Dirac fermions as a cause of unusual Quantum Hall Effect in Graphene

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S.G. Sharapov, V.P. Gusynin, H. Beck, PRB **69**, 075104 (04);

V.P. Gusynin, S.G. Sharapov, PRB **71**, 125124 (05), PRL **95**, 146801 (05);

cond-mat/0512157; V.P. Gusynin, S.G. Sharapov, J.P. Carbotte, cond-mat/0603267.

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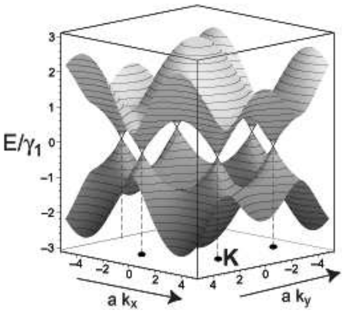
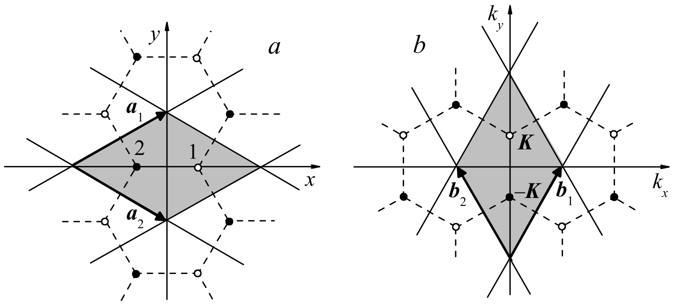
Planar graphite or graphene

$$H = t \sum_{\mathbf{n},\delta} a^\dagger_{\mathbf{n}} b_{\mathbf{n}+\delta} + h.c.,$$

where $t \approx 3eV$, $a = \sqrt{3}a_{CC} = 2.46$Å is the lattice constant of 2D graphite.



BZ



Diabolo

(Left) Two bands touch each other and cross the Fermi level in six $\mathbf{K}$ points located at the corners of the hexagonal 2D Brillouin zone (BZ). (a) Rhombic primitive cell with two non-equivalent positions for carbon atoms. (b) Rhombic extended BZ. Two non-equivalent $\mathbf{K}$ points in the extended BZ, $\mathbf{K}' = -\mathbf{K}$. (Left) $\mathbf{K}$ points also called Dirac or diabolical points due to a toy *Diabolo*.

Hamiltonian for graphene

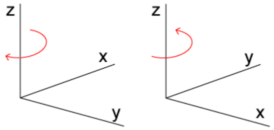
The low-energy excitations at K, K' points labelled as $j = 1, 2$ have a linear dispersion $E_k = \pm v_F k$ with $v_F = (\sqrt{3}/2)ta \approx 10^6 m/s$. These excitations are described by a pair of two-component (Weyl) spinors $\psi_{j\sigma}$, which are composed of the Bloch states residing on the two different sublattices of the bi-particle hexagonal lattice.

$$H = v_F \sum_{\sigma=\uparrow,\downarrow} \int \frac{d^2k}{(2\pi)^2} \bar{\psi}_{1\sigma}(t, \mathbf{k})(\tilde{\gamma}^1 k_x + \tilde{\gamma}^2 k_y)\psi_{1\sigma}(t, \mathbf{k}),$$

where the momentum $\mathbf{k} = (k_x, k_y)$ is already given in a local coordinate system associated with a chosen K point, $\bar{\psi}_{1\sigma} = \psi^\dagger_{1\sigma}\tilde{\gamma}^0$ and $\tilde{\gamma}^{0,1,2} = (\sigma_3, i\sigma_2, -i\sigma_1)$.

P.R. Wallace, PR **71**, 622 (1947); G.W. Semenoff, PRL **53**, 2449 (1984).

Lagrangian for planar graphite

Local coordinate system for the point $\mathbf{K}'$ is related to the system associated with the point $\mathbf{K}$ by a parity transformation: 

$\Longrightarrow$ these two spinors can be again combined in one four-component Dirac spinor $\Psi_\sigma = (\psi_{1\sigma}, \psi_{2\sigma})$. The number of spin components $N_f = 2$. The Lagrangian density

$$\mathcal{L}_0 = \sum_{\sigma=1}^{N_f} v_F \bar{\Psi}_\sigma(t,\mathbf{r}) \left(\frac{i\gamma^0(\partial_t - i\mu)}{v_F} - i\gamma^1\partial_x - i\gamma^2\partial_y \right) \Psi_\sigma(t,\mathbf{r}),$$

where $\bar{\Psi}_\sigma = \Psi_\sigma^\dagger \gamma^0$ and 4×4 γ-matrices belonging to a reducible representation Dirac algebra $\{\gamma^\mu, \gamma^\nu\} = 2\hat{I}_4 g^{\mu\nu}$, $g^{\mu\nu} =$ diag$(1,-1,-1)$.

Final QED$_3$ form of Lagrangian

An external field $\mathbf{B} \perp$ to the plane $\mathbf{A}^{\text{ext}} = (-By/2, Bx/2)$

$$\mathcal{L} = \sum_{\sigma=\pm1} \bar{\Psi}_\sigma(t,\mathbf{r}) \Big[i\gamma^0(\hbar\partial_t - i(\mu - \sigma g/2\mu_B B)) \quad \text{–Zeeman term}$$

$$+ i v_F \gamma^1 \left(\hbar\partial_x + i\frac{e}{c}A_x^{\text{ext}}\right) + i v_F \gamma^2 \left(\hbar\partial_y + i\frac{e}{c}A_y^{\text{ext}}\right) - \Delta \Big] \Psi_\sigma(t,\mathbf{r})$$

$\mu \propto \text{sgn}(V_g)\sqrt{|V_g|} \in [-3600\text{K}, 3600\text{K}]$ when $V_g \in [-100\text{V}, 100\text{V}]$; $\mu > 0$ – electrons.

Δ is a possible excitonic gap (Dirac mass) generated due to Coulomb interaction. Magnetic field favors gap opening:

V.P. Gusynin *et al.*, PRL **73**, 3499 (1995), PRB **66** (02); D.V. Khveshchenko, PRL **87** (01).

Although for simplicity one often says “linear dispersion” all results are valid for Dirac-like spectrum:

$E(\mathbf{k}) = \pm\sqrt{\Delta^2 + v_F^2\mathbf{k}^2}$!

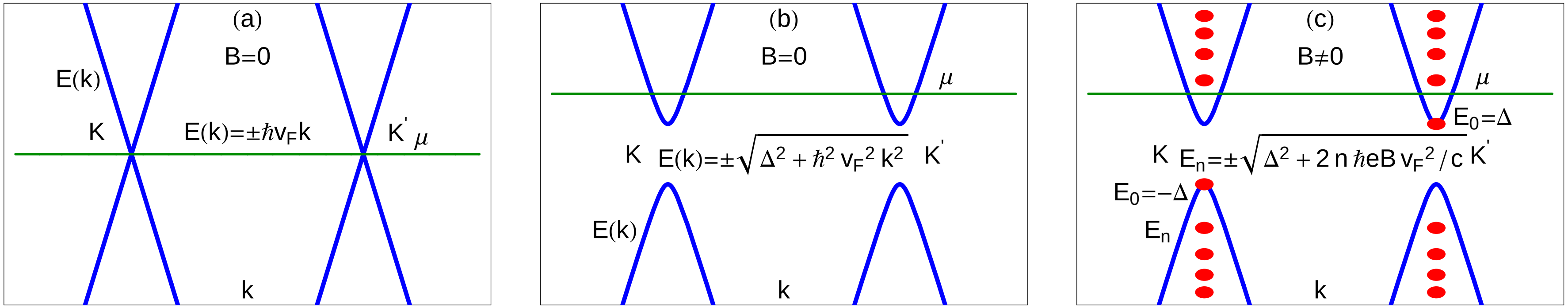
Dirac Landau levels

In nonrelativistic case the distance between LL coincides with the cyclotron energy, $\hbar\omega_c[\text{K}] = \frac{e\hbar B}{m_e c} \sim 1.35B[\text{T}]$ and is the same as Zeeman splitting in graphene.

For the relativistic system the energy scale, characterizing the distance between Landau levels, is

$\hbar\omega_L = v_F\sqrt{\frac{\hbar 2eB}{c}}[\text{K}] \sim 400K \cdot \sqrt{B[\text{Tesla}]}$ for $v_F \approx 10^6$m/s!

This is why SdH effect is observed at room temperature!



(a)The low-energy linear-dispersion . (b) A possible modification of the spectrum by the finite gap Δ. μ is shifted from zero by the gate voltage. (c) Landau levels E_n.

Scheme of analytical calculation

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- Use the Kubo formula for both $\sigma_{xx}(\Omega)$ and $\sigma_{xy}(\Omega)$.
- But before recall quantum magnetic oscillations...

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Shubnikov de Haas effect (in 2D)

$$\sigma_{xx} = \frac{\sigma_0}{1+(\omega_c\tau)^2}\left[1 + 2\sum_{k=1}^{\infty}(-1)^k \cos\left(2\pi k\frac{\mu}{\hbar\omega_c}\right) R_T(k,\mu) R_D(k)\right],$$

where $\omega_c = \frac{eB}{m_e c}$ cyclotron frequency (for $\xi(\mathbf{k}) = \frac{\hbar^2\mathbf{k}^2}{2m_e} - \mu$),

$R_T(k) = \frac{2\pi^2 kT/\hbar\omega_c}{\sinh 2\pi^2 kT/\hbar\omega_c}$ is *temperature amplitude factor*,

$R_D(k) = \exp\left(-2\pi k\frac{\Gamma}{\hbar\omega_c}\right)$ is *Dingle factor*

(amplitude reduction due to impurities, *Dingle temperature* $T_D = \frac{\Gamma}{\pi}$). Here $k_B = 1$ and below $\hbar = 1$ as well.

m_e is extracted from the dependence $\sigma_{xx}(T)$. If this formula is used for graphene, the cyclotron mass m_e would depend on the gate voltage!

Oscillations of DOS

DOS is the sum over LL (Still hard to see oscillations?)

$$D_0(\epsilon) = \frac{N_f eB}{2\pi}\left[\delta(\epsilon-\Delta)+\delta(\epsilon+\Delta)+2\sum_{n=1}^{\infty}\left(\delta(\epsilon-M_n)+\delta(\epsilon+M_n)\right)\right],$$

with $M_n = \sqrt{\Delta^2+2eBn}$. Using the Poisson summation formula $\ldots$ (Mathematical miracle. It oscillates!)

$$D(\epsilon) = \frac{N_f}{2\pi}\mathrm{sgn}(\epsilon)\frac{d}{d\epsilon}\left\{\theta(\epsilon^2-\Delta^2-\Gamma^2)\left[\epsilon^2-\Delta^2-\Gamma^2\right.\right.$$
$$\left.\left.+\;2eB\sum_{k=1}^{\infty}\frac{1}{\pi k}\sin\frac{\pi k(\epsilon^2-\Delta^2-\Gamma^2)}{eB}\exp\left(-\frac{2\pi k|\epsilon|\Gamma}{eB}\right)\right]\right\}.$$

$N_f = 2$ is the number of the spin components

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Oscillating conductivity

$$\sigma_{\rm osc} = \frac{\sigma_0\theta(\mu^2-\Delta^2)}{1+(\omega_c\tau)^2}\sum_{k=1}^{\infty} \cos\left[\frac{\pi k(\mu^2-\Delta^2)}{|eB|}\right] R_T(k,\mu)R_D(k,\mu),$$

where *Dingle and temperature amplitude factors*

$$R_D(k,\mu)=\exp\left(-2\pi k\frac{|\mu|\Gamma}{eB}\right), \qquad R_T(k,\mu)=\frac{2\pi^2kT|\mu|/(eB)}{\sinh\frac{2\pi^2kT|\mu|}{eB}}.$$

R_T, R_D depend on μ - there is a dependence on carrier concentration. In nonrelativistic (NR) case it was:

$$R_D^{\rm NR}(k)\propto \exp\left(-2\pi k\frac{\Gamma}{\hbar\omega_c^{\rm NR}}\right), \qquad R_T^{\rm NR}(k)=\frac{2\pi^2kT/\hbar\omega_c^{\rm NR}}{\sinh 2\pi^2kT/\hbar\omega_c^{\rm NR}},$$

$$\omega_c^{\rm NR}=\frac{eB}{m_ec}$$

For the relativistic system:

$$\hbar\omega_c=\frac{e\hbar Bv_F^2}{c|\mu|}=\frac{e\hbar B}{cm_c}, \qquad m_c=\frac{|\mu|}{v_F^2}\propto\sqrt{|V_g|}!$$

There is temptation to say that the effective carrier mass $= m_c$ in graphene is very small,

but this does not help, due to the phase shift by π of oscillations.

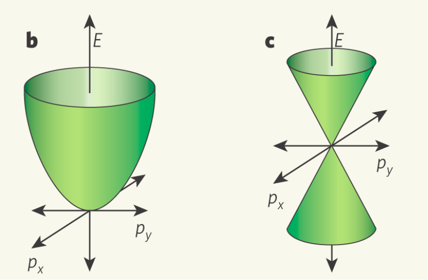
Manifestation of Berry's Phase

Oscillating conductivity

$$\sigma_{xx} \propto \sum_{k=1}^{\infty} \cos\left[2\pi k\left(\frac{B_f}{B}+\frac{1}{2}+\boxed{\beta}\right)\right] R_T(k) R_D(k) R_s(k),$$

where β is Berry's phase. If $\beta = 0 \Rightarrow \times(-1)^k$ in LK formula.

$$\left.\begin{array}{c} E_n = \frac{e\hbar B}{m_e c}\left(n+\frac{1}{2}\right); \\ \beta = 0. \end{array}\right\} \Leftarrow$$



Dirac $\Rightarrow \left\{\begin{array}{c} E_n = v_F\sqrt{\frac{\hbar 2neB}{c}}; \\ \beta = -1/2. \end{array}\right.$

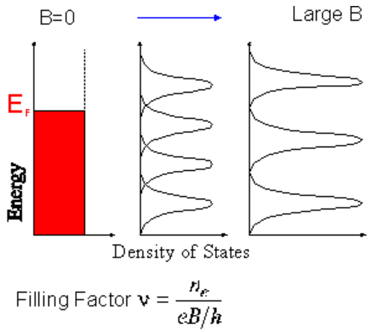
Cross-sectional area of the orbit in k-space:

nonrelativistic $S(\varepsilon) = 2\pi m\varepsilon$ and relativistic $S(\varepsilon) = \pi\varepsilon^2/v_F^2$

Semiclassical quantization condition:

$S(\varepsilon) = \frac{2\pi\hbar eB}{c}\left(n+\frac{1}{2}+\beta\right)$ G.P. Mikitik and Yu. V. Sharlai, PRL **82**, 2147 (1999).

Oscillating conductivity towards QHE



$$\cos\left[\pi k(\mu^2 - \Delta^2)/|eB|\right] \to \cos\left(\pi \nu_B\right).$$

Unusually the minima of σ_{osc} at the odd fillings

$$\nu_B = 2n+1, \quad \text{where} \quad \nu_B^{\text{classic}} \equiv \frac{2\pi\hbar|\rho|}{N_f|eB|}, \qquad \rho = \frac{N_f(\mu^2 - \Delta^2)}{2\pi\hbar^2 v_F^2}\operatorname{sgn}\mu,$$

$\rho \propto V_g$ is the carrier imbalance; $\rho \equiv n - n_0 = n_+ - n_-$, where n_+ and n_- are the densities of electrons and holes.

This inspired experimentalists K.S. Novoselov, *et al.*, Nature **438**, 197 (2005); Y. Zhang, *et al.*, Nature **438**, 201 (2005) to look for an unconventional Quantum Hall Effect predicted in:

V.P. Gusynin, S.G.Sh., PRL **95**, 146801 (2005);

N.M.R. Peres, F. Guinea, A.H. Castro Neto, cond-mat/0506709.

Quantum Hall effect in graphene

Clean limit $\Gamma \to 0$ [compare with 2DEG, M. Jonson, S. Girvin, PRB **29**, 1939(1984)]:

$$\sigma_{xy} = -\frac{e^2}{h}\mathrm{sgn}(eB)\left[\tanh\frac{\mu+\Delta}{2T} + \tanh\frac{\mu-\Delta}{2T}\right.$$
$$\left. + 2\sum_{n=1}^{\infty}\left(\tanh\frac{\mu+\sqrt{\Delta^2+2eBn}}{2T} + \tanh\frac{\mu-\sqrt{\Delta^2+2eBn}}{2T}\right)\right].$$

When $\Delta = 0$ and $T \to 0$ we obtain

$$\sigma_{xy} = -\frac{2e^2}{h}\mathrm{sgn}(eB)\mathrm{sgn}\mu\left[1 + 2\sum_{n=1}^{\infty}\theta\left(|\mu| - \sqrt{2eBn}\right)\right]$$
$$= -\frac{2e^2}{h}\mathrm{sgn}(eB)\mathrm{sgn}\mu\left(1 + 2\left[\frac{\mu^2 c}{2\hbar|eB|v_F^2}\right]\right).$$

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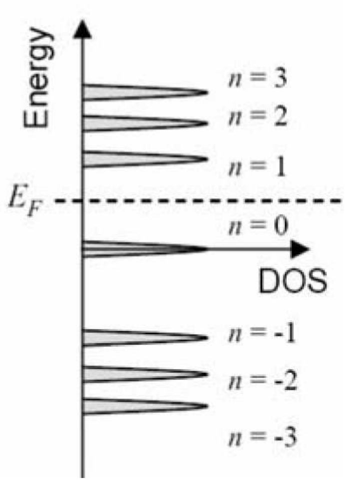
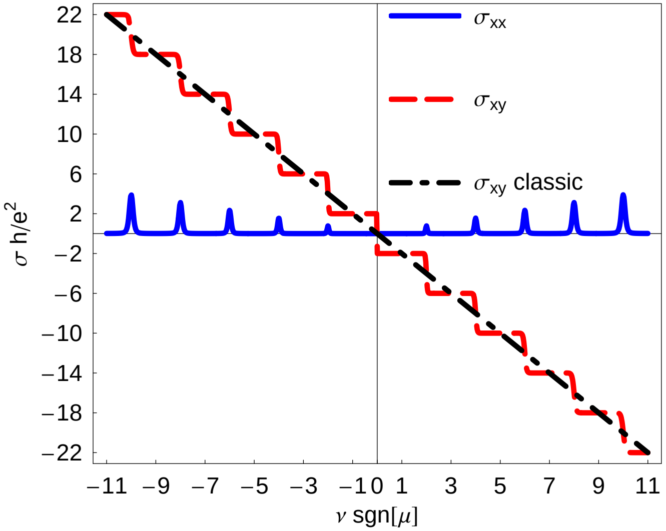




The Hall conductivity σ_{xy} and the diagonal conductivity σ_{xx} measured in e^2/h units as a function of the filling ν_B. The straight line – classical dependence $\sigma_{xy} = ec|\rho|/B$.

$$\sigma_{xy} = -\frac{2e^2}{h}(2n+1), \qquad n = \ldots, -1, 0, 1, \ldots$$

The $n = 0$ Landau level is special: $E_n = \sqrt{2nv_F^2\hbar|eB|}/c \Rightarrow$ $E_0 = 0$ and its degeneracy is half of the degeneracy of LL with $n \geq 0$.

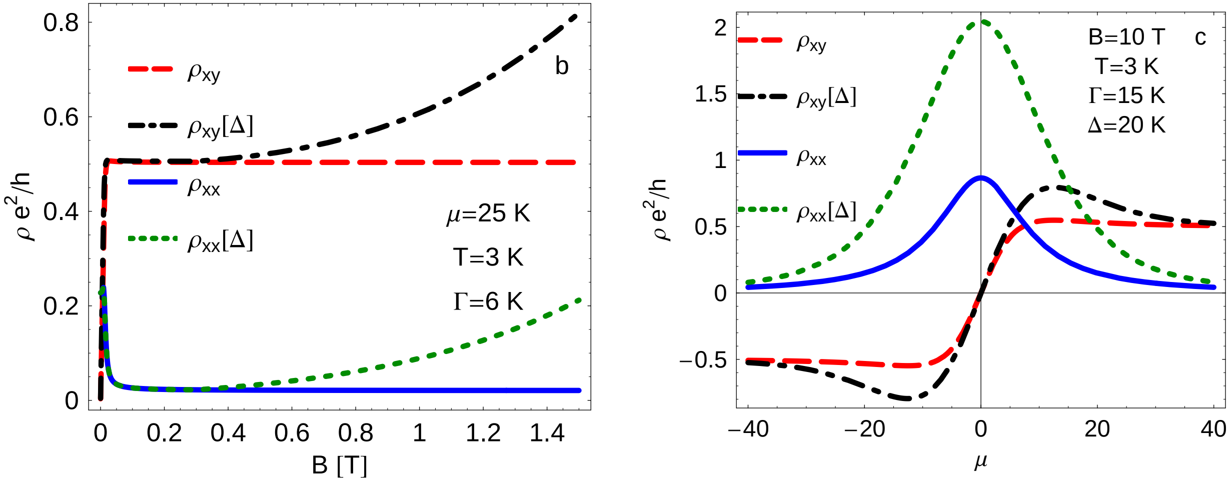
Towards new physics near $\mu = 0$

Strong field, clean limit $\Gamma \to 0$ ($\mu > 0$, $eB > 0$, $\Delta > 0$):

$$\sigma_{xy} = -\frac{e^2}{h}\left[\tanh\frac{|\mu| + \Delta(B)}{2T} + \tanh\frac{|\mu| - \Delta(B)}{2T}\right] \to \frac{2e^2}{h}\theta(|\mu| - \Delta(B)), T \to 0.$$

Assume $\Delta(B) = c\sqrt{B - B_c}\theta(B - B_c)$, B_c – critical field

$\Rightarrow$ A new Quantum phase transition instead of FQHE?!

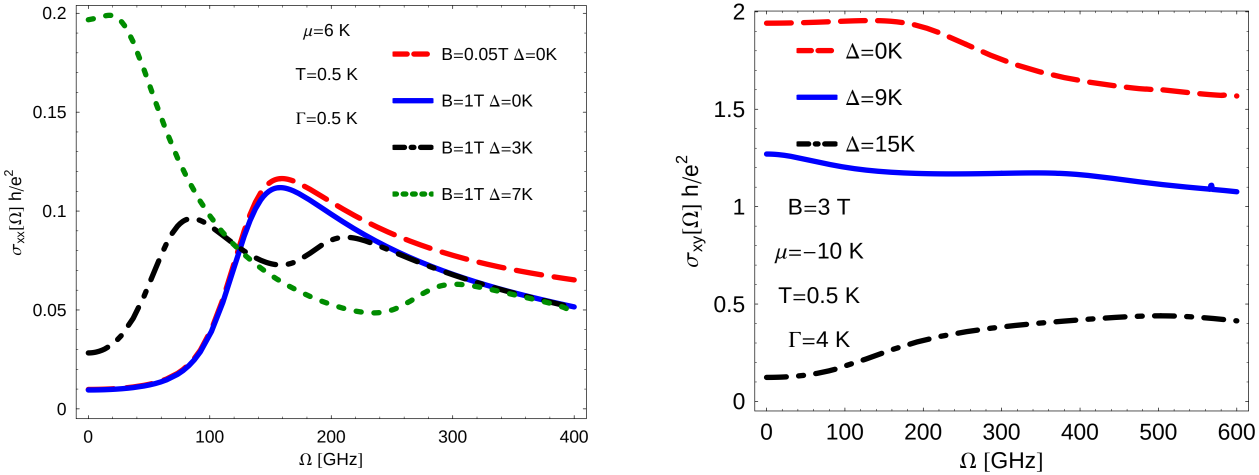


(Left) The Hall resistivity ρ_{xy} and the resistivity ρ_{xx} measured in e^2/h units as a function of field B. (Right) The same as left, but as a function of μ. The dotted (black and green) lines are calculated using $\Delta(B)$. See cond-mat/0512157.

Other possible probes

Can we use microwaves to study graphene?

The gap Δ should also be seen in microwave response in the strong field and indeed we expect some features at $\Omega \simeq |\mu| \pm \Delta$.



(Left) The microwave conductivity $\sigma_{xx}(\Omega, T)$ in units e^2/h vs frequency Ω in GHz.

(Right) The microwave Hall conductivity $\sigma_{xy}(\Omega, T)$ in units e^2/h vs frequency Ω in GHz. See cond-mat/0603267.

Summary. Three effects observed

- Unusual thermal and Dingle factors in MO:

$$R_T = \frac{2\pi^2 kTc|\mu|/(v_F^2 e\hbar B)}{\sinh \frac{2\pi^2 kTc|\mu|}{v_F^2 e\hbar B}}, \qquad R_D = \exp\left(-2\pi k \frac{c|\mu|\Gamma}{v_F^2 e\hbar B}\right).$$

  Density dependent cyclotron mass $m_c = |\mu|/v_F^2$

  Only this $\sim \sqrt{|\rho|}$ dependence proofs that $\Delta = 0$!

- Phase shift of π of oscillations of $\sigma_{xx}(1/B)$
- Uneven (or half-) Integer Quantum Hall effect:

  $\sigma_{xy} = -\frac{2e^2}{h}(2n+1), \qquad n = \ldots, -1, 0, 1, \ldots$

- Open questions: minimal value of $\sigma_{xx}(\mu = 0)$; excitonic gap and a new insulating state near $\mu = 0$ in high fields; role of spin splitting and spin-orbit coupling; bilayer graphene, other probes, . . .

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