s_{-i}, r)_r \geq \alpha_r + 1$. Thus $P(c(s_{-i}, r)_r) \geq P_N$, which means agent i pays weakly more when selecting r than s_i. Therefore, there is no profitable deviation from s. □

Combining lemmas 3.1, 3.2 and 3.3 we have that:

Corollary 2 *For any SCG exists a natural K, $P_N \in \mathbb{R}^+$ and a set of natural numbers $\{\alpha_r\}_{r\in R}$ derived from P_N, as described in Definitions 5 and 6, such that a strategy profile s is a Pure NE iff in s there are K resources with the cost is P_N and if $P_r(c(s)_r) \neq P_N$ then $c(s)_r = \alpha_r$.*

Remark 3.1 *Let e and e' be two strategy profiles of G, where $c(e) = c(e')$. If e can be attained by greedy behavior then due to agents symmetry e' also can be attained by greedy behavior.*

Theorem 1 (Greedy EQ) *Let G be a SCG. Any Pure NE Strategy Profile of G is attained by greedy behavior*

Proof:

Let $e \in NE(G)$. We will show that a profile e' where $c(e') = c(e)$ can be attained by greedy behavior.

Let us denote $X = \{r : P_r(\alpha_r + 1) = P_N\}$.

Let us denote X_e the set of resources with cost P_N in e. From Lemma 3.1 $|X_e| = K > 0$. Clearly $X \supseteq X_e$.

Let us denote the congestion vector attained after the first $n - K$ greedy agents chose their resources as e^*. Note that $\sum_{r \in R} \alpha_r = n - K$. Thus the first $n - K$ chose resources with cost less than P_N (when they chose it) and we can say that:

$$e^*_r = \alpha_r \; \forall r \in R \tag{3.8}$$

The last K greedy agents chose resources with the minimal cost. Due to Definition 5 $P_r(\alpha_r + 1) \geq P_N$. Clearly $|X| > |X_e| = K$. Thus there are at least K resources satisfying:

$$P_r(\alpha_r + 1) = P_N \tag{3.9}$$

Therefore the last K greedy agents will chose different resources in X, due to monotonicity of P. Let us denote the profile e' as a profile where the last K greedy agents chose the resources in X_e. Note that:

$$c(e) = \alpha_r + 1 = c(e^*) + 1 = c(e') \;\; \forall r \in X_e \tag{3.10}$$

$$c(e) = \alpha_r = c(e^*) = c(e') \;\; \forall r \in R \setminus X_e \tag{3.11}$$

Therefore, $c(e) = c(e')$. From Remark 3.1 we get that e can be attained by greedy behavior. ■

Corollary 3 *Let G be a SCG. The profile e of G is a pure NE iff it is attained by greedy behavior*

From Theorem 1 we have one direction. The other direction is Proven by Fotakis et al. in [5], Theorem 1, where they show that a greedy behavior of agent leads to a Pure NE strategy profile of G.

Corollary 4 *Let G be a SCG. If $P_r(m) \neq P_x(m') \quad \forall m, m' \in \mathbb{N}$ and $\forall r, x \in R$ then $|c(NE(G))| = 1$.*

Proof:

The cost of P_N is attained on a unique resource - r. Thus from Lemma 3.1:

$$c(e)_r = \alpha_r + 1 \ \ \forall e \in NE(G) \tag{3.12}$$

From Corollary 2:

$$c(e)_x \geq \alpha_x \ \ \forall x \in R \setminus \{r\} \ \ \forall e \in NE(G) \tag{3.13}$$

Combined with Lemma 3.1 and the fact that that $P_x(\alpha_x + 1) > P_N \ \forall x \neq r$ we get:

$$c(e)_x = \alpha_x \ \ \forall x \neq r \ \ \forall e \in NE(G) \tag{3.14}$$

Thus, all $e \in NE(G)$ have the same congestion vector. □

Corollary 5 *If $c(NE(G) = \{u\}$ then $P_r(u_r + 1) > P_N \ \forall r \in R$*

Proof:

From P_N definition we know that there is a resource x satisfying $P_x(u_x) = P_N$. Clearly $P_x(u_x + 1) > P_N$. Assume that exists a resource r satisfying $P_r(u_r + 1) \leq P_N$. Due to Corollary 2 we can say that $P_r(u_r + 1) = P_N$. Let s be a strategy profile satisfying $c(s) = u$. Let us denote the strategy profile derived from s by moving one agent from x to r as t. Obviously $c(s) \neq c(t)$. Note that t is a greedy strategy profile as s, since the difference between s and t is in the behavior of the last greedy agent who selects r or x, since in both cases the agent pays P_N. According to Theorem 1, $t \in NE(G)$ and $c(t) \in c(NE(G))$ in contradiction to the fact that $c(NE(G)) = u$. □

Corollary 6 *If $|c(NE(G))| > 1$ then for any resource r exists a pure NE strategy profile s such that $c(s)_r = \alpha_r$. Moreover, in any Pure NE strategy profile s exists a resource r such that $P_r(c(s)_r + 1) = P_n$*

Proof:

If $P_r(\alpha_r + 1) > P_N$ then it is obvious. If $P_r(\alpha_r + 1) = P_N$ note that in G there are at least $K + 1$ resources satisfying $P_x(\alpha_x + 1) = P_N$ because there is more than one way to arrange the last K greedy agents. One of those ways leaves α_r agents on resource r.

The second part is obvious since there are more than K resources satisfying $P_r(\alpha_r + 1) = P_n$. One of those has the congestion of α_r. □

Now we provide an important property of SCG, which will hold also for CCG and will be used later in this paper.

Lemma 3.4 (P_N in all profiles) *Let G be a SCG with n agents and P_N as described in Definition 5. In any strategy profile of G there is an agent paying P_N or more.*

Proof:

Assume there is a strategy profile s in which all agents pay less than P_N. Any resources r is selected at most by α_r agents. Thus, all the total number of agents selecting resources is at most $\sum_{r \in R} \alpha_r = n - K$ agents. From Lemma 3.1 we know that $K > 0$. So, we have less than n agents selecting resources in the strategy profile s which is a contradiction. □

Corollary 7 *The following is a restatement of Lemma 3.4: In any strategy profile of a CCG with an underlying game G and any partition C there is a sub agent paying P_N or more.*

Lastly, we provide an interesting property of congestion vectors corresponding to Pure NE strategy profiles:

Lemma 3.5 *Let G be a SCG with n agents. $c(NE(G))$ is convex*

Proof:

Let $u, v \in c(NE(G))$ and t the NE strategy profile such that $c(t) = v$. From Corollary 2 we know that:

$$u_r - 1 \leq v_r \leq u_r + 1 \;\; \forall r \in R, \;\; \forall u, v \in c(NE(G)) \tag{3.15}$$

Let us denote $U^+ = \{r : u_r = v_r + 1\}$ and $V^+ = \{r : v_r = u_r + 1\}$.

We use induction on $d(u,v)$ to prove this result. If $d(u,v) \leq 1$ then it is obvious. Assume that exists an integer $m \geq 1$ such that for any $u, v \in NE(G)$ with $d(u,v) \leq m$ there is a direct congestion path in $c(NE(G))$.

Let u, v be two congestion vectors in $c(NE(G))$ such that $d(u,v) = m+1$. Since the sum of coordinates is the same in u and v we have $|U^+| = |V^+|$. Since $\sum_{r \in R} |u_r - v_r| = \sum_{r \in U^+ \cup V^+} |u_r - v_r| = |U^+| + |V^+|$, we have that $d(u,v) = |U^+| = |V^+|$, therefore if $d(u,v) > 0$ then $|U^+| = |V^+| > 0$. Note that from Corollary 2 and Equation 3.15 we know that:

$$u_x > v_x \Rightarrow u_x = \alpha_x + 1 \; and \; v_r > u_r \Rightarrow v_r = \alpha_r + 1 \qquad (3.16)$$

Let us denote two resources $r \in V^+$ and $x \in U^+$. Let us denote strategy profile derived from t when we move one agent from r to x as s. We know that:

$$c(s)_r = u_r = \alpha_r + 1 \; and \; c(s)_x = c(t)_x - 1 = \alpha_x \qquad (3.17)$$

Combining Equations 3.16 and 3.17 we have that $c(s)_x = u_x = \alpha_x + 1$. Since $u \in c(NE(G))$ we know that $P_x(c(s)_x) = P_x(u_x) = P_N$, from Lemma 3.1. In s resource r is selected by α_r agents and x by $\alpha_x + 1$, whereas in t it is reversed. From the definition of s, $c(s)_z = c(t)_z \; \forall z \neq r, x$. Combined with Corollary 2 we get that s is a Pure NE strategy profile. Clearly $d(c(t), c(s)) = 1$ and $d(u, c(s)) = m$. Thus, from induction hypothesis there is a direct congestion path in $c(NE(G))$ from u to $c(s)$ and from $c(s)$ to v. When the two paths are combined we get a direct path from u to v. $\square$.

Chapter 4

Congestion Games and CCG Properties

A central observation due to Rosenthal [18] is that Congestion Games have a Pure NE. In Monderer and Shapley [13] it is shown that those games have an Exact Potential. An exact potential leads to the existence of a pure NE. Monderer and Shapley [13] also introduce the Finite Improvement Path (FIP) property, which is weaker than Exact Potential, but still leads to the existence of a Pure NE.

A natural question is whether those properties of Congestion Games hold in CCG. We will show that in the general case CCG do not have any of these properties. We will show some necessary conditions on G and C to ensure some of these properties and some negative conditions.

4.1 Pure NE

One of the more important characteristics of congestion games, as proven in Rosenthal [18], is that such games posses a Pure NE. An important question is whether this property holds in CCG. A partial answer to this is provided by Hayrapetyan [7] who show that if the CCG underlying game is a SCG and the costs are weakly convex ($\Delta_r(m) \leq \Delta_r(m+1) \;\; \forall r \in R, m \in [1, \ldots, n-1]$) the game has a pure NE. We provide some conditions on the partition. For example we show that any CCG with $MC(C) = 2$ posses a Pure NE, but this result cannot be extended to the general case when $MC(C) \geq 3$.

4.1.1 Positive Pure NE Results

For the positive result we will need some preliminary steps:

Lemma 4.1 *Let G be a SCG and C a partition of N. Let $\overline{G^C}$ be a Restricted CCG with the underlying game G and the partition C. Let s be a strategy profile of $\overline{G^C}$ (where $s_{k,i} \neq s_{k,j} \ \ \forall k \in N^C, \forall i, j \in C_k$). If $c(s) \in c(NE(G)) \Rightarrow s \in NE(\overline{G^C})$*

Proof:

Let us denote the set of all the best reply strategies to s_{-k} as $BR(s_{-k})$. Let us denote the closest best reply strategy to s_k in $BR(s_{-k})$ as t_k, meaning $d(c(t'_k), c(s_k)) \geq d(c(t_k), c(s_k))$ for any best reply strategy $t'_k \in BR(s_{-k})$.

Assume $c_k(t_k) \neq c_k(s_k)$. Since s_k and t_k are both strategy profiles of $\overline{G^C}$, agent k select each resource once both in t_k and s_k. Thus exist resources r and x such that: $c_k(t_k)_r = 1, c_k(s_k)_r = 0, c_k(s_k)_x = 1$ and $c_k(t_k)_x = 0$. Let us denote the sub agent of k who selects r in t_k as i. Note that agent k select each resource at most once (Restricted CCG) and all other agents behave the same in s and (s_{-k}, t_k), thus:

$$c(s)_r + 1 = c(t_k, s_{-k})_r \textit{ and } c(s)_x = c(t_k, s_{-k})_x - 1 \tag{4.1}$$

Let t'_k be agent k strategy derived from t_k by moving sub agent i from r to x. From the lemma condition we know that $c(s) \in NE(G)$, thus $P_x(c(s)_x) \leq P_r(c(s)_r + 1)$. From Equation 4.1 and t'_k definition we know that $c(s_{-k}, t'_k)_x = c(s)_x$ and $c(s_{-k}, t'_k)_r = c(s)_r$. Thus:

$$P_x(c(s_{-k}, t'_k)_x) = P_x(c(s)_x) \leq P_r(c(s)_r + 1) = P_r(c(s_{-k}, t_k)_r) \tag{4.2}$$

Therefore, sub agent i pays weakly less in (t'_k, s_{-k}) than in (t_k, s_{-k}). Since i is the only sub agent of agent k selecting r or x in t_k and t'_k agent k pays weakly less too. Note that all other sub-agents of k pay the same in (t_k, s_{-k}) and (t'_k, s_{-k}). Since $t_k \in BR(s_{-k})$ we have that $t'_k \in BR(s_{-k})$. Obviously $d(c(t_k, s_{-k}), c(s)) = 1 + d(c(t'_k, s_{-k}), c(s))$, which contradicts the fact that t_k is the closest best reply strategy to s_k in $BR(s_{-k})$. Therefore, $c_k(t_k) = c_k(s_k)$

From Remark 2.2, s_k is also a best reply strategy to s_{-k}. Since k is an arbitrary agent, s is a Pure NE strategy profile because it is attained in best reply strategies. $\square$.

The following Theorem provides us with a structural condition on the partition for Pure NE existence in CCG. This condition depends only on the structure and the congestion vector of a Pure NE of the underlying game.

Theorem 2 (Layer NE) *Let G be a SCG and C a partition of N. Let G^C be a CCG with the underlying game G and the partition C. Let s be a strategy profile of G^C where $s_{k,i} \neq s_{k,j} \ \forall k \in N^C, \forall i, j \in C_k$. If $c(s) \in c(NE(G)) \Rightarrow s \in NE(G^C)$.*

Proof:

Let $\overline{G^C}$ be a restricted CCG with the same underlying game and partition as G^C.

Let s be a profile as described in the Theorem. Let t_k be the best reply strategy for agent k to s_{-k}. We show that $c_k(t_k)_r \leq 1 \ \forall k \in N^C$ and $\forall r \in R$.

Assume this is not true and agent k has a profitable deviation from s_k to t_k, where $t_{k,i} = t_{k,j} = r$. Since $c_k(s_k)_r \leq 1$ we know that $c(s_{-k}, t_k)_r \geq c(s)_r + 1$. Since $c(s_{-k}, t_k)_r > c(s)_r$ exists a resource x such that $c(s_{-k}, t_k)_x + 1 \leq c(s)_x$.

Let t'_k be a strategy profile derived from t_k by moving sub agent i from r to x. In the strategy profile (s_{-k}, t'_k) agent i pays $P_x(c(s_{-k}, t'_k)_x)$. Note that:

$$c(s_{-k}, t'_k)_x = c(s_{-k}, t_k)_x + 1 \leq c(s)_x \tag{4.3}$$

From Equation 4.3 and the fact that $c(s) \in NE(G)$ we get that:

$$P_x(c(s_{-k}, t'_k)_x) = P_x(c(s_{-k}, t_k)_x + 1) \leq P_x(c(s_x)_x) \leq P_r(c(s)_r + 1) \tag{4.4}$$

Using monotonicity of the cost functions and the fact that $c(s)_r + 1 \leq c(s_{-k}, t_k)_r$ we get:

$$P_r(c(s)_r + 1) \leq P_r(c(s_{-k}, t_k)_r) \tag{4.5}$$

Combining Equations 4.4 and 4.5 we get that

$$P_x(c(s_{-k}, t'_k)_x) \leq P_r(c(s_{-k}, t_k)_r) \tag{4.6}$$

Thus, sub agent i pays in (s_{-k}, t'_k), where he chose x, no more than than in (s_{-k}, t_k), where he chose r. Note that sub agent j, who selects r both in (s_{-k}, t'_k) and (s_{-k}, t_k), pays strictly less in (s_{-k}, t'_k) than in (s_{-k}, t_k), because $c(s_{-k}, t'_k)_r < c(s_{-k}, t_k)_r$. From definition of x, $c_k(t_k)_x < c_k(s_k)_x$. Since $c_k(s_k)_x = 1$ and $c_k(s_k)_x > c_k(t_k)_x$, we get that

$c_k(t_k)_x = 0$. All sub agents who choose $R \setminus \{r, x\}$ pay the same in (s_{-k}, t_k) and (s_{-k}, t'_k). To conclude, agent k, who doesn't select x, but selects r in t_k pays strictly less in (s_{-k}, t'_k) than in (s_{-k}, t_k). This contradicts the fact that t_k is a best reply to s_{-k}.

Thus, agent k's best reply strategy to s_{-k} is a strategy that is allowed also in $\overline{G^C}$. Moreover, due to Theorem assumptions all other agents also play strategies allowed in $\overline{G^C}$. Therefore, due to Lemma 4.1 we can say that s_k is also a best reply strategy to s_{-k}. ■

Corollary 8 *Let G be a SCG. Let C be a partition of N with $MC(C) = 2$. Let G^C be the CCG with the underlying game G and the partition C. If there exists a pure NE strategy profile e of G where $\forall r \in R \;\; c(e)_r \leq n^c$ then $NE(G^C) \neq \emptyset$.*

Proof:

We show that it is possible to arrange the agents in a profile s such that $c(s) = c(e)$, where all compound agents choose two different resources.

Assume there are ν compound agents - $1, \ldots, \nu$ and $n^c - \nu$ singleton agents $\nu + 1, \ldots, n^c$ in C. We will denote the two sub agents of compound agent k as k_f and k_s. Let us arrange the sub agents in the following order: We first allocate a single sub agent from each compound agent $(1_f, \ldots \nu_f)$. Then we allocate the singleton agents - $\nu + 1, \ldots n^c$. Lastly, we allocate the remaining sub agents, by the same order we put their partners $(1_s, \ldots \nu_s)$. We will use $y_1, \ldots, y_n$ to denote the agents in this order.

Let us define a strategy profile s of G^C as follows: Take the sub agents $y_1, \ldots, y_{c(e)_1}$ and allocate them on resource 1. Take the sub agents $y_{c(e)_1+1}, \ldots, y_{c(e)_2}$ and allocate them on resource 2 and so on. Obviously, $c(s) = c(e)$. Note that in y any compound agent has exactly $n^c - 1$ places between his sub-agents. Since $c(e)_r \leq n^c$, they will be allocated on different resources. Applying Theorem 2 on s we get that $s \in NE(G^C)$. □

Now we concentrate on CCG with partitions satisfying $MC(C) = 2$ and show that such CCG have a Pure NE. We will check what deviations can be profitable in such games and show that exists a strategy profile where all the possible deviations are unprofitable.

Lemma 4.2 *Let G^C be a CCG with an underlying SCG G and a partition C. Let $|C_k| = 2$. Let s be a strategy profile of G^C, where agent k selects two different resources. If there is no profitable deviation to any sub agent of agent k then there is no profitable deviation to agent k.*

Proof:

Assume that agent k has deviates from s_k to t_k. Let us distinguish between the various sizes of $|s_k \cap t_k|$ - 0,1,2 and show that in all cases it is an unprofitable one.

- If $|s_k \cap t_k| = 2$ then from Remark 2.2 agent k pays the same both in s and (s_{-k}, t_k).
- If $|s_k \cap t_k| = 1$ then $c_k(t_k)$ can be attained from $c_k(s_k)$ by a deviation of a single sub agent. From Remark 2.2 agent k pays the same in (s_{-k}, t_k) and (s_{-k}, t'_k) if $c_k(t_k) = c_k(s'_k)$. Thus WLOG t_k is attained form s_k by deviation of one sub agent. By our assumption any deviation of a sub agent from s is unprofitable. Therefore, the sub agent deviating from s_k to t'_k pays weakly more. Moreover, the other sub agent pays weakly more too, since agent k chose two different resources in s_k. Thus, agent k pays weakly more in (s_{-k}, t_k) than in s.
- If $|s_k \cap t_k| = 0$ then t_k is attained after two unprofitable deviations from s_k and each of the sub agents pays more in t_k then in s_k, meaning agent k pays more in t_k than in s_k.

Therefore, if there is no profitable deviation to a sub agent of k there is no profitable deviation for agent k. $\square$.

Definition 7 *Let C be a partition with $MC(C) = 2$. Let G^C be a CCG with the partition C. Let s be a strategy profile of G^C. We will denote $SA(s)$ the set of singleton agents of G^C, $DR(s)$ the set of compound agents who select different resources in s and $UR(s)$ the set of compound agents who select a unique resource twice in s.*

Lemma 4.3 *Let G^C be a CCG with an underlying SCG G and a partition C with MC(C)=2. Let $s \notin NE(G^C)$, where all agents select resource b at least once. Let us denote one of the resources in $R \setminus \{b\}$ satisfying $P_r(c(s)_r + 1) \geq P_x(c(s)_x + 1)\ \forall x \in R \setminus \{b\}$ as r. Let us denote an agent $k \in UR(s)$. Assume that all agents in $SA(s)$ and $DR(s)$ play their best reply strategies in s and $t_k = (r, b)$ is a profitable deviation for agent k. Then all agents in $SA(s_{-k}, t_k) \cup DR(s_{-k}, t_k)$ play their best reply strategies in (s_{-k}, t_k).*

Proof:

Let us check all possible deviations for the relevant agents in (t_k, s_{-k}):

Agents in $SA(t_k, s_{-k})$won't deviate, since in (s_{-k}, t_k) the congestion, thus also the cost of b is strictly lower than in s whereas the congestion and cost of all other resources weakly higher and already in s they played their best reply strategies.

Consider an agent in $DR(t_k, s_{-k})$ We check the consequences of possible deviations:

1. Moving a sub agent between resources in $R \setminus \{b, r\}$ is not strictly profitable since doing it was unprofitable in s and their congestion is the same in s and (s_{-k}, t_k) it is not profitable also in (s_{-k}, t_k).

2. Moving a sub agent from r to $R \setminus \{b, r\}$ is unprofitable in s. Since the cost of r is the only resource whose cost in s is strictly less than in (s_k, t_k) it is unprofitable in (s_k, t_k).

3. Due to the definition of r we can say that for any $x \neq b$:

$$P_r(c(s_{-k}, t_k)_r) = P_r(c(s)_r + 1) \geq P_x(c(s)_x) = P_x(c(s_{-k}, t_k)_x) \tag{4.7}$$

   Thus moving a sub agent from a resource in $R \setminus \{b\}$ to r is unprofitable for the sub agent in (s_{-k}, t_k).

4. Since the sub agents in $R \setminus \{r\}$ had no profitable deviations in s we know that:

$$P_x(c(s_{-k}, t_k)_x) = P_x(c(s)_x) \leq P_x(c(s)_r + 1) = P_r(c(s_{-k}, t_k)_r) \; \forall x \in R \setminus \{b\} \tag{4.8}$$

   And since the deviation from s to (s_{-k}, t_k) was profitable for agent k, we can say that for any resource $x \neq b$:

$$P_x(c(s_{-k}, t_k)_x) + P_b(c(s_{-k}, t_k)_b) \leq P_r(c(s_{-k}, t_k)_r) + P_b(c(s_{-k}, t_k)_b) < 2P_b(c(s)_b) \tag{4.9}$$

   Combining Equation 4.8 and 4.9 we conclude that moving a sub agent from a resource in $R \setminus \{b\}$ back to b is unprofitable.

Combining all the steps so far with Lemma 4.2 we get that for agents in $DR(t_k, s_{-k})$ there is no profitable deviation. from (s_{-k}, t_k). □

Lemma 4.4 *Let C be a partition with $MC(C) = 2$. Let G^C be a CCG with the partition C and an underlying SCG G. Let $s \notin NE(G^C)$ where the resource b is selected by all agents. Assume $P_b(c(s)_b) \leq P_N$ and $P_r(c(s)_r) \geq P_N \; \forall r \in R \setminus \{b\}$. If $s_k = (b, b)$. Then in any best reply strategy t_k to s_{-k} $c_k(t_k)_b > 0$.*

Proof:

Assume exist resources $r, x \neq b$ such that $w_k = (r, x)$ when $r, x \neq b$. Note that $c(w_k, s_{-k})_r = c(s)_r + 1$ and $c(w_k, s_{-k})_x = c(s)_x + 1$. Therfore:

$$P_r(c(w_k, s_{-k})_r) + P_x(c(w_k, s_{-k})_x) \geq 2P_N \geq 2P_b(c(s)_b) \tag{4.10}$$

Therefore, in (w_k, s_{-k}) agent k pays at least as in s. Therefore, if w_k is a best reply strategy to s_{-k} then s_k is too. Due to Lemma 4.3 if s_k is a best reply strategy to s_{-k} then s is a Pure NE, since agent in $UR(s)$ has no profitable deviation. This contradicts the assumption that $s \notin NE(G^C)$. □

Corollary 9 *Let s be a strategy profile satisfying the conditions of Lemma 4.4. Then s is a Pure NE of G^C iff for agent $k \in UR(s)$ there is no profitable deviation with one sub agent.*

Now we turn to to prove that all CCG games with $MC(C) = 2$ have a pure NE.

Theorem 3 *Let G be a SCG, C a partition where $MC(C) \leq 2$ and G^C a CCG with the underlying game G and the partition C. Then $NE(G^C) \neq \emptyset$*

Proof:

G is a SCG. Therefore, due to Rosenthal [18], $NE(G) \neq \emptyset$.

Let $e \in NE(G)$. If $c(e)_r \leq n^c \; \forall r \in R$ then from Corollary 8 $NE(G^C) \neq \emptyset$. Therefore, we can assume that:

$$\exists b \;\; such \; that \; c(e)_b > n^c \tag{4.11}$$

Note that such resource b is unique since $n^c \geq \frac{n}{2}$. Clearly it is possible to construct a strategy profile s^0 of G^C satisfying the following restrictions:

1. All singleton agents choose b.

2. Among all compound agents, at least one sub agent chooses b.

3. The other sub agents are spread among the resources, so that $c(e)_r = c(s^0)_r \; \forall r \in R$.

Obviously, $c(s^0) = c(e)$ thus:

$$\begin{aligned} c(s^0) = c(e) \in NE(G) \\ \Downarrow \\ P_b(c(s^0)_b \leq P_N \leq P_r(c(s^0)_r + 1) \\ P_r(c(s^0)_r + 1) \geq P_x(c(s^0)_x)\ \forall r, x \in R \end{aligned} \tag{4.12}$$

Due to Theorem 2 if we would refer to all compound agents in $UR(s^0)$ as two singleton agents we would have a Pure NE. Therefore, all agents in $SA(s^0)$ and $DR(s^0)$ play their best reply strategies in s^0. From this if $UR(s^0) = \emptyset$ than s^0 is a Pure NE. To conclude, we can assume that s^0 is not a Pure NE, otherwise $NE(G^C) \neq \emptyset$, therefore $|UR(s^0)| > 0$.

Let m be a positive integer.

Due to Equations 4.11 and Corollary 9 if s^{m-1} is not a Pure NE and $P_b(c(s^{m-1})_b \leq P_N \leq P_r(c(s^{m-1})_r + 1)$ then the deviation described in Lemma 4.3 is profitable in s^{m-1}. For such s^{m-1} (that can be s^0 due to Equation 4.12) let s^m be the strategy profile derived from s^{m-1}, when an agent in $UR(s^{m-1})$ deviates as described in Lemma 4.3 - with one sub agent who deviates greedily.

Note that $|UR(s^m)| = |UR(s^{m-1})| - 1$ and due to Lemma 4.3 all agents in $n \setminus UR(s^m)$ play their best reply strategies in s^m. Clearly $P_b(c(s^m)_b \leq P_N \leq P_r(c(s^m)_r + 1)\ \forall r \neq b$, thus Corollary 9 holds for s^m. Therefore, if agent $k \in UR(s^m)$ cannot profitably deviate with one sub agent s^m is a Pure NE.

If exists $m < |UR(s)|$ such that s^m is a Pure NE we are done. On the other hand, if s^m is not a Pure NE, we can define s^{m+1}, as described above.

If $s^0, s^1, \ldots s^{|UR(s^0)|-1}$ aren't Pure NE, then we can define $s^{|UR(s^0)|}$. Clearly $UR(s^{|UR(s^0)|}) = \emptyset$ thus due to Lemma 4.3 $s^{|UR(s^0)|}$ is a Pure NE. ■

Corollary 10 *Let G be a SCG, C a partition with $MC(C) = 2$. Let G^C a CCG and $\overline{G^C}$ a restricted CCG, both with the same underlying game G and same partition C. If $c(NE(G)) \setminus c(\overline{G^C}) \neq \emptyset$ then exists a Pure NE s of G^C where b is selected by all agents of G^C at least once and $c(s)_b < c(t)_b\ \forall t \in NE(G)$.*

4.1.2 Non-Existence of Pure NE in Wider Classes

Now we provide a counter example, to show that the $MC(C) = 2$ condition is essential. In Example 7 we will show that if the underlying game is not a SCG then no Pure NE exist.

Example 2 *A Coalitional Congestion Game with $MC(C) > 2$ may not posses a Pure NE.*

Consider a game with two identical resources A and B and four sub agents, with the following P functions:

Resource / Agents #:	1	2	3	4
A:	0	**12**	16	18
B:	0	**12**	16	18

From Theorem 1 the profile where agents split equally among resources is a Pure NE. For example, Agents 1 and 2 choose resource A and agents 3 and 4 choose resource B.

When $C = [\{1, 2, 3\}, \{4\}]$ then G^C is the following 2 player game:

G^C	A	B
A,A,A	-54, -18	-48, 0
A,A,B	-32, -16	-36, -12
A,B,B	-36, -12	-32, -16
B,B,B	-48, 0	-54, -18

Note that this game has no pure NE, despite the fact that the SCG had a pure NE. This is since AAB and ABB are dominating strategies for the compound agent. If we omit the other two strategies we will get a matching pennies games, that posses no Pure NE.

4.2 Exact Potential

An Exact Potential defined by Monderer and Shapley [13] is a function from the strategy space (of all agents) to $\mathbb{R}$ which satisfies:

$$\mathbb{P}(s) - \mathbb{P}_(s_{-i}, t_i) = U_i(s) - U_i(s_{-i}, t_i) \forall i \in N, \forall t_i \in S_i, \forall s \in S_1 \times S_2 \ldots \times S_n \quad (4.13)$$

Monderer and Shapley in [13] show that games without a Pure NE do not posses an Exact Potential. As previously noted, Rosenthal showed that Congestion Games posses a Pure NE in [18].

Definition 8 *A CCG will be called linear if the resource cost functions are linear.*

In this section we analyze conditions under which games posses an Exact Potential. Fotakis et al. showed in Theorem 6 in [6] that a CCG is linear possesses an Exact Potential. We will show the other direction under minimal demands from the partition C. First note that the condition $MC(C) = 2$ which due to Theorem 3 is sufficient to prove existence of Pure NE does not guarantee existence of Exact Potential, as the following lemma shows:

Lemma 4.5 *Let G be a SCG with 2 resources, 3 sub agents and $C = [\{1, 2\}\{3\}]$. Let G^C be a CCG. G^C has an Exact Potential iff the CCG is linear*

Proof:

Let G^C be the following game when we denote the costs functions P_A and P_B of G as follows:

Resource / Agents #:	1	2	3
A:	a_1	a_2	a_3
B:	b_1	b_2	b_3

The game G^C will look as follows:

G^C	A	B
A,A	$2a_3, a_3$	$2a_2, b_1$
A,B	$a_2 + b_1, a_2$	$a_1 + b_2, b_2$
B,B	$2b_2, a_1$	$2b_3, b_3$

From Monderer and Shapley [13], Theorem 2.8, Exact Potential exists iff following equations are true:

$$a_2 + b_1 - 2a_3 + a_3 - b_1 + 2a_2 - a_1 - b_2 + b_2 - a_2 = 0$$
$$a_2 + b_1 - 2b_2 + a_1 - b_3 + 2b_3 - a_1 - b_2 + b_2 - a_2 = 0 \quad (4.14)$$
$$2a_3 - 2b_2 + a_1 - b_3 + 2b_3 - 2a_2 + b_1 - a_3 = 0$$

After simplifying the equations we get:

$$2a_2 = a_1 + a_3$$
$$2b_2 = b_1 + b_3$$

Which can happen iff the cost functions are linear. Thus, iff the cost functions are linear the game G^C possesses an Exact Potential. □

Now we extend this result to a larger set of CCG:

Lemma 4.6 *Let G be a SCG with $n \geq 3$. Let C be a partition of N that has at least one element of size 1 and at least one element of size 2. Let G^C be a CCG with a partition C and the underlying game G. If the cost functions aren't linear ($\exists r \in R$ and $m \in \mathbb{N}$ such that $P_r(m+1) \neq P_r(m) + P_r(m+2)$) G^C will not posses an Exact Potential.*

Proof:

Let us denote the resource that has a non-linear cost functions as A, meaning that exists a natural $0 < m < n-1$ such that $2P_A(m+1) \neq P_A(m) + P_A(m+2)$.

Let us denote the singleton agent as i and the agent with $|C_k| = 2$ as k. Let us denote a strategy profile where $m-1$ of the sub agents of agents in $N^C \setminus \{k, i\}$ select A as $s_{-\{k,i\}}$. Let us denote as $l-1$ the number of sub agents of agents in $N^C \setminus \{k, i\}$ who select B in $s_{-\{k,i\}}$.

Let us denote the following game derived from G^C as G': All agents strategies except i and k have one strategy - the one they play in $s_{-\{k,i\}}$. Agents i and k will be allowed to choose only resources A and B. Note that G' looks exactly as the game in Lemma 4.5.

Due to Lemma 4.5 G' has no Exact Potential since P_A is not linear: $2P_A(m+1) \neq P_A(m) + P_A(m+2)$.

From Theorem 2.8 in Monderer and Shapley [13] we know that if G' doesn't posses Exact Potential then also G^C does not posses an Exact Potential. Therefore, G^C doesn't posses an Exact Potential. $\square$.

Combining Lemma 4.6 with Theorem 6 in Fotakis et al. [6] we get:

Theorem 4 *Let G be a SCG, C a partition that has at least one element of size 1 and at least one element of size 2. Let G^C be a CCG with the underlying game G and partition C. G^C will posses an Exact Potential iff the CCG is linear.*

4.3 FIP Property

If $MC(C) = 2$ a Pure NE exists, though an exact Potential may not, Here we wish to check an interim property - FIP. Games that posses an Exact Potential posses the FIP property, which leads to the existence of a Pure NE, as shown in Monderer and Shapley [13].

FIP Property as defined in Monderer and Shapley, [13], section 2:
A path is a sequence $\gamma = (s^0, s^1 \ldots)$ of strategy profiles, such that for every $k \geq 1$ there exists a unique agent, say agent i, such that $s^k = (s^{k-1}_{-i}, x)$ for some $x \neq y^{k-1}_i$ in S_i. $\gamma = (s^0, s^1, \ldots)$ is an *improvement path* with respect to the game G if $u_i(s_k - 1) < u_i(s_k)$ $\forall k \geq 1$, where i is the unique deviator at step k. A game G has the FIP property if every improvement path is finite.

More formally, FIP property says that any deviation sequence is of finite length.

Example provided by Fotakis et al. [6] that that exist CCG with $MC(C) = 2$ that do not have the FIP property. Our example is similar to it, but has strictly increasing cost functions. Note that in this example three resources have linear costs and one resource with non-linear cost, by changing only one element of the cost function ($P_A(1)$).

Example 3 *A coalitional congestion game G^C with four resources, four sub agents and a partition $C = [\{1, 2\}\{3, 4\}]$, without the FIP Property.*

Consider the following example, with two resources and four agents, each representing a coalition of two sub agents. The P functions are:

Resource / Agents #	1	2	3	4
A	0	7	8	9
B	4	9	14	19
C	8	9	10	11
D	2	5	8	11

Following is a cycle of unlimited profitable deviation, from which it is clear that G^C doesn't have the FIP Property:

Agent 1 Strategy	A A	A A	B C	B C	AA
Agent 1 Pays	16	14	13	17	16
Agent 2 Strategy	A B	C D	C D	A B	A B
Agent 2 Pays	12	10	11	9	12

Remark 4.1 *Note that that due to Theorem 3 G^C posses a Pure NE, despite the fact that it doesn't posses the FIP property.*

Chapter 5

Small Coalitions CCG NE

In this chapter we concentrate on the case when the underlying game is a SCG and partition C with $MC(C) = 2$, since there a Pure NE always exists, as shown in Theorem 3. We will look deeper into those Pure NE strategy profiles and on their congestion vectors.

5.1 Congestion Vectors in Small Coalition CCG NE

Here we check what congestion vectors can be attained in Pure NE profiles in CCG and Restricted CCG in this case. Then we provide some results regarding the congestion vectors sets of Pure NE strategy profiles. The main result will refer to convex case, where we show that all Pure NE strategy profiles of G^C are close to the ones of G.

Definition 9 *Let G be a SCG. Let us denote the following set of Congestion vectors:* $ANE(G) = \{u : \exists! b \; s.t : u_b < v_b \;\; \forall v \in c(NE(G))\}$.

From Rosenthal [18] we know that $c(NE(G)), c(NE(\overline{G^C})) \neq \emptyset$, since $\overline{G^C}$ is a Congestion Game, as shown in Remark 2.3. Also, from Theorem 3 we know that $NE(G^C) \neq \emptyset$, since $MC(C) = 2$ and the underlying game is a SCG.

We show that congestion vector of a Pure NE strategy profile of G^C, with largest coalition of a pair, is either in $NE(G)$ or $ANE(G)$ and in many ways it has congestion vectors similar to those of NE strategy profile of SCG.

Lemma 5.1 *Let G be a SCG. Let u be a congestion vector. Then if $v_b > u_b \ \forall v \in NE(G)$ and $\exists v \in NE(G) : v_r < u_r \Rightarrow P_r(u_r) > P_b(u_b + 1)$.*

Proof:

From Corollary 2 and Lemma 3.1, $u_r \geq \alpha_r + 1$. Also from Lemma 3.1 $u_b \leq \alpha_b$. Therefore, we can say that $P_b(u_b + 1) \leq P_N \leq P_r(u_r)$. We will distinguish between two cases and in each we show that the inequality is strict:

- If $|c(NE(G)| = 1$ then due to Corollary 5 $P_r(u_r) > P_n$.
- If $|c(NE(G)| > 1$ then due to Corollary 6 $u_b < \alpha_b \Rightarrow P_b(u_b + 1) < P_N$.

□

Lemma 5.2 *Let G be a SCG, C a partition of N with $MC(C) = 2$ and G^C a CCG with the underlying game G and partition C. If exists three resources r, x, b that satisfy $P_r(u_r) > P_b(u_b + 1)$ and $P_r(u_r) > P_x(u_x + 1)$ then $u \notin c(NE(G^C))$*

Proof:

Assume the contrary. Let s be a Pure NE strategy profile of G^C satisfying $c(s) = u$. Let k be an agent who chooses r in s, with sub agent i. Then $c_k(s_k)_b = 0$ or $c_k(s_k)_x = 0$ (or both) since agent k has only two sub agents. Assume WLOG:

$$c_k(s_k)_x = 0 \tag{5.1}$$

Let us denote agent k's strategy derived from s_k by moving sub agent i from r to x as t_k. Sub agent i pays in (s_{-k}, t_k) less than in s, since $P_r(u_r) > P_x(u_x + 1)$. Note that the other sub agent j of agent k chooses the same resource x' in both s and t. From Equation 5.1 $x' \neq x$, but r may be equal to x. Thus sub agent j pays weakly more in s than in (s_{-k}, t_k). Therefore, agent k pays strictly more in s than in (s_{-k}, t_k), making the deviation from s_k to t_k strictly profitable for agent k in contradiction to the fact that s is a Pure NE strategy profile. □

Lemma 5.3 *Let G be a SCG and C a partition of N where $MC(C) = 2$. Let $NE(G^C)$ be a CCG with the underlying game G and partition C. Let $v \in NE(G^C)$. Then exists at most one resources b that satisfy $c(t)_b < v_b \ \forall v \in NE(G)$.*

Proof:

Assume that in a Pure NE strategy profile t of G^C there are two resources b and b' satisfying:

$$c(t)_b < v_b, c(t)_{b'} < v_{b'} \;\; \forall v \in c(NE(G)) \tag{5.2}$$

Let s be a Pure NE strategy profile of G. Exists a resource r satisfying $c(s)_r > c(t)_r$, since $c(s)_b < c(t)_b$ and the number of sub-agents remains the same in all profiles. Then from Lemma 5.1 we get:

$$P_r(c(t)_r) > P_b(c(t)_b + 1) \; and \; P_r(c(t)_r) > P_{b'}(c(t)_{b'} + 1) \tag{5.3}$$

From Lemma 5.2 we have that $c(t) \notin c(NE(G^C))$ in contradiction to the fact that $t \in NE(G^C)$. □

Lemma 5.4 *Let G be a SCG with $|c(NE(G))| > 1$ and C a partition of N where $MC(C) = 2$. Let G^C be a CCG with the underlying game G and partition C. Let $u \in NE(G^C)$ where $u_x \geq \alpha_x \; \forall x \in R$ and $\exists r \in R$ such that $P_r(u_r) > P_N$. Then exists (at least) two resources x_1, x_2 satisfying $P_{x_m}(u_{x_m} + 1) \leq P_N$.*

Proof:

Let us define $X = \{r : P_r(\alpha_r + 1) = P_N\}$. Since $|c(NE(G))| > 1$ we know that $|X| > K$.

Let u be a congestion vector in $c(NE(G^C))$ such that $u_x \geq \alpha_x \;\; \forall r \in R$ and exists a resource r such that $P_r(u_r) > P_N$.

If $r \notin X$ then from the definition of r, $u_r \geq \alpha_r$. The total congestion of all resources except r in u satisfies:

$$\sum_{x \in R \setminus \{r\}} u_x \leq n - \alpha_r - 1 \tag{5.4}$$

There are at most $n - \alpha_r - 1$ sub-agents choosing resources in $R \setminus \{r\}$ and each resource $x \in R \setminus \{r\}$ is chosen at least by α_x agents. When we count the number of agents we need in order to allocate at least α_x agents on all resources except r and $u_r \geq \alpha_r + 1$ on r, as done in u, we get:

$$\left(\sum_{r \in R \setminus \{r\}} \alpha_r \right) + u_r \geq n - K - \alpha_r + \alpha_r + 1 \geq n - K + 1 \tag{5.5}$$

Therefore, at most $K-1$ sub agents are left to allocate in after we have allocated α_x sub agents on each of the resources $x \in R \setminus \{r\}$ and $\alpha_r + 1$ on r. Thus from Pigeonhole principle, in u at least two of the $K+1$ resources in X, denote two of them x_1, x_2, satisfy: $u_{x_m} = \alpha_{x_m}$. Consequently we will have that at least two resources in X that satisfy $P_x(u_x + 1) = P_N$.

If $r \in X$ then from definition of r, $u_r \geq \alpha_r + 2$. Similarly to the case when $r \notin X$ we will have at most $K-2$ sub agents to allocate on the K resources in $X \setminus \{r\}$. From Pigeonhole Principle we will have that at least two resources in X that satisfy $P_x(c(s)_x + 1) = P_N$. □

Remark 5.1 *Let $\overline{G^C}$ be a Restricted CCG with the same underlying game G and partition C as G^C. Then Lemmas 5.2, 5.3 and 5.4 with identical proofs holds also in $\overline{G^C}$.*

Theorem 5 *Let G be a SCG and C a partition of N where $MC(C) = 2$. Let G^C be a CCG with the underlying game G and partition C, with $MC(C) = 2$. Then $c(NE(G^C)) \subseteq ANE(G) \cup c(NE(G))$*

Remark 5.2 *The same result hold also for the Restricted CCG $\overline{G^C}$, with identical proof.*

Proof:

What we will show is if $u \in c(NE(G^C))$ and $u \notin c(NE(G)) \Rightarrow u \in ANE(G)$.

Note that according to Lemma 5.3 we know that such resource b is unique.

If $c(NE(G)) = \{v\}$ then obviously if $u \notin c(NE(G))$, there is at least one resource b satisfying $u_b < v_b$.

Let G be a SCG with $|c(NE(G))| > 1$ and $u \in c(NE(G^C)) \setminus c(NE(G))$. Let G^C be a CCG as described in the Theorem.

If $\exists b$ such that $c(u_b) < \alpha_b$ then due to corollary 2 $u_b < v_b \; \forall v \in NE(G)$.

To complete the proof we show the existence of such resource b when $u \in c(NE(G^C)) \setminus c(NE(G))$ and $u_x > \alpha_x \; \forall x \in R$:

Assume that $u \in NE(G^C)$, and there is no such resource b. From Corollary 2 we know that if $u_r \geq \alpha_r \; \forall r \in R$ and $P_r(u_r) \leq P_N \;\; \forall r \in R$ then $u \in c(NE(G))$. Since we know that $u \notin c(NE(G))$, we know that exists a resource r satisfying $P_r(u_r) > P_N$ (otherwise due to Corollary

Lemma 5.5 *Pure NE Identify we have a Pure NE).*

According to Lemma 5.4, in u two resources b, b' satisfy $\min(P_b(u_b+1), P_{b'}(u_{b'}+1)) \leq P_N$ *when* $P_r(u_r) > P_N$. *From Lemma 5.2 we get a contradiction to the fact that* $u \in c(NE(G^C))$.

Therefore, also in the case when $|NE(G)| > 1$ *for* $u \in NE(G^C) \backslash NE(G)$ $\exists b : u_b < v_b \; \forall v \in c(NE_S)$. ■

Remark 5.3 *We will denote the unique resource satisfying the condition in Theorem 5 as b.*

Now we show another connection between $NE(G)$ and $NE(G^C)$. Congestion vectors in $NE(G^C) \setminus NE(G)$ have a congestion vector of a SCG NE, when we "remove" the under-selected resource b and split coalitions, as the following lemma says:

Lemma 5.6 *Let G be a SCG, C a partition of N with* $MC(C) = 2$. *Let* $NE(G^C)$ *be a CCG with the underlying game G and with partition C. Assume* $v \in c(NE(G^C)) \setminus G(NE(G)$. *Then for any two resources* $r, x \neq b$ *(where b is selected less than in any NE strategy profile of G)* $P_x(v_x+1) \geq P_r(v_r)$

Remark 5.4 *This result with same proof also holds for* $NE(\overline{G^C})$

Proof:

Assume that $v \in c(NE(G^C)) \setminus c(NE(G))$. From Theorem 5 in v exists a unique resource b that $v_b < c(t)_b \; \forall t \in NE(G)$. Assume that exist two resources $r, x \neq b$ such that:

$$P_r(v_r) > P_x(v_x+1) \tag{5.6}$$

We can assume WLOG that $P_r(v_r) \geq P_{x'}(v_{x'}) \;\; \forall x' \in R$.

Let t be a Pure NE strategy profile of G. Since $v_b < c(t)_b$ exists a resource x' such that $v_{x'} > c(t)_{x'}$. From Lemma 5.1 we get:

$$P_r(v_r) \geq P_{x'}(v_{x'}) > P_b(c(t)_b) \geq P_b(v_b+1) \tag{5.7}$$

Due to Equations 5.6, 5.7 and Lemma 5.2 applied on resources b, x, r we get a contradiction to the fact that $v \in c(NE(G^C))$. □

Lemma 5.7 *Let G be a SCG, C a partition with $MC(C) = 2$ and G^C a CCG. Let $v \in NE(G) \setminus NE(G)$, and b the under-selected resource, as described in Theorem 5. Let x be one of the resources with the maximal cost in t. Assume that $P_N \leq P_x(v_x) \leq P_b(v_b+2)$. Then exists $u \in NE(G)$ such that $d(u, v) = 1$.*

Proof:

Let t be a Pure NE strategy profile of G^C satisfying $c(t) = v$.

From assumption $\alpha_b + 1 \leq v_b + 2$, meaning $\alpha_b \leq v_b + 1$. From Theorem 5 and Corollary 2 $v_b \leq \alpha_b$. Therefore:

$$\alpha_b - 1 \leq v_b \leq \alpha_b \tag{5.8}$$

We divide the proof into two cases. We show show that in both cases exists a Pure NE of G adjacent to v.

1. If $P_b(v_b + 2) = P_x(v_x) = P_N$, let us define u as the following congestion vector:

   $$\left\{ \begin{array}{c} u_z = v_z \ \ \forall z \neq b, x \\ u_b = v_b + 1 \\ u_x = v_x - 1 \end{array} \right\}$$

   Obviously, $d(u, v) = 1$.

   Let s be a SCG profile such that $c(s) = u$. Due to the definition of x no agent pays more than P_N in t. Note that $P_b(u_b) < P_N$ thus no agent pays more than P_N in s too. Which means:

   $$P_{x'}(u_{x'}) \leq P_N \ \forall x' \in R \tag{5.9}$$

   From Lemma 5.6 we know that:

   $$P_N = P_x(v_x) \leq P_{x'}(v_{x'} + 1) \ \ \forall x' \in R \setminus \{b\} \tag{5.10}$$

   $P_b(v_b + 2) = P_b(u_b + 1) = P_N \Rightarrow u_b = \alpha_b$. Combined with Equations 5.10 we get that:

   $$c(v)_r \geq \alpha_r \ \forall r \in R \tag{5.11}$$

Applying Equations 5.9 and 5.10 in Lemma 3.3 we get that s is a Pure NE strategy profile in G.

2. If $P_b(v_b+2) > P_N$ then in all SCG NE strategy profiles b will be selected by exactly $v_b + 1$ agents due to Lemma 3.1, Theorem 5 and our assumption.

   Let us look on a strategy profile q of G, satisfying $c(q) = v$. From Corollary 3 and Lemma 5.6 we know that on $R \setminus \{b\}$ the congestion in q is as of $N - v_b$ greedy agents, one more than chose those resources in any NE strategy profile of G.

   Let us denote the strategy profile derived from q by moving an agent from x to b as q'. In q' the number of agents who select b is as in any NE of G. Since we removed one agent who paid the highest cost outside b we will have a greedy formation in $R \setminus \{b\}$ with the number of agents as in any NE of G. Due to Corollary 3 q' is a Pure NE strategy profile of G and clearly $d(c(q', v) = 1$.

In both cases there is a Pure NE s of G which satisfies $d(c(t), c(s)) = 1$. □

Now we turn to provide the main result regarding the Pure NE of G^C, in the case that the cost functions P are weakly convex.

Note that convex cost functions is equivalent to weakly increasing Δ's in the cost functions because weak convexity implies $\Delta_r(m) = P_r(m+1) - P_r(m) \leq P_r(m+2) - P_r(m+1) = \Delta_r(m+1)$ for $m > 1$ and the definition of $\Delta_r(1) = 0$ keeps this inequality, since the cost functions are strictly increasing.

Theorem 6 *Let G be a SCG and C is the partition of N where $MC(C) = 2$. Let G^C be a CCG with the underlying game G and partition C. If in G the cost functions are weakly convex then for any $t \in NE(G^C)$ exists $s \in NE(G)$ such that $d(c(s), c(t)) \leq 1$.*

Proof:

This theorem is trivial for he case when $c(t) \in c(NE(G))$. Therefore we will concentrate on the case when $c(t) \notin NE(G)$.

Let us denote the resource with the highest cost in t as x.

Assume that $v \in c(NE(G^C)) \setminus c(NE(G))$. Let t be the Pure NE strategy profile of G^C satisfying $c(t) = v$. From Theorem 5 exists a unique resource b such that:

$$c(t)_b < c(s)_b \; \forall s \in NE(G) \tag{5.12}$$

Let s be a NE of G. Since the resource b is selected by more agents in s than in t exists a resource r such that $c(t)_r > c(s)_r$.

From Lemma 5.1 sub agent can deviate from r to b and pay less, thus all agents selecting r in t must also select b. Since $P_r(v_r) \leq P_x(v_x)$ we can say the same for x too. Moreover, deviating from (x, b) to (b, b) must be unprofitable in t. thus:

$$P_b(v_b) + P_x(v_x) \leq 2P_b(v_b + 1)$$
$$P_r(v_x) \leq 2P_b(v_b + 1) - P_b(v_b)$$

From convexity and weakly increasing Δ's we get that:

$$P_x(v_x) \leq P_b(v_b + 1) + \Delta_b(v_b + 1) \leq P_b(v_b + 1) + \Delta_b(v_b + 2) = P_b(v_b + 2) \qquad (5.13)$$

Combining Equation 5.13 and Lemma 3.4 we get:

$$P_N \leq P_x(v_x) \leq P_b(v_b + 2) \qquad (5.14)$$

To complete the proof we apply Lemma 5.7 on v. ■

Corollary 11 *If $c(NE(G)) \subseteq c(NE(G^C))$ then $c(NE(G^C))$ is connected. The proof is direct from Theorem 6 and Lemma 3.5.*

5.2 Double Selected Resources

As we have shown in Lemma 5.6, Theorem 5 and Theorem 6 there is much in common between NE congestion vectors of G and G^C. Moreover, when comparing between G^C and $\overline{G^C}$ even more common properties appear. Here we look on strategy profiles where a compound agent selects a resource twice, which can happen only in G^C. We analyze when in NE a resource will be selected twice by the same agent and show that allowing such strategies do not extend our strategy space by much.

Definition 10 *Let s be a profile of G^C. We denote the set of resources that exists a compound agent selecting this resource with both his sub agents as $MA(s)$. Meaning: $MA(s) = \{r : \exists k \text{ and } i, j \in C_k \text{ such that } s_{k,i} = s_{k,j} = r\}$.*

The most basic question is what are the conditions where an agent may even consider selecting a resource twice in a NE. The following Lemma provides the answer to this question.

Lemma 5.8 *Let G be a SCG and C a partition of N with $MC(C) = 2$. Let $NE(G^C)$ be a CCG with the underlying game G and partition C. Let s be a Pure NE strategy profile of G^C and $r \in MA(s)$. Then $c(s)_r \leq c(t)_r \;\; \forall t \in NE(G)$.*

Proof:

Let us denote the congestion vector $v = c(s)$.

If $c(NE(G)) = v$ - the condition of the Lemma is met trivially for every resource $r \in R$.

In all other cases we will show that $r \in MA(s) \Rightarrow v_r \leq \alpha_r$. Note that from Corollary 2 this condition leads to $v_r \leq u_r \;\; \forall u \in c(NE(G))$. Thus showing that $v_r \leq \alpha_r$ is sufficient.

A resource r can be chosen by an agent in s twice only if:

$$P_r(v_r) < P_x(v_x + 1) \;\; \forall x \in R \tag{5.15}$$

Otherwise, the agent can move one of his sub agents from r to x. The deviating sub agent pays less weakly, while the other sub agent pays less strictly, since P_r is strictly increasing.

We will distinguish between two cases and show that in both of them we show that $r \in MA(s) \Rightarrow v_r \leq \alpha_r$.

1. If $v \notin c(NE(G))$ then from Theorem 5 $v \in ANE(G)$. Thus $\exists! b : P_b(v_b + 1) \leq P_N$. If $r \neq b$ in $MA(s)$ then due to Equation 5.15 $P_r(v_r) < P_b(v_b + 1) \leq P_N$ meaning - $v_r \leq \alpha_r$. If $r = b$ then from $ANE(G)$ definition and Lemma 3.1 $v_b \leq \alpha_b$.
2. If $v \in c(NE(G))$ we can assume that $|c(NE(G))| > 1$ (the other case was proven). From Corollary 6 there is a resource $x : P_x(v_x + 1) = P_N \Rightarrow v_x = \alpha_x$, meaning - x can be in $MA(s)$. If $r \neq x$ is in $MA(s)$ then due to Equation 5.15 $P_r(v_r) < P_N = P_x(v_x + 1)$, meaning - $v_r \leq \alpha_r$.

Therefore, when $v \neq c(NE(G))$ $r \in MA(s) \Rightarrow v_r \leq \alpha_r$. □

This result provides us with an interesting corollary regarding the sets of NE:

Corollary 12 *Let G be a CCG, C a partition of N with $MC(C) = 2$. Let $NE(G)$ be a CCG and $\overline{G^C}$ be a Restricted CCG both with the same underlying game G and partition C. $c(NE(G)) \cap c(\overline{G^C}) \neq \emptyset \Rightarrow c(NE(G^C)) \subseteq c(\overline{G^C})$*

Proof:

Assume the contrary. Let there be a congestion vector $v \in c(NE(G^C)) \setminus c(\overline{G^C})$ and a Pure NE strategy profile s of G^C satisfying $c(s) = v$. Clearly:

$$\exists r : v_r > n^c \Rightarrow r \in MA(s) \tag{5.16}$$

Therefore, $e_r \leq n^c < v_r$. So, we have that $v_r < e_r$ and $e_r \in c(NE(G))$, which is a contradiction to Lemma 5.8. □

Another legitimate question is how many of such double selected resources may exist in NE. The next Lemma shows that if we look only on Congestion Vectors of NE, we can decrease this number significantly.

Lemma 5.9 *Let G be a SCG, C a partition of N with $MC(C) = 2$. Let $NE(G^C)$ be a CCG with the underlying game G and partition C. Let us denote $x \neq r$ two resources in R. Assume $s \in NE(G^C)$ where agent k's strategy is (r, r) and agent l's is (x, x). Let us denote the strategy profile derived from s when agents k and l change their strategies to (x, r) as t. Then $t \in NE(G^C)$.*

Proof:

All agents in $N^C \setminus \{k, l\}$ will have no reason to deviate in t, since the congestion on all resources is the same in s and t ($c(s) = c(t)$) and they choose the same strategies in s and t. Agent k didn't wanted to deviate from r to any other resource z in the strategy profile s, when he had both of his sub agents in r. This can only happen if:

$$2P_r(c(s)_r) \leq P_r(c(s)_r - 1) + P_z(c(s)_z + 1) \tag{5.17}$$

From monotonicity of P and Equation 5.17 we get:

$$P_r(c(s)_r) < P_z(c(s)_z + 1) \tag{5.18}$$

Thus moving a sub agent from r to z is not profitable in t. Same with x instead of r and agent l instead of k. Due to Lemma 4.2 there is no profitable deviation to agents k and l, since in t there is no profitable deviation for sub agents of agents k and l, who select two different resources in t. Therefore, t is a Pure NE strategy profile. $\square$

Corollary 13 *Let G be a SCG and C a partition of N with $MC(C) = 2$. Let $NE(G^C)$ be a CCG with the underlying game G and partition C. Let $s \in NE(G^C)$ such that $|MA(s)| > 1$.*

Proof:

Using Lemma 5.9 in induction we get that $\exists t \in NE(G^C)$ such that $|MA(t)| \leq 1$ and $c(s) = c(t)$. $\square$

Chapter 6

CCG Pure NE Total Cost

In this chapter we check the total cost of NE in CCG and compare it with total cost of NE strategy profiles of the underlying SCG. Troughout this chapter we will assume that the CCG G^C has a Pure NE and that $P_N > 0$.

Definition 11 *Let G be a Congestion Game or a CCG. We will define the total cost of a profile e of the game G as $TC(e) = \sum_{r \in R} c(e)_r \cdot P_r(c(e)_r)$. For a SCG (CCG) G we also define $MaxTC(G) = max_{e \in NE(G)} TC(e)$ and $MinTC(G) = min_{e \in NE(G)} TC(e)$.*

Let G be a SCG. Let us define:

$$\mu(G) = \max_{C \text{ } is \text{ } a \text{ } partition \text{ } of \text{ } n} MaxTC(G) \tag{6.1}$$

Price of Collusion (PC) is defined in Hayrapetyan [7] as the cost ratio $\frac{\mu(G)}{TC(e_s)}$.

Price of Collusion shows how the cost may increase due to formation of coalitions. We would like to add more criteria to compare increase in the total cost, by comparing also the best NE strategy profiles, or any two NE strategy profiles of G and G^C. We will look on specific CCG, rather than on all possible partitions and check how this cost ratio can increase, or decrease, using the following definition:

Definition 12 *Let G be a SCG and s be a pure NE strategy profile of G. Let G^C be a CCG with the underlying game G. Let t be a Pure NE strategy profile of G^C. We define* *a function* $TCR : NE(G^C) \times NE(G) \rightarrow \mathbb{R}^+$ *as follows:*

$$TCR(t, s) = \frac{TC(t)}{TC(s)} \tag{6.2}$$

We will refer to $MaxTCR(G, C) = \max_{t \in NE(G^C), s \in NE(G)} \frac{TC(t)}{TC(s)}$ *and* $MinTCR(G^C, G) = \min_{t \in NE(G^C), s \in NE(G)} \frac{TC(t)}{TC(s)}$.

For TCR to be well defined we will assume that the CCG Game G^C has a Pure NE and that $P_N > 0$. (otherwise due to Lemma 3.1 the denominator will be zero).

TCR measures not only the increase in the total cost of NE caused by collusion, but also the increase. As we will see in some cases the collusion increases the Total Cost while in others it decreases the Total Cost.

Remark 6.1 $\mu(G) \geq \max_{e_c in NE(G^C), e_s \in NE(G)} TCR(e_c, e_s)$.

6.1 Lower Bound on NE Cost Ratio

Here we provide a general tight lower bound on $MinTCR$ since we assume nothing of the partition. The bound depends only on the number of agents (n) and is equal to 1/n. If $n > 2$ it is unattained and if $n = 2$ it can be attained.

The proof is divided into three parts: First we show that $MinTCR \geq 1/n$, then we show that the inequality must be strict if $n > 2$ and finally we provide two examples that this bound is tight, one for the case of $n > 2$ and another for $n = 2$.

Lemma 6.1 *Let G be a SCG with n agents, C a partition of n and* G^C *a CCG with an underlying game G.* $MinTCR(G^C, G) \leq n$

Proof:

From Lemma 3.1 the total cost in any NE strategy profile of G is at most $n \cdot P_N$. From Lemma 3.4 in any strategy profile of G^C exists a sub agent who pays P_N or more thus $TC(t) \geq P_N$ for any strategy profile t of G^C. Therefore $MinTCR(G^C, G) \geq \frac{P_N}{nP_N} = 1/n$. □

Lemma 6.2 *Let G be a SCG, G^C a CCG with an underlying game G. If $n > 2$ then $MinTCR(G^C, G) > 1/n$.*

Proof:

Assume that $\forall e_c \in NE(G^C)\ TC(e_c) > P_N$. Due to Lemma 3.1, $TC(e_s) \leq n \cdot P_N\ \forall e_s \in NE(G)$. Therefore, $TCR(e_c, e_s) > 1/n$.

Therefore, we can concentrate on the case where exists a Pure NE strategy profile of G^C with total cost of P_N, denote it t. From Lemma 3.4 in t one agent pays P_N and $n-1$ agents pay zero. Since the costs are strictly increasing and non-negative $P_r(m) = 0 \Rightarrow m = 1$. Thus all sub-agents select different resources in t.

Let s be a Pure NE of G. From Theorem 1 s can be attained by greedy strategy profile. In any greedy strategy profile the first $n-1$ greedy agents will choose different resources with cost zero and the last agent will choose resource r with cost of P_N.

If no agent of the first $n-1$ agents chose r then $TC(s) = P_N$. If one agent of the first $n-1$ agents chose r then $TC(s) = 2P_N$. Since the first $n-1$ greedy agents chose different resources, after the last agent chose r its is at most two. Thus $TC(s) \leq 2P_N$ and $TCR(G^C, G) \geq 1/2$. Concluding, if $n > 2 \Rightarrow MinTCR(G^C, G) > 1/n$. □.

Definition 13 *Let us denote the partition $\widehat{C} = \{1, \ldots, n\}$ - the partition of N into the grand coalition.*

Example 4 *For any $0 < \varepsilon < 1$ and $n > 2$ $MinTCR(G^C, G) = (1+\varepsilon)/n$ in the following G and $C = \widehat{C}$*

Let there be a fixed n and a small positive $0 < \varepsilon$. Let G be a SCG with n agents, two resources and the following P functions:

Resource / Agents #:	1	2	...	n-1	n
A:	$\frac{\varepsilon}{n(n-1)}$	$\frac{2\varepsilon}{n(n-1)}$	...	$\frac{(n-1)\varepsilon}{n(n-1)}$	1
B:	$1+\frac{\varepsilon}{n}$	2	...	n-1	n

From Theorem 1, in a Pure NE strategy profile s of G all the n agents will choose A thus $TC(s) = n$.

If $C = \widehat{C}$ and $t \in NE(G^C)$ then $c(t)_A = n - 1$ and $c(t)_B = 1$. In t the cost of resource A is $\frac{(n-1)\varepsilon}{n(n-1)}$ and of B is $1 + \frac{(n-1)\varepsilon}{n(n-1)}$. Summing it up we get that $TC(t) = 1 + \varepsilon$. Therefore: $TCR(G^C, G) = \frac{1+\varepsilon}{n}$

Example 5 *For the case that $n = 2$ the following games have a $MinTCR$ of 2:*

Consider a SCG G with two agents and two resources - A and B, where $P_A(1) = 0, P_A(2) = 1, P_B(1) = 1, P_B(2) = 2$.
The strategy profile where both agents choose A is a Pure NE strategy profile with Total Cost of 2.

Let G^C be a CCG with the underlying game G and the grand coalition. Easy to see that a strategy profile where one agent chooses A and the other choose B is a NE strategy profile of G^C with the total cost of 1. Thus, $MinTCR \leq 1/2$. Combined with Lemma 6.1 we get that $MinTCR = 1/2$

Combining Lemmas 6.1 and 6.2 with Examples 4 and 5 we get the following Theorem:

Theorem 7 *Let G be a SCG with n agents and C a partition of n. Let G^C be a CCG with the underlying game G and partition C. If $n > 2$ then $MinTCR(G^C, G) > 1/n$ is a tight bound for $MinTCR$. If $n = 2$ then 2 is the tight bound for $MinTCR$.*

6.1.1 SCG Price of Anarchy

Let G be a SCG. The Price of Anarchy (PA), as described by Koutsoupias and Papadimitriou [10] is the total cost ratio between the Pure NE of G with the highest total cost and the profile that minimizes the total cost:

$$PA(G) = \frac{MaxTC(G)}{MinTC(G^{\widehat{C}})} \tag{6.3}$$

From this we get the following corollaries:

Corollary 14 *n is the tight upper bound on the Price of Anarchy in SCG. If $n > 2$ it is unattained n and if $n = 2$ it can be attained. The proof is immediate from Theorem 7 and Equation 6.3.*

Corollary 15 *For any SCG G we can say that:*

$$n > 2 \Rightarrow \frac{TC(s)}{TC(t)} < n \;\; \forall s,t \in NE(G) \tag{6.4}$$

$$n = 2 \Rightarrow \frac{TC(s)}{TC(t)} \leq n \;\; \forall s,t \in NE(G) \tag{6.5}$$

6.2 Upper Bound on NE Cost Ratio

Here we provide an upper bound on TCR. Similarly to the case with MinTCR, we will show that $MaxTCR(G^C, G) \leq n$, then we will show that value of n cannot be attained. Here, however, we do not show that the bound is tight. Hayrapetyan [7] showed that $PC \leq 2$ when all the resources have weakly convex cost functions or all resources have weak concave cost functions, in Theorems 3.5 and 3.12. Therefore, the convex or concave case the bound is 2.

To provide the result we provide a bound on the maximal payment an agent can have in a Pure NE of G^C, as follows:

Lemma 6.3 *Let $s \in NE(G^C)$. Then $U_k(s) \leq |C_k| \cdot P_N$. In words: agent k doesn't pay more than $|C_k| \cdot P_N$ in any Pure NE strategy profile.*

Proof:

Assume that there is an agent k that pays in s more than $|C_k| \cdot P_N$. Then we will propose a different strategy to agent k where he doesn't pay more than $|C_k| \cdot P_N$

Let us denote $u = c(s)$ and $v \in NE(G)$. Let us denote w as the congestion vector u when we removed agent k. Clearly $\sum_{r \in R} w_r = n = C_k$. Therefore:

$$\sum_{r \in R} \max(v_r - w_r, 0) = \sum_{v_r > w_r} u_r - w_r \geq \sum_{r \in R} v_r - w_r = \sum_{r \in R} v_r - \sum_{r \in R} w_r = n - (n - |C_k|) = |C_k| \tag{6.6}$$

Therefore, you can reallocate the $|C_k|$ sub agents of agent k on resources $r : v_r > w_r$, such that if $c_k(r) > 0$ then $w_r + c_k(r) \leq v_r$. Note that $v \in c(NE(G))$, thus due to Lemma 3.1 no one of agent k sub agents will pay more than P_N.

Agent k who pays more than $|C_k| \cdot P_N$ can deviate to this strategy pay at most $|C_k| \cdot P_N$, in contradiction to the fact that we were in a Pure NE strategy profile. □

Corollary 16 *For any* $t \in NE(G^C)$ $TC(t) \leq n \cdot P_N$. *From Lemma 6.3 any agent* $k \in N^c$ *pays at most* $|C_k| \cdot P_N$.

Lemma 6.4 *Let* G^C *be a CCG. If exists* r *such that* $P_r(1) = 0$ *then* $c(t)_r \geq 1$ $\forall t \in NE(G^C)$.

Proof:

Let t be a Pure NE of G^C and r such that $P_r(1) = 0$ and $c(t)_r = 0$. From Lemma 3.4 exists a sub agent i of agent k that pays P_N or more in t. Let us denote the profile derived from t by changing the strategy of sub agent i to r as s.

In s sub agent i pays 0, which is strictly less than in t where he pays P_N. Moreover all sub agents pay weakly less in s than in t since i selects a new different resource. Therefore agent k pays in s less than in t in contradiction to the fact that t is a Pure NE of G^C. □

Lemma 6.5 *Let* G *be a SCG with* n *agents. Let* G^C *be a CCG with the underlying game* G. *Then* $MaxTCR(G^C, G) \leq n$

Proof:

From Lemma 6.3 we know that in any Pure NE of G^C agent pays at most $|C_k| \cdot P_N$. Therefore, if $t \in NE(G^C)$ we can say that:

$$TC(t) \leq \sum_{k \in N^C} |C_k| \cdot P_N = P_N \cdot \sum_{k \in N^C} |C_k| = n \cdot P_N. \tag{6.7}$$

In any NE strategy profile of G at least one sub agent pays P_N, from Lemma 3.1. Combined with Equation 6.7 we get that $TCR(t, s) \leq \frac{nP_N}{P_N} = n$. Since we took a general NE both of G^C and G we can conclude that $MaxTCR(G, C) \leq n$ □

Remark 6.2 *Note that if* $n = 2$ *the only coalition formation is* $\widehat{C}$, *which will leave us with one agent. Then* $MaxTCR \leq 1$ *since the compound agent dominating strategy is to minimize the total cost. If fact for any* n *and any SCG* G, $MaxTCR(G, \widehat{C}) \leq 1$

For the case $n > 2$ similarly to the first case, we have an unattained bound lemma:

Lemma 6.6 *If* $n > 2$ *then* $MaxTCR(G^C, G) < n$

Proof:

Assume that $\forall e_s \in NE(G^S)$ $TC(e_s) > P_N$. Due to Lemma 6.3 $TC(e_c) \leq n \cdot P_N$ $\forall e_c \in NE(G^C)$. Therefore, $TCR(e_c, e_s) < n$.

Therefore, we can concentrate on the case when exists a Pure NE strategy profile of G^C, with total cost of P_N, denote it s. Similarly to Lemma 6.2 in s there are $n-1$ agents who pay 0 and one agent pays P_N. Moreover, all agents select different resources, since the costs are strictly increasing.

Let t be a Pure NE strategy profile of G^C. Due to Lemma 6.4 we know that $n-1$ sub agents select the $n-1$ resources satisfying $P_r(1) = 0$. The n^{th} sub agent, denote him i, chooses resource r. Clearly $c(t)_r \leq 2$ thus resource r is selected by at most one more sub agent - j.

From the congestion vector of s we know that exists resource x such that $P_x(1) = P_N$. Therefore, if in t sub agent j pays more than P_N he can deviate profitably to a unselected resource x, contradicting the fact that t is a Pure NE.

Thus, $TC(t) \leq 2P_N$, since all sub agents except i and j pay zero, and i and j pay at most P_N. To conclude $TCR(t, s) \leq 2$. Therefore if $n > 2$ $MaxTCR(G^C, G) < n$ $\square$.

Combining Lemmas 6.5, 6.6, Remark 6.2 and Corollary 16 we get the following Theorem:

Theorem 8 *Let* G *be a SCG with* n *agents and* C *a partition. Let* G^C *be a CCG with the underlying game* G *and any partition* C*. Then* $MaxTCR(G^C, G) < n$.

Corollary 17 *If* $n > 2$ *then:*

$$PC < n \cdot \frac{min_{e \in NE(G)} TC(e)}{max_{e' \in NE(G)} TC(e')} \tag{6.8}$$

Proof:

Note that if $n > 2$ then:

$$PC = \frac{\max_{C\ partition\ of\ N} MaxTC(G^C)}{MaxTC(G)} = \tag{6.9}$$

$$= \frac{\max_{C\ partition\ of\ N} MaxTC(G^C)}{MinTC(G)} \cdot \frac{MinTC(G)}{MaxTC(G)} = \tag{6.10}$$

$$= \max_{C\ partition\ of\ N} MaxTCR(G, C) \cdot \frac{MinTC(G)}{MaxTC(G)} < \tag{6.11}$$

$$< n \cdot \frac{min_{e \in NE(G)} TC(e)}{max_{e' \in NE(G)} TC(e')} \tag{6.12}$$

Therefore an upper bound on MaxTCR provides a bound on PC. □

Corollary 18 *Price of Anarchy of G^C is tightly bounded by the number of sub agents.*

Proof:

From Corollary 16 the total cost in any Pure NE strategy profile in G^C is at most $n \cdot P_N$. From Lemma 3.4 in any strategy profile the total cost is at least P_N. Therefore, the price of anarchy is bounded by n. The bound is tight since SCG is a private case of a CCG, and from 14 the bound on the price of anarchy of SCG is n. □

6.3 Small Coalition Case

For the case when MC(C)=2, we can provide a better upper bound for TCR and in some cases this bound is tight.

Lemma 6.7 *Let G be a SCG and G^C a CCG with a partition C satisfying $MC(C) = 2$ and underlying game G. Then $\frac{MaxTC(G^C)}{MaxTC(G)} \leq 2$*

Proof:

Let e be a Pure NE strategy profile satisfying $TC(e) = MaxTC(G^C)$ and q a Pure NE strategy of G such that $TC(q) = MaxTC(G)$.

If $c(e) \in NE(G)$ then clearly $\frac{MaxTC(G^C)}{MaxTC(G)} < 1$.

If $c(e) \notin NE(G)$ - from Theorem 5 in e there is a resource b selected by less sub agents than in any Pure NE Strategy profile in G, including q.

Let us denote $P_b(c(q)_b) = D$. $MaxTCR = TC(q) \geq c(q)_b \cdot D$ since there are $c(q)_b$ sub agents selecting b, each paying D and other agents pay weakly more than zero.

Let us denote the set of resources r satisfying $c(e)_r > c(q)_r$ as X. (since $c(e)_b < c(q)_b$ we know that $X \neq \emptyset$). From Lemma 5.1 sub agents who chooses $r \in X$ in e must also choose b. Thus in the strategy profile e at most $c(e)_b$ sub agents choose resources in X. Since $e \in c(NE(G^C)$ it is unprofitable for agents to deviate in e from (b, r) to (b, b). Therefore:

$$P_r(e_r) + P_b(e_b) \leq 2P_b(e_b + 1) \leq 2P_b(q_b) = 2D \; \forall r \in X \tag{6.13}$$

Since only compound agents select resources in X, we know that there are at most $c(e)_b$ such compound agents. Thus the total cost of sub agents who select $X \cup \{b\}$ in e is at most $2D \cdot c(q)_b$. On other resources the congestion in e is at most as in q, which we will denote β, we get:

$$\begin{aligned} TC(q) &= D \cdot c(q)_b + \beta \\ TC(e) &\leq 2D \cdot c(e_b) + \beta \\ &\Downarrow \\ TCR(e, q) \leq 2 \Rightarrow \frac{MaxTCR(G^C)}{MaxTCR(G)} &\leq 2 \end{aligned} \tag{6.14}$$

□

We now turn to provide a tight bound on TCR when we take the opposite NE, with the highest total cost:

Lemma 6.8 *Let G be a SCG and a partition C with $MC(C) = 2$. Let G^C a CCG with the partition C and underlying game G. Than $\forall e$ such that $c(e) \in NE(G) \setminus c(NE(G^C))$ $\exists q \in NE(G^C) : TCR(q, e) \leq \frac{n}{n^c+1}$, where n^c is the number of agents in G^C.*

Proof:

If $c(e) \notin c(NE(G^C))$ then from Corollary 8 we can say that $c(e) \notin c(\overline{G_C})$. This can only happen if there is a resource b such that $c(e)_b = n^c + 1 > \frac{n}{2}$. Let us denote the cost of resource b in e as D.

Since more than n^c agents select b where each pay D and the rest pay zero or more we know that:

$$TC(e) \geq (n^c + 1)D \tag{6.15}$$

Let us arrange the agents in a profile t of G^C where $c(t) = c(e)$, as described in Theorem 3:

1. All singleton agents choose b.
2. Among all compound agents, at least one sub agent chooses b.
3. The other sub agents are spread among the resources, so that each resource r, will be chosen $c(e)_r$ times.

We know that $c(t) = c(e) \in c(NE(G) \setminus c(NE(G^C))$. From Corollary 10 we know that exists a Pure NE of G^C, q such that;

1. $c(q)_b < c(t)_b$
2. In q all agents select resource b at least with one sub agent.

Thus in q all singleton agents pay less than D. Moreover we can say that:

$$P_b(c(q)_b + 1) \leq D \tag{6.16}$$

Thus, if in e compound agent k will deviate from (r, b) to (b, b), he will pay at most $2D$. Since this deviation is unprofitable, any compound agent pays in q at most $2D$.

To conclude we have that $TC(e) \leq nD$, since the singleton agents pay less than D and the compound at most $2D$. Combined with Equation 6.15 we can say that $TCR(e, q) \leq \frac{n}{n^c+1}$ □

Implementing Lemma 6.8 on a Pure NE strategy profile e of G, such that $TC(e) = MinTCR(G$ we get the following result:

Corollary 19 *If $c(e) \in c(NE(G^C))$ then obviously $\frac{MinTC(G^C)}{MinTC(G)} = 1$.*

If $c(e) \notin c(NE(G^C))$ then $\frac{MinTC(G^C)}{MinTC(G)} \leq \frac{n}{n^c+1}$. Moreover - if in C there is a singleton agent the inequality is strict.

Example 6 *Now we provide an example similar to example 3.13 in Hayrapetyan et al. [7], which shows that* $\frac{MinTC(G^C)}{MinTC(G)} \geq \frac{n}{n^c+1} - \varepsilon$*. Therefore,* $\frac{n}{n^c+1}$ *is a tight bound, but it may be unattained.*

Let there be $0 < \varepsilon < 0.1$. Let G be a SCG with two resources, four agents, two compound agents (each with 2 sub agents) and the following P functions:

Resource / Agents #	1	2	3	4
A	**0**	1	1+2ε	1+3ε
B	1-3ε	1-2ε	**1-0.1**ε	1+ε

Note that $n = 4$ and $n^c = 2$. What we need to show is that:

$$\frac{MinTC(G^C)}{MinTC(G)} > \frac{4}{3} - \varepsilon = \frac{n}{n^c+1} - \varepsilon \tag{6.17}$$

From Theorem 1, in all NE strategy profiles of G 3 agents choose B and the one A, where agents pay less than 1. Thus $MinTC(G) = 3 - 0.3\varepsilon$

G^C in strategic form will looks as follows, when the tabular entries represent costs:

G^C	A,A	A,B	B,B
A,A	$2+6\varepsilon, 2+6\varepsilon$	$2+4\varepsilon, 2-\varepsilon$	$2, 2-4\varepsilon$
A,B	$2-\varepsilon, 2+4\varepsilon$	$2-\varepsilon, 2-\varepsilon$	$1-\varepsilon, 2-0.2\varepsilon$
B,B	$2-4\varepsilon, 2$	$2-0.2\varepsilon, 1-\varepsilon$	$2+\varepsilon, 2+2\varepsilon$

The unique NE strategy profile is $\{AB, AB\}$, where two sub agents chose each of the resources, with the total cost of $4 - 2\varepsilon$. Since this is the unique NE of G^C up to agents permutation, we can say that $MinTC(G^C) = 4 - 2\varepsilon$ To conclude, for $\varepsilon < 0.1$ we have:

$$\frac{MinTC(G^C)}{MinTC(G)} = \frac{4-2\varepsilon}{3-0.3\varepsilon} \geq \frac{4}{3} - \frac{1.6\varepsilon}{3-0.3\varepsilon} > \frac{4}{3} - \varepsilon = \frac{n}{n^c+1} - \varepsilon \tag{6.18}$$

Chapter 7

Future Research and Extensions

In this chapter we show similar models to ours and provide some preliminary results regarding them. First we alter the coalition cost and analyze Pure NE in this model. Then we extend the possible underlying games and show that a Pure NE may not exist there. Lastly, we shift from the cost model to utility model, and show that our Total Cost Ratio bounds fail to work in this case, as any bound except the trivial ones - unattained 0 and ∞.

7.1 Different Coalitions Approach

Here we turn to a model similar to the one defined in Fotakis et al. [6] and provide some results regarding the congestion vectors of NE profiles. Unlike in the cost approach CCG dealt trough the paper, here the setting is of time approach CCG, defined as follows:

CCG_∞ is a Coalitional Congestion Game, with the same set of agents and strategies as defined earlier. However, agents utility differs from before and is defined as follows: $\forall s^c \in S^c$ $\widehat{U_k^c}(s^c) = \min_{i \in C_k} U_i(s^c)$ and $\widehat{U^c} = \{\widehat{U_k^c}\}_{k \in C}$.

Such definition is reasonable when the costs are in context of time and you wish to know when the last sub agent of your agent will complete his path on the graph. The original model is reasonable in the money context, that the road has an actual monetary cost, which each of the sub agents pay.

Let G be a SCG, G_∞^C be a CCG_∞. We will denote as before the set of NE profiles in G_∞^C

as $NE(G^C_\infty)$ and we will use the congestion function c as defined earlier. Note that all results regarding SCG from chapter 3 and many others, are valid in this section too.

Lemma 7.1 *Let G be a SCG. Let G^C_∞ be a CCG with the underlying game G. Let t be a profile in G^C_∞. $c(t) \in c(NE(G)) \Rightarrow t \in NE(G^C_\infty)$*

Proof:

Let us denote an arbitrary agent k in G^C_∞.

Let s be a pure NE strategy profile of G such that $c(t) = c(s)$.

Let us denote a possible deviation of agent k from t_k as t'_k. We will show that this deviation is not profitable.

If $c(t_{-k}, t'_k) = c(t)$ agent k just rearranged his sub agents, leaving the highest cost for his sub agents as it was and this deviation is not strictly profitable.

If $c(t_{-k}, t'_k) \neq c(t)$ then:

$$\exists x, r \in R : c(t_{-k}, t'_k)_r > c(t)_r \text{ and } c(t_{-k}, t'_k)_x < c(t)_x \tag{7.1}$$

Since agent k was the only agent deviating from t and the congestion of r increased due to it we know that $c_k(t'_k) > 0$. Let us denote one of the sub agents of k who chooses r in t'_k as i. Note that the lower bound on agent k's cost is the cost of sub agent i. Since $c(t) \in NE(G)$ and due to Corollary 2 we know that:

$$c(t_{-k}, t'_k)_r > c(t)_r = c(s)_r \geq \alpha_r \Rightarrow c(t_{-k}, t'_k)_r \geq P_N. \tag{7.2}$$

Therefore agent k pays in (t'_k, t_{-k}) at least P_N. From Lemma 3.1 in the profile t agent k paid at most P_N. Meaning - the deviation to t'_k wasn't strictly profitable. Since t'_k is an arbitrary deviation, agent k has no profitable deviations from t. Since k was an arbitrary agent we have that $t \in NE(G^C_\infty)$. □

Lemma 7.2 *Let G be a SCG. Let G^C_∞ be a CCG with the underlying game G. In any NE strategy profile G^C_∞ the lowest cost for any agent is P_N.*

Proof:

Assume that t is a NE strategy profile of G^C_∞, with a resource r satisfying $P_r(c(t)_r) > P_N$ and $P_r(c(t)_r \geq P_x(c(t)_x) \;\; \forall x \in R$. Let us denote the agent who chooses a resource r with one of his sub agents as k.

Let us remove agent k and put instead of him $|C_k|$ greedy singleton agents, as done in Lemma 6.3. Due to Lemma 6.3 we get that no one of those greedy agents pays more than P_N. Since agent k pais more than P_N he can deviate profitably from t to the strategy of the greedy sub agents and pay P_N. This contradicts the the fact that t was a Pure NE strategy profile of G^C_∞.

From Lemma 3.4, in any profile exists a sub agent paying P_N or more, therefore, the highest cost for an agent in any strategy profile cannot be lower than P_N. Thus, in any strategy profile of G^C_∞ the highest cost of an agent is P_N. □

Corollary 20 *If $c(NE(G)) = 1$ then $c(NE(G^C_\infty)) = c(NE(G))$, since this is the only congestion vector where the cost of all resources is at most P_N.*

7.2 Extended Underlying Game

In this section we look on CCG with a non-simple Congestion Game as the underlying game. We provide an example that such CCG possess no Pure NE even if the underlying game has only three agents and three resources.

Example 7 *Exists a CCG with three sub agents and three resources without Pure NE, when the underlying game is not a SCG.*

Let G be a Congestion Game with three identical resources and three agents. Each agent of G chooses two of the three resources. The cost of each resource is $P(n) = 6 - \frac{6}{n}$.

Let $C = [\{1, 2\}\{3\}]$, and G^C is the CCG with underlying game G and partition C. After omitting identical strategies due to sub agents symmetry G^C looks as follows:

G^C	AB	AC	BC
AB,AB	-16,-8	-14,8	-14,-4
AC,AC	-14,-4	-16,4	-14,-4
BC,BC	-14,-4	-14,-4	-16,-8
AB,AC	-11,-7	-11,-7	-12,-6
AB,BC	-11,-7	-12,-6	-11,-7
AC,BC	-12,-6	-11,-7	-11,-7

Note that the compound agent's strategies (AB,AB) , (AC,AC) and (BC,BC) are dominated by the other three strategies. In the remaining three strategies it's a matching pennies game, thus posses no Pure NE. Therefore, in case of a non-Simple Congestion Game as the Underlying Game, the CCG may not has a pure NE, even for the simplest examples.

Less sub agents or less resources will lead to a trivial CCG - in case of two sub agents the CCG will have one agent and in case of two resources sub agents have only one strategy. However, narrowing down the sub agents strategy space may leave us with a CCG that posses a Pure NE.

7.3 Extended Resource Functions

Here we assume that the agents gain utility when selecting resources. Here, similarly to the cost case, we assume that $P_N > 0$ and G^C posses a Pure NE. The Total Cost (TC) in such model is the Social Welfare, but we will keep our old notation of TC and TCR when we will refer to Social Welfare and Social Welfare Ratios.

Remark 7.1 *Note that all the results regarding NE of SCG and CCG proven earlier in this paper are valid here. This is since adding/subtracting a fixed constant to the costs, which doesn't affect the NE of the game.*

Since TCR is the ratio between two positive numbers R^+ is the maximal possible image of TCR. As the following examples shows TCR are tightly bounded by 0 and ∞, thus any possible value of TCR may be attained, as the following examples show:

Example 8 *Consider the following example with two resources (A, B) and two sub agents. The resource utilities functions are as follows, when M is an arbitrary large number:*

Resource / Agents Number	1	2
A	M	2
B	1	0

Note that from Theorem 1 in any Pure NE strategy profile s of G both agents choose A making $TC(s) = 4$. Let C be a grand coalition of the two agents ($C = \{1, 2\}$) and G^C a CCG.

In any pure NE strategy profile t of G^C one sub agent will choose A and another B, making $TC(t) = M + 1$. Thus $TCR(t, s) = \frac{M+1}{4}$, which is unbounded when $M \to \infty$. Since this is a two sub agents case, it can be extended to a model with increasing or decreasing Δ's, linear, non linear or any cost functions.

Example 9 *Consider the following example with two resources* (A, B) *and four sub agents. The resource utilities functions are as follows, when* M *is an arbitrary large number:*

G_S	1	2	3	4
A:	12	10	**8**	6
B:	**M**	7	3.5	0

From Theorem 1 in any Pure NE strategy profile s of G there are three agents who choose A and one who chooses B, making the $TC(s) = M + 3 \cdot 8$. Let G^C be the CCG with underlying game G and the partition $C = [\{1, 2\}\{3, 4\}]$. In strategic form G^C looks as follows:

G_C	AA	AB	BB
AA:	12, 12	16, M+8	20, 14
AB:	M+8, 16	**17, 17**	15.5, 7
BB:	14, 20	7, 15.5	0, 0

In the unique NE strategy profile t (marked in bold) both agents play (A,B) thus $TC(t) = 34$. Therefore, $TCR(t, s) = \frac{34}{M+24}$, which goes to zero when $M \to \infty$.

Combining those two examples we get that in the utility model TCR can receive any positive value, and the only bounds on it are 0 and ∞. This means that the social welfare ratios between NE strategy profiles of G and G^C can be anything. From this we can also say that the Price of Anarchy for SCG is unbounded in the utility case.

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משחקי צפיפות קואליציוניים

סרגיי קוניאבסקי

משחקי צפיפות קואליציוניים

חיבור על מחקר

לשם מילוי חלקי של הדרישות לקבלת התואר

מגיסטר למדעים בחקר ביצועים וניתוח מערכות

סרגיי קוניאבסקי

הוגש לסנט הטכניון – מכון טכנולוגי לישראל

פברואר 2006 חיפה שבט התשס"ז

המחקר נעשה בהנחיית פרופ' רן סמורודינסקי ופרופ' אורי רוטבלום בפקולטה לתעשיה וניהול.

בזכות סבלנותם הרבה, הכוונתם ועצתם אתם מחזיקים ביד תיזה זאת. בזכותם היה המחקר למעניין, פורה ומשמעותי, ויכלתי להגיע לכדי הגשתה.

אני מודה לקרן גוטווירט ולטכניון על התמיכה הכספית הנדיבה בהשתלמותי.

תוכן עניינים

תקציר:

משפחת משחקי הצפיפות (Congestion Games) היא אחת מהמשפחות המעניינות מבין המשחקים הלא שיתופיים, שכבר בשנת 1952, נעשה בו שימוש לתיאור בעיית תחבורה, ב-[21]. משחקים אלו מתוארים על ידי מודל פשוט, והם יכולים לתאר מגוון רחב של בעיות, כגון בעיות תחבורה, תעבורה, נצילות. בנוסף, יש להם תכונות רבות המובילות לתוצאות רבות. משחק צפיפות, שהוגדר באופן רשמי ב-[13], נתון ע״י אוסף שחקנים N, אוסף משאבים R. כל שחקן בוחר משאב או תת קבוצה של משאבים. תשלום על שימוש במשאב תלוי במשאב ובכמות השחקנים שבחרו משאב זה, ללא קשר לזהות השחקנים. גובה התשלום מתואר ע״י פונקציות תשלום (או תועלת) P_r, המוגדרת לכל משאב r. כמו כן התשלום של שחקן במשחק צפיפות הוא סכום התשלומים על המשאבים שבחר. תת קבוצה חשובה של משחקים אלו היא משפחת משחקי הצפיפות הפשוטים (Simple Congestion Games) שבהם כל אחד מהשחקנים בוחר משאב יחיד מ-R, ותשלומו תלוי רק במשאב שבחר ובמספר השחקנים שבחרו משאב כמוהו.

לאורך התיזה נדון במודל שרק לאחרונה הופיע בספרות (ב-[6] ו-[7]), שנקרא לו משחקי צפיפות קואליציוניים. מודל זה מרחיב את מודל משחקי הצפיפות ע״י הוספת מבנה קואליציות על השחקנים, המוגדר ע״י חלוקה (Partition). השחקנים (החברים בקואליציות) משחקים את משחק צפיפות, כשחברי אותה קואליציה משתפים פעולה ביניהם והתשלום של הקואליציה הוא סכום התשלומים של חברי הקואליציה, שנגזר מהמשחק הרגיל.

במודל משחקי צפיפות הרגיל השחקנים המשחקים את המשחק הם ישויות עצמאיות, כשכל אחד מהם משחק לבדו מול כל השחקנים האחרים. אבל במציאות ישנם מקרים רבים עם מבנה חברות מעל לרמת השחקנים, כשכל אחת מהן שולטת במספר שחקנים המשחקים את המשחק. במצב זה לא השחקן הוא שמקבל את ההחלטה, אלא החברה, ובהחלטה זאת יש התחשבות באינטרסים של שחקנים אחרים של אותה החברה. למשל, נוכל להסתכל על משחק שיבוץ, שבו יש מספר משימות שצריכים לעבור עיבוד על אחת מבין המכונות. אם נסתכל על המשחק שבו כל משימה בוחר את המכונה בעצמה נקבל משחק צפיפות רגיל. לעומת זאת אם כל משימה היא חלק מפרוייקט המכיל מספר משימות לעיבוד במקביל, והפרוייקט מקבל את ההחלטה על השיבוץ. אזי המודל שאנו נקבל הוא של משחק צפיפות קואליציוני.

כאמור, מודל משחקי הצפיפות שימושי במגוון רחב של בעיות בתחומי תחבורה, מדעי המחשב, כלכלה ועוד. בכדי להמחיש זאת נביא דוגמא, שתבהיר את הצורך במודל הקואליציוני, מעולם התחבורה:

במודל יש שתי חברות הובלה מתחרות. לראשונה (חברה א׳) שתי משאיות ולשניה (חברה ב׳) משאית אחת. ניתן להוביל סחורה מכל אחד משני המחסנים שנסמנם 1 ו-2. כל משאית נשלחת ע״י החברה לאחד משני המחסנים, והתשלום עליה תלוי במחסן שבחרה, וכמה משאיות נוספות בחרו אותו מחסן כמוה. אם אנו נתייחס לנהג המשאית בתור השחקן המחליט ולמחסן בתור משאב נקבל משחק צפיפות פשוט. אבל בדוגמא שלנו מקבלי ההחלטות אינם הנהגים, אלא חברות ההובלה הן שמחליטות איזה למחסן תיסע כל משאיות שברשותה. כמו כן התשלום של החברה הוא סך התשלומים על המשאיות בחברה. שימו לב כי האסטרטגיה שבה חברה א׳ בוחרת

את מחסן 1 לשתי המשאיות שברשותה אינה חוקית במשחק צפיפות רגיל, שכן שחקן יכול לבחור כל משאב פעם אחת לכל היותר. לעומת זאת במשחק קואליציוני היא מותרת, כאשר שתי המשאיות של חברה א׳ מופנות למחסן I. לכן במודל הרגיל, כשחברות הם השחקנים ומחסנים הם המשאבים, לא ניתן לממש משחק זה, ואנו נדרשים להשתמש במודל של משחקי צפיפות קואליציוניים.

ניתן מספרים בכדי להמחיש את הדוגמא בצורה מספרית כשנגדיר את התשלומים למשאית כדלקמן:

מחסן / מס׳ משאיות	**1**	**2**	**3**
מחסן 1	1	3	5
מחסן 2	1	2	4

למשל, אם רק משאית אחת נוסעת למחסן מס׳ 2 אזי היא תשלם יחידת כסף אחת. אם יהיו שתי משאיות – כל אחת תשלם שתי יחידות כסף. בתיזה אנו נראה שהמשחק עם התשלומים המתוארים אינו משחק צפיפות לכל סט משאבים ושחקנים שנבחר.

מבנה הקואליציות במשחק שלנו הוא קואליציה אחת היא זוג (לחברה ראשונה שתי משאיות) והאחרת שחקן בודד (לחברה השנייה משאית יחידה), למשל באופן הבא: $C=\{1,2;3\}$.
לכן, התשלומים במשחק הצפיפות הקואליציוני המתאים יהיו כדלקמן:

סטר׳ של א׳ / סטר׳ של ב׳	**1**	**2**
1,1	5+5,5	3+3,1
1,2	3+1,3	1+2,2
2,2	2+2,1	4+4,4

לדוגמא, אם חברה א׳ בחרה לשלוח את שתי המשאיות שלה למחסן 2 וחברה ב׳ בחרה לשלוח למחסן 1, אזי חברה א׳ תשלם 2+2=4 יחידות, שתי יחידות על כל משאית ששלחה, וחברה ב׳ תשלם יחידת כסף אחת על המשאית שלה.

בספרות של משחקי צפיפות, שהגדרתם הבסיסית הופיע ב-[13], יש התייחסות רבה לשתי התכונות הבאות:

1. נסמן ב-s_i אסטרטגיה של שחקן i ונסמן ב- $U_i(s_1,...,s_n)$ את התשלום של i בפרופיל $(s_1,...,s_n)$. פוטנציאל מדויק היא פונקציה PT, ממרחב פרופילי האסטרטגיות לממשיים, כך שלכל שחקן i ולכל שתי אסטרטגיות כלשהן s_i, t_i של שחקן i המשוואה הבאה מתקיימת:
$$PT(s_1, \dots s_n) - PT(s_1, \dots s_{i-1}, t_i, s_{i+1}, \dots s_n) = U_i(s_1, \dots s_n) - U_i(s_1, \dots s_{i-1}, t_i, s_{i+1}, \dots s_n)$$
נאמר שלמשחק יש פוטנציאל מדויק (Exact Potential) אם במשחק קיימת פונקציה PT כנ״ל.

2. מסלול שיפור הוא סדרה של פרופילי אסטרטגיות, כאשר הפרופיל הבא בסדרה מתקבל מקודמו ע״י שינוי האסטרטגיה של שחקן אחד בלבד, המרוויח כתוצאה מהשינוי. נאמר למשחק G יש את תכונת FIP (Finite Improvement Path property) אם כל מסלול שיפור ב-G הוא באורך סופי.

ב-[13] הוכח כי למשחקי צפיפות יש את תכונת FIP וישנה שקילות שבין משחקי צפיפות למשחקים שיש להם פוטנציאל מדויק. כרבע מאה קודם לכן, ב-[18] הוכח כי יש להם שווי משקל טהור, תוך כדי שימוש עקיף בפוטנציאל מדויק.
השאלה הבסיסית היא האם תכונות אלו מתקיימות במשחקי צפיפות קואליציוניים. תשובות חלקיות לשאלות אלו ניתן למצוא כבר בספרות. למשל, ב-[6] הוכח כי במידה ופונקציות התשלום הן ליניאריות אזי יש למשחק פוטנציאל מדויק, וב-[7] הוכח כי במידה ופונקציות תשלום על המשאבים הן קמורות אזי יש למשחק שווי משקל נאש טהור. אנו נוסיף תנאים הכרחיים ומספיקים לקיום תכונות אלו ונראה, בין היתר, קיום שיווי משקל נאש טהור במקרה בו הקואליציות קטנות ודוגמאות למשחקים בהם כל שלושת התכונות אינן מתקיימות. בנוסף, אנו נאפיין את המקרים בהם יש למשחק פוטנציאל מדויק, תחת דרישות מינימליות ממבנה הקואליציות.

תחום נוסף שנחקור הוא התועלת החברתית (Social Welfare) בשווי משקל , המוגדר כסך כל התשלומים (או התועלות) של כלל השחקנים במשחק. נראה מהו במחיר האנרכיה (Price of Anarchy), שהוגדר ב-[10] כיחס התועלת החברתיות של פרופיל האסטרטגיות הממקסם את התועלת החברתית ופרופיל שיווי משקל נאש הממזער אותה. כמו כן נרחיב את התוצאות של [2] ו-[3], שדנו במחיר האנרכיה במשחק צפיפות שבו עלות המשאבים היא ליניארית או פולינומיאלית למקרה הכללי. כמו כן אנו נבחן מהי ההשפעה של מבנה קואליציות על התועלת החברתית, ונראה חסמים על היחס בין התועלת החברתית של שווי משקל נאש טהור במודלים השונים, כאשר חלק מהחסמים יהיו הדוקים.