OPTIMAL GRID $L(2,\vec{1_k})$ COLORINGS

Amrita Pati
(email: **f1999013@bits-pilani.ac.in**)
Birla Institute of Technology and Science
Pilani-333 031 (Rajasthan)

Devyani Gupta
(email: **f1999057@bits-pilani.ac.in**)
Birla Institute of Technology and Science
Pilani-333 031 (Rajasthan)

Shashank R
(email: **f1999025@bits-pilani.ac.in**)
Birla Institute of Technology and Science
Pilani-333 031 (Rajasthan)

Abstract

In this paper, we investigate the problem of assigning channels (frequencies) to the stations of a wireless network so that interfering stations are assigned channels with a given separation and the number of channels used is minimized. Since transmitters and receivers in wireless networks correspond to the nodes in a bi-dimensional grid, this paper focuses on coloring the nodes of a bi-dimensional grid. We deal with a special type of coloring called $L(\mathbf{2},\vec{\mathbf{1}_k})$ coloring. Coloring the grid with this constraint models the assignment of channels to very *dense* networks (which will be the case in the near future), where reuse distances are high. We find lower bounds for the coloring and present and prove an algorithm for *optimally* coloring the grid, corresponding to the above mentioned constraint. This algorithm can be used to assign channels to specific, but realistic network topologies.

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1 Introduction

The enormous growth of wireless networks has made the efficient use of the scarce radio spectrum important. A "Frequency Assignment Problem" (FAP) models the task of assigning frequencies (channels) from a radio spectrum to a set of transmitters and receivers, satisfying certain constraints [MPR99]. The main constraint or difficulty against an efficient use of radio spectrum is *interference*, caused by unconstrained simultaneous tranmissions. Interferences can be eliminated (or at least reduced) by means of suitable *channel assignment* techniques, which partition the given radio spectrum into a set of disjoint channels that can be used simultaneously by the stations while maintaining acceptable radio signals. The same set of channels can be used a certain distance later. This is because radio signals get attenuated over distance. Thus, two stations in a network can use the same channel without interferences (*co-channel stations*), provided that they are spaced sufficiently apart. The minimum distance at which co-channels can be reused with no interferences is called *co-channel reuse distance* σ.

In a *dense* network – a network where there is a large number of transmitters and receivers in a small area – interference is more probable to occur. Thus, the reuse distance is high in such areas. In such a topology, it might not be sufficient to assign to adjacent stations, just differing channels. Although, with increasing distance, the constraint on the separation between assigned channels would relax. In other words, channels assigned to nearby stations must be separated by at least a gap which is inversely proportional to the distance between the two stations. A minimum *channel separation* δ_i is required between channels assigned to stations at distance i, with $i < \sigma$, such that δ_i decreases when i increases [Hal80]. The purpose of channel assignment algorithms is to assign channels to transmitters in such a way that the co-channel reuse distance and the channel separation constraints are verified and the difference between the highest and the lowest channels assigned is as small as possible [BPT00].

Formally, the *channel assignment problem with separation* (CAPS) can be modelled as an appropriate coloring problem on an undirected graph $G = (V,E)$ representing the network topology, whose vertices in V correspond to stations, and edges in E correspond to pairs of stations that can hear each other's transmission. [BPT00] Let the set of channels C be a set of non-negative integers $\{0,\ldots,g\}$, and let the distance $d(u,v)$ between stations u and v of the network be the number of edges on a shortest path between the corresponding vertices u and v of G [Har72]. Given the co-channel reuse distance σ and the channel separation vector (δ_1,δ_2,...,$\delta_{\sigma-1}$) of nonnegative integers, an $L(\delta_1,\delta_2,\ldots,\delta_{\sigma-1})$-coloring of the graph G is a function f from the vertex set V to the set C of channels, or colors, such that

$$|f(u) - f(v)| \geq \delta_i \quad d(u,v) = i \quad 1 \leq i \leq \sigma - 1 \tag{1}$$

The same color can be used for vertices which are at distance σ or greater, while colors which are at least δ_i apart must be used for vertices at distance i, with $i \leq \sigma - 1$. For example, an $L(2,1)$-coloring is an assignment of colors from the set $\{0,\ldots,g\}$ to the vertices of the graph G (V,E) such that *adjacent* vertices get colors which differ by *at least two* and vertices at distance two get at least *different* colors. Similarly, we have the colorings $L(2,1,1)$, $L(2,1,1,1)$ and so on.

A g-$L(\delta_1,\delta_2,\ldots,\delta_{\sigma-1})$-coloring is an $L(\delta_1,\delta_2,\ldots,\delta_{\sigma-1})$ coloring of the graph G such that $C = \{0,\ldots,g\}$. Such a coloring is *optimal* if it uses the smallest possible value for g. The largest color used by an *optimal* L $(\delta_1,\delta_2,\ldots,\delta_{\sigma-1})$-coloring is denoted by $\lambda(G)$. Since the set C contains 0, the overall number of colors involved by an optimal coloring f is in fact $\lambda(G)$+1 (although, due to the channel separation constraint, some colors in $\{1,\ldots,\lambda(G)\text{-}1\}$ might not be actually assigned to any vertex). Finally, given a network G, the co-channel reuse distance σ and the channel separation vector $(\delta_1,\delta_2,\ldots,\delta_{\sigma-1})$, the CAPS problem does correspond to finding an optimal $L(\delta_1,\delta_2,\ldots,\delta_{\sigma-1})$-coloring of G.

In this paper, we introduce the notion of $L(2,\vec{1_k})$ colorings, and study them for bi-dimensional grids. An $L(2,\vec{1_k})$-coloring of a graph is simply an $L(\delta_1,\delta_2,\ldots,\delta_{\sigma-1})$-coloring where $\delta_1 = 2$ and $\delta_i = 1$ where $2 \leq i \leq \sigma - 1$. The subscript k denotes the *number of pairs of ones* that follow the 2 in the constraint. $k \geq 2$. So, we have $L(2,1,1,1,1)$ colorings, $L(2,1,1,1,1,1,1)$ colorings and so on as instances of this coloring.

The $L(2,\vec{1_k})$ problem deals with coloring graphs in such a way that adjacent vertices are assigned colors that differ by at least 2 and vertices at distance greater than 1 but less than σ are assigned colors that differ by at least 1. The reason for choosing such a complex coloring of a bi-dimensional grid is that this coloring very closely models the assignment of channels to transmitters and receivers in a very dense wireless network. The vertices of the graph can represent the transmitters in a network and the colors assigned correspond to the channels assigned to these transmitters such that there is minimum interference.

Optimal $L(2,1)$ and $L(2,1,1)$ colorings have been proposed by Bertossi et al [BPT00], for bi-dimensional grids and other topologies. Approximated solutions for outerplanar, permutation and split graphs are presented by Bodlaender et al [BKTvL00]. In this paper, we present the optimal colorings for bi-dimensional grids subject to the $L(2, \vec{1_k})$ constraint.

2 Notation and Terminology

Following are a few definitions of terms used in the paper:

- Clique [BPT00]: A clique is a graph (possibly a subgraph of some other graph also) in which each vertex is connected to all the remaining vertices.

- Augmented graph, $G_{N,\sigma}$ [BPT00]: If N is the graph corresponding to the network, the augmented graph $G_{N,\sigma}$ is obtained as follows: The vertex set of this graph is the same as that of N. But two vertices p and q in this graph are connected by an edge *iff* the distance between the vertices in N, *i.e* $d(p,q) \leq \sigma - 1$.

- σ [BPT00]: This is the reuse distance: the distance after which the same channel can be assigned without interference. For a g-$L(2, \vec{1_k})$ coloring, we have
$$\sigma = 2(k+1).$$

- Bi-Dimensional grid, N or (G,r,c) [BPT00]: A planar grid with r rows and c columns in which the degree of every vertex that is not at the boundary of the grid is 4, and every boundary vertex has a degree of 3 or 2.

3 $L(2, \vec{1_k})$ colorings of (G, r, c)

In Section 1, we discussed informally the notion of $L(2, \vec{1_k})$ colorings. Formally,

- an $L(\delta_1, \delta_2, \ldots, \delta_{\sigma-1})$-coloring [BPT00]: is a function f from the vertex set V of the graph $G(V,E)$ to the set C of channels, or colors, such that
$$|f(u) - f(v)| \geq \delta_i \quad d(u,v) = i \quad 1 \leq i \leq \sigma - 1$$

- a g-$L(\delta_1, \delta_2, \ldots, \delta_{\sigma-1})$-coloring: is an $L(\delta_1, \delta_2, \ldots, \delta_{\sigma-1})$-coloring of the graph G such that the set of colors used is $C = \{0, \ldots, g\}$.

3.1 Lower bounds on $L(2, \vec{1_k})$ colorings of (G, r, c)

The lower bound on an optimal coloring of a graph is the minimum number of colors required to color the vertices so that they violate no constraints. Clearly, the value of this lower bound depends on the constraint specified. A common technique to find a lower bound on the constrained coloring of N is to find the augmented graph of N corresponding to the constraint.

The channel assignment problem with no separation constraint and co-channel reuse constraint σ, namely the $L(1, 1, \ldots, 1)$-coloring problem on the graph N, can be reduced to a *simple* coloring problem simply as follows: Color the augmented graph $G_{N,\sigma}$! Suppose we find a clique in $G_{N,\sigma}$ of size s. Then coloring this graph takes *at least* s colors. This is also the lower bound for the coloring of N. Thus, the size of the largest clique in $G_{N,\sigma}$ is a lower bound on the number of channels required to solve the channel assignment problem without channel separation constraint. Clearly, in the presence of both the channel separation and the channel-reuse constraints, at least as many channels are required as in the presence of only the reuse constraint only. Formally, a lower bound for the $L(1, 1, \ldots, 1)$-coloring problem is also a lower bound for the $L(\delta_1, 1, \ldots, 1)$-coloring problem, and for the $L(2, 1, \ldots, 1)$-coloring problem in particular. [BPT00]

Theorem 1. *If there is a g-$L(2, 1, 1, \ldots, 1)$ coloring of (G, r, c) with $r \geq \sigma - 1$ and $c \geq \sigma$ or $r \geq \sigma$ and $c \geq \sigma - 1$, then $g \geq \lceil \frac{\sigma^2}{2} \rceil$.*

Proof. We prove this by showing the existence of a clique of size $\lceil \frac{\sigma^2}{2} \rceil$ in the augmented graph corresponding to the constraint. Let the reuse distance of the $L(2, 1, 1, \ldots, 1)$ coloring be σ. Construct the clique corresponding to an $L(1, 1, 1, \ldots, 1)$ constraint, which also has its reuse distance σ. The distance between any two points in this clique must be not more than $\sigma - 1$. We demonstrate an inductive procedure for identifying cliques, with the reuse distance σ increasing in steps of 1. Consider the smallest clique in Figure 1. This corresponds to a reuse distance of 3. Clearly, no two points inside the clique are at distance more than 2 from each other. We construct a clique corresponding to a reuse distance of 4 from this clique by adding a set of points parallel to *each lower edge* of the clique. (Figure 1.) The new clique will have these sets of points as its lower edges. No points inside the new clique are now at distance more than 3 from each other. Figure 1 shows the additional vertices that have to be added to the cliques of sizes 4 and 5 to construct the cliques corresponding to reuse distance of 5 and 6.

We notice that the pattern in which new vertices are added to the clique is as follows. When $\sigma - 1$ for a clique is even, we add σ vertices to obtain the clique of the next reuse distance. When $\sigma - 1$ is odd, we add $\sigma + 1$ vertices to

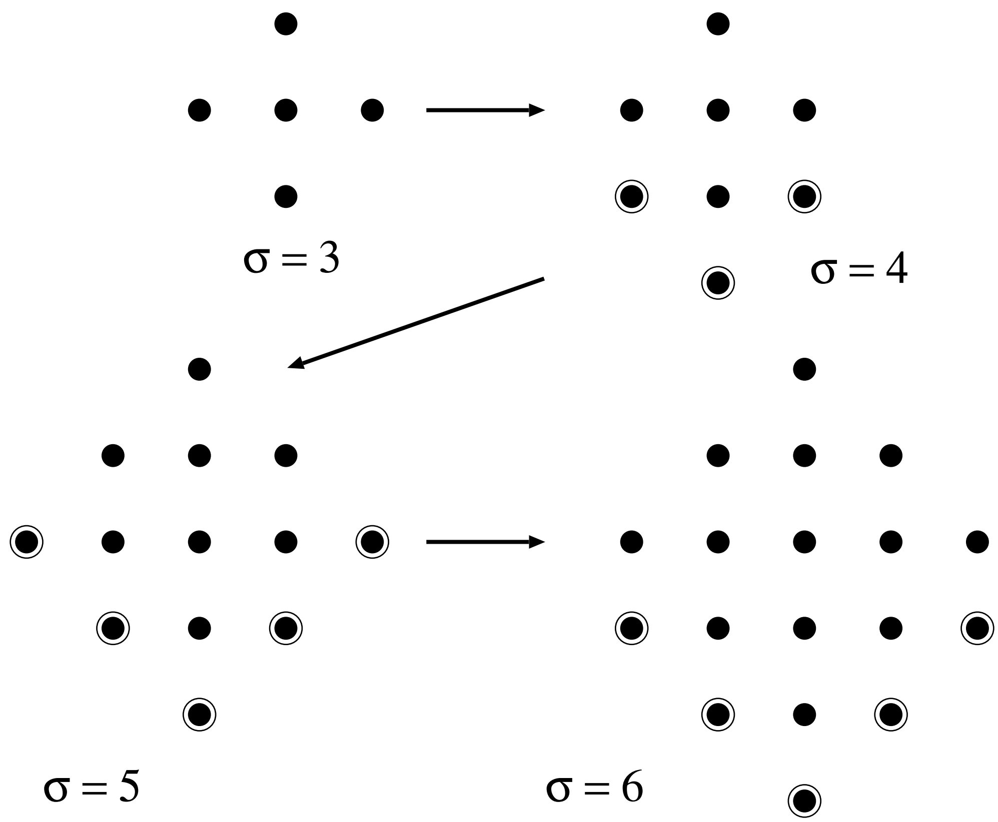

Figure 1: Cliques in $G_{N,\sigma}$ corresponding to increasing values of σ. The marked circles represent the new vertices added at each level.

obtain the next clique. We also notice that a clique corresponding to reuse distance σ contains $\lceil \frac{\sigma^2}{2} \rceil$ vertices. This can be verified from the cliques shown in Figure 1. Let v indicate the number of vertices in the clique.

$$\begin{array}{lll} \text{For} & \sigma = 3, & v = 5, \\ \text{For} & \sigma = 4, & v = 8, \\ \text{For} & \sigma = 5, & v = 13, \\ \vdots & & \\ \text{For} & \sigma = n, & v = \lceil \frac{n^2}{2} \rceil \end{array}$$

By induction, we prove that the formula holds good for any σ. The base case of the induction is when $\sigma = 3, 4$. The formula clearly holds in the base case. Lets assume that the formula is true for all $\sigma \leq n$. We have to show that this is true for $\sigma = n + 1$.

Case 1:($\sigma - 1$ is even) We know that in this case, we add σ new vertices to the present clique. Thus the resulting number of vertices in the new clique is

$$\lceil\frac{\sigma^2}{2}\rceil + \sigma = \frac{\sigma^2 - 1}{2} + 1 + \sigma = \frac{(\sigma+1)^2}{2} = \lceil\frac{(\sigma+1)^2}{2}\rceil \quad (2)$$

Case 2:($\sigma - 1$ is odd) A similar analysis shows that the formula holds true in this case as well.

Thus, the lower bound for a g-$L(2, 1, 1\ldots1)$ is $\lceil\frac{\sigma^2}{2}\rceil$. This proves that the theorem holds good for any σ. □

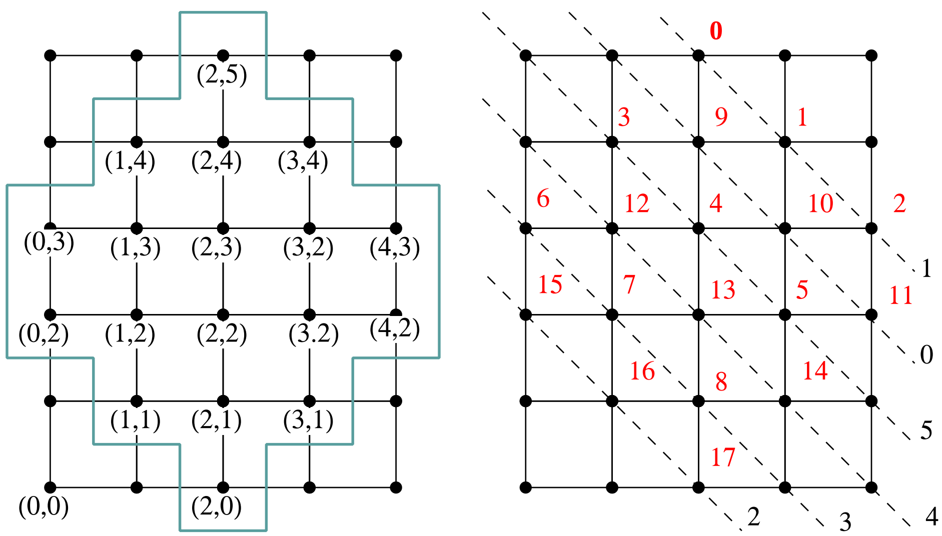

Figure 2: $L(2, 1, 1, 1, 1)$ Coloring of a grid: The clique consists of vertices inside the colored polygon. The colors assigned to the vertices are marked in red. The diagonals are marked with their mod values. Since $\sigma = 6$, the mod values range from 0 to 5.

3.2 Optimal Grid-g-$L(2, \vec{1_k})$ colorings

We now aim to find a coloring that uses a number of colors that is as close to the lower bound (deduced earlier) as possible. We proceed by finding a coloring for the vertices of the clique in the augmented graph corresponding to the g-$L(2,\vec{1_k})$ constraint. If the vertices outside the clique require no more colors, then we have found an optimal coloring.

Consider the g-$L(2,\vec{1_k})$ coloring of a bi-dimensional grid when k=2. As discussed earlier, its augmented graph has a clique of size 18. So we would require at least 18 colors. A coloring of the clique corresponding to this constraint is shown in Figure 2.

If we represent each vertex by its coordinates (i,j) in a plane, then we observe that along each diagonal, the quantity $(i+j)\text{mod}(2(k+1))$ remains constant. Hence we can identify each diagonal by its *mod* value. We follow a strategy of coloring alternate diagonals with consecutive colors as shown in Figure 2. This strategy ensures that adjacent vertices are separated by a minimum of two colors, and vertices assigned consecutive colors always occur along a diagonal, so they are separated by a distance of two.

Observing this coloring scheme, we present an algorithm to color the vertices of a bi-dimensional grid satisfying the L(2,1,1,1,1) constraint.

$$f(u) = \begin{cases} 0 & (i+j) \equiv 1 \bmod 6 \quad i \bmod 3 = 2 \\ 1 & (i+j) \equiv 1 \bmod 6 \quad i \bmod 3 = 0 \\ 2 & (i+j) \equiv 1 \bmod 6 \quad i \bmod 3 = 1 \\ & \\ 3 & (i+j) \equiv 5 \bmod 6 \quad i \bmod 3 = 1 \\ 4 & (i+j) \equiv 5 \bmod 6 \quad i \bmod 3 = 2 \\ 5 & (i+j) \equiv 5 \bmod 6 \quad i \bmod 3 = 0 \\ & \\ 6 & (i+j) \equiv 3 \bmod 6 \quad i \bmod 3 = 0 \\ 7 & (i+j) \equiv 3 \bmod 6 \quad i \bmod 3 = 1 \\ 8 & (i+j) \equiv 3 \bmod 6 \quad i \bmod 3 = 2 \\ & \\ 9 & (i+j) \equiv 0 \bmod 6 \quad i \bmod 3 = 2 \\ 10 & (i+j) \equiv 0 \bmod 6 \quad i \bmod 3 = 0 \\ 11 & (i+j) \equiv 0 \bmod 6 \quad i \bmod 3 = 1 \\ & \\ 12 & (i+j) \equiv 4 \bmod 6 \quad i \bmod 3 = 1 \\ 13 & (i+j) \equiv 4 \bmod 6 \quad i \bmod 3 = 2 \\ 14 & (i+j) \equiv 4 \bmod 6 \quad i \bmod 3 = 0 \\ & \\ 15 & (i+j) \equiv 2 \bmod 6 \quad i \bmod 3 = 0 \\ 16 & (i+j) \equiv 2 \bmod 6 \quad i \bmod 3 = 1 \\ 17 & (i+j) \equiv 2 \bmod 6 \quad i \bmod 3 = 2 \end{cases}$$

Figure 3: Algorithm to assign colors with the $L(2,1,1,1,1)$ constraint

Algorithm Grid-18-L(2,1,1,1,1)-coloring for (G,r,c): If $r \geq 6$ and $c \geq 5$ or $r \geq 5$ and $c \geq 6$, then assign to each vertex $u=(i,j)$ the color $f(u)$ as shown in Figure 3.

Here, we present the algorithm to optimally color a bi-dimensional grid. It uses a number of colors equal to the lower bound which we found out.

Algorithm Grid-g-$L(2,\vec{1}_k)$-coloring for (G,r,c): If $r \geq 2(k+1)$ and $c \geq 2k+1$ or $r \geq 2k+1$ and $c \geq 2(k+1)$, then assign to each vertex (i,j) the color

$$f(i,j) = (k+1) \cdot p(i,j,k) + q(i,j,k) \tag{3}$$

where the functions p and q are defined as follows:

$$p(i,j,k) = \rho((i+j) \bmod (2(k+1)), k) \tag{4}$$

where

$$\rho(x,k) = \begin{cases} \frac{x-1-k}{2} \bmod (k+1) & (k+x) \bmod 2 = 1 \\ \rho(x+1,k) + (k+1) & (k+x) \bmod 2 = 0 \end{cases} \tag{5}$$

$$q(i,j,k) = (i \bmod (k+1) + p(i,j,k) + 1) \bmod (k+1) \tag{6}$$

Theorem 2. *The Grid-g-$L(2,\vec{1}_k)$-coloring algorithm is optimal for a bi-dimensional grid B of size $r \times c$, with $r \geq 2(k+1)$ and $c \geq 2k+1$ or $r \geq 2k+1$ and $c \geq 2(k+1)$.*

Proof. Since the maximum values that p and q can take are $2k+1$ and k respectively, in this case:

$$f(i,j) = (k+1)(2k+1) + k = 2(k+1)^2 - 1.$$

This is the largest color that can be assigned to any vertex in the scheme. In other words, a maximum of $2(k+1)^2$ colors are used by the scheme. Since we showed earlier that this was the lower bound for $L(2,\vec{1}_k)$ coloring, this function is an optimal coloring, if it satisfies the channel separation and reuse constraints.

Adherence to the channel reuse constraint: Suppose two distinct vertices (i_1,j_1) and (i_2,j_2) have the same color. Then,

$$\begin{aligned} p(i_1,j_1) = p(i_2,j_2) &\Rightarrow x(i_1,j_1) = x(i_2,j_2) \\ &\Rightarrow (i_1+j_1) \bmod 2(k+1) = (i_2+j_2) \bmod 2(k+1) \\ &\Rightarrow ((i_1-i_2)+(j_1-j_2)) \bmod 2(k+1) = 0. \end{aligned}$$

Thus, two distinct vertices get the same color only when their distance is a multiple of $2(k+1)$.

Adherence to the channel separation constraint: Consider two adjacent vertices (i,j) and $(i+1,j)$. The colors assigned to them would be $f(i,j)$ and $f(i+1,j)$.

$$f(i+1,j) - f(i,j) = (k+1)\{p(i+1,j,k) - p(i,j,k)\} + \{q(i+1,j,k) - q(i,j,k)\} \tag{7}$$

Consider the first term in the RHS of the previous equation. If we define $x = (i + j + 1) \bmod 2(k + 1)$, then we have

$$\begin{aligned} p(i+1,j,k) - p(i,j,k) &= \rho(x,k) - \rho(x-1,k) \\ &= -(k+1) \end{aligned}$$

This would mean that in equation 7, we would have:

$$f(i+1,j) - f(i,j) = -(k+1)^2 + q(i+1,j,k) - q(i,j,k) \tag{8}$$

Since the maximum and minimum values of $q(i+1,j,k) - q(i,j,k)$ are k and 0, we have

$$\begin{aligned} \max |f(i+1,j) - f(i,j)| &= (k+1)^2 \\ \min |f(i+1,j) - f(i,j)| &= |-(k+1)^2 + k| \\ &= k(k+1) + 1 \geq 7 \quad \text{since } k \geq 2 \end{aligned}$$

Thus, we see that any two horizontally adjacent vertices are colored with colors differing by *at least seven*. Similarly, it can be proved that two vertically adjacent vertices are colored with colors differing by *at least five*. Hence, the channel separation constraint is also verified. □

4 Conclusions

In this paper, we have defined an optimal $L(2, \vec{1_k})$ coloring scheme for bi-dimensional grids. An interesting class of open problems is devising similar schemes for other planar topologies such as hexagonal grids, rings etc. as well as for these topologies in higher dimensions.

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References

[BKTvL00] Hans L. Boedlander, Ton Kloks, Richard B. Tan, and Jan van Leeuwen. λ-coloring of graphs. *STACS*, (1770):395–406, 2000.

[BPT00] Alan A. Bertossi, Cristina M. Pinotti, and Richard B. Tan. Efficient use of radio spectrum in wireless networks with channel separation between close stations. *Journal of the ACM*, pages 18–27, 2000.

[Hal80] W.K. Hale. Frequency assignment: Theory and application. *Proceedings of the IEEE*, 68:1497–1514, 1980.

[Har72] Frank Harary. *Graph Theory*. Addison-Wesley series in Mathematics. Addison-Wesley Publishing Company, Inc., October 1972.

[MPR99] R.A. Murphey, P.M. Pardalos, and M.G.C. Resende. Frequency assignment problems. In Z. Du and P.M. Pardalos, editors, *Handbook of Combinatorial Optimization*. Kluwer Academic Publishers, 1999.