5 Prove $x^2 - 7y^2 = 2$ has infinitely many solutions.

$x_0 = \sqrt{7}$ $\qquad a_0 = 2.$

$x_1 = \dfrac{1}{\sqrt{7}-2} = \dfrac{\sqrt{7}+2}{3}$. $\qquad a_1 = 1$

$x_2 = \dfrac{1}{\frac{\sqrt{7}+2}{3} - 1} = \dfrac{3}{\sqrt{7}+2-3} = \dfrac{3}{\sqrt{7}-1} = \dfrac{3(\sqrt{7}+1)}{6} = \dfrac{\sqrt{7}+1}{2}$; $a_2 = 1$

$x_3 = \dfrac{1}{\frac{\sqrt{7}+1}{2} - 1} = \dfrac{2}{\sqrt{7}+1-2} = \dfrac{2}{\sqrt{7}-1} = \dfrac{2(\sqrt{7}+1)}{6} = \dfrac{\sqrt{7}+1}{3}$; $a_3 = 1$

$x_4 = \dfrac{1}{\frac{\sqrt{7}+1}{3} - 1} = \dfrac{3}{\sqrt{7}+1-3} = \dfrac{3(\sqrt{7}+2)}{3} = \sqrt{7}+2$; $a_4 = 4$

$x_5 = \dfrac{1}{\sqrt{7}+2-4} = \dfrac{1}{\sqrt{7}-2} = \dfrac{\sqrt{7}+2}{3} = x_1$

so $\sqrt{7} = [2; \overline{1, 1, 1, 4}]$

0	1			
1	0			
2	1	2	0	$2^2 - 7(1^2) = -3$
3	1	1	1	$9 - 7(1^2) = 2$
5	2	1	2	$25 - 28 = -3$
8	3	1	3	$64 - 7 \cdot 9 = 1$
37	14	4	4	$1369 - 1372 = -3$
45	17	1	5	$2025 - 2023 = 2$
82	31	1	6	
127	48	1	7	
590	223	4	8	
717	271	1	9	$514089 - 514087 = 2$

By Claim 2 in Thm 4.2 ; if ~~r_k is a solution~~ $r_k = (p_k, q_k)$ is a solution to $x^2 - 7y^2 = 2$ then so is r_{k+5}. Hence there are infinitely many solutions to $x^2 - 7y^2 = 2$.