The second quantization principle

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I have formulated the second quantization principle with the help and consideration of the aspects and analysis of my recent papers, "A Theoretical Study of ψ - waves" and "A Fundamental Treatise on Continuity Equation". These papers helped me formulating the basic mathematical aspects of second quantization principle.

KEY WORDS: Quantum description of systems; poisson bracket; kinetic energy operator; continuity equation; commutator.

1. Introduction

I have considered the *quantum description of systems* and the *Heigenberg equation of motion* to replace the classical poisson bracket to quantum one.

The replacing of the classical poisson brackets by the quantum ones enables us to change a quantum mechanical description of system which has counterpart, that is, a system described in classical mechanics by a Hamiltonian expressed in terms of canonical variables p_α and q_α. Such a transition is called a *quantization rule* for the classical system.

I have formulated such a quantization rule which may be called the *second quantization principle*. It describes the relation between the quantum mechanical poisson bracket and the commutator of the canonical operators.

2. Quantum mechanical equations of motion

Let we consider the *quantum mechanical equations of motion*;

$$\frac{\partial\left\langle \hat{A} \right\rangle}{\partial t} = \left\langle \frac{\partial \hat{A}}{\partial t} \right\rangle + \frac{i}{\hbar}\left\langle [\hat{H}, \hat{A}]_- \right\rangle \tag{1}$$

$$\frac{d\left\langle \hat{A} \right\rangle}{dt} = \left\langle \frac{d\hat{A}}{dt} \right\rangle - \frac{i}{\hbar}\left\langle [\hat{L}, \hat{A}]_- \right\rangle \tag{2}$$

$$\nabla_\alpha \left\langle \hat{A} \right\rangle = \left\langle \nabla_\alpha \hat{A} \right\rangle - \frac{i}{\hbar}\left\langle [\hat{P}_\alpha, \hat{A}]_- \right\rangle \tag{3}$$

(See my first paper, "A theoretical study of ψ-waves", eq (197), eq (203) and eq (209)).
and in the consideration of eq (1) and eq (2), I get

$$\left\langle \frac{d\hat{A}}{dt} \right\rangle = \left\langle \frac{\partial \hat{A}}{\partial t} \right\rangle + \frac{i}{\hbar} \left\{ \left\langle [\hat{L}, \hat{A}]_{-} \right\rangle + \left\langle [\hat{H}, \hat{A}]_{-} \right\rangle \right\} + \frac{d\left\langle \hat{A} \right\rangle}{dt} - \frac{\partial \left\langle \hat{A} \right\rangle}{\partial t} \tag{4}$$

And considering the commutator rule;

$$[\hat{B}, \hat{A}]_{-} + [\hat{C}, \hat{A}]_{-} = [(\hat{B} + \hat{C}), \hat{A}]_{-} \tag{5}$$

and considering it for eq (4), I get

$$\left\langle \frac{d\hat{A}}{dt} \right\rangle = \left\langle \frac{\partial \hat{A}}{\partial t} \right\rangle + \frac{i}{\hbar} \left\langle [(\hat{H} + \hat{L}), \hat{A}]_{-} \right\rangle + \frac{d\left\langle \hat{A} \right\rangle}{dt} - \frac{\partial \left\langle \hat{A} \right\rangle}{\partial t} \tag{6}$$

And considering the kinetic energy operator;

$$\hat{T} = \frac{i\hbar}{2} \left\{ \frac{\partial}{\partial t} - \frac{d}{dt} \right\} \tag{7}$$

and considering,

$$2\hat{T} = \left\{ \hat{H} + \hat{L} \right\} \tag{8}$$

(See my first paper, "A theoretical study of ψ-waves", eq (59) and eq (65)).
and in the consideration of eq (6), eq (7) and eq (8), I get

$$\left\langle \frac{d\hat{A}}{dt} \right\rangle = \left\langle \frac{\partial \hat{A}}{\partial t} \right\rangle + \frac{i}{\hbar} \left\{ \left\langle [2\hat{T}, \hat{A}]_{-} \right\rangle + 2\hat{T} \left\langle \hat{A} \right\rangle \right\} \tag{9}$$

And considering the commutator rule;

$$[n\hat{A}, \hat{B}]_{-} = n[\hat{A}, \hat{B}]_{-} \tag{10}$$

for an arbitrary n.
and in the consideration of eq (9) and eq (10), I get

$$\left\langle \frac{d\hat{A}}{dt} \right\rangle = \left\langle \frac{\partial \hat{A}}{\partial t} \right\rangle + 2\frac{i}{\hbar}\left\{ \left\langle [\hat{T}, \hat{A}]_{-} \right\rangle + \hat{T}\left\langle \hat{A} \right\rangle \right\} \tag{11}$$

3. The canonical function-second quantization principle

We consider a canonical function defined as,

$$A \equiv A(q_\alpha, p_\alpha, t) \tag{12}$$

which may be an eigenvalue of a canonical operator $\hat{A}$. The canonical function A obeys its differential form;

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \{H, A\}_{p_\alpha, q_\alpha} \tag{13}$$

which is the *Heisenberg equation of motion*.
and considering the operators in eq (13);

$$\hat{f}_1 = \frac{d}{dt} \tag{14}$$

$$\hat{f}_2 = \frac{\partial}{\partial t} \tag{15}$$

and

$$\Lambda = \{H, \quad\}_{p_\alpha, q_\alpha} \tag{16}$$

which is the poisson bracket operator (classical poisson bracket operator) (See appendix-1, eq (42))

And considering the eigenvalue equation for the canonical function;

$$\hat{A}\psi(q_\alpha, t) = A(q_\alpha, p_\alpha, t)\psi(q_\alpha, t) \tag{17}$$

and considering eq (14), eq (15) and eq (16) for eq (13), I get

$$\hat{f}_1 A = \hat{f}_2 A + \Lambda A \tag{18}$$

and considering the eigenvalue equation (17) for eq (18) and the poisson bracket eigenoperator equation (See appendix-1, eq (46)) , I get

$$\hat{f}_1\hat{A} = \hat{f}_2\hat{A} + \hat{\Lambda}\hat{A} \tag{19}$$

and considering eq (14), eq (15) and eq (16) for eq (19), I get

$$\frac{d\hat{A}}{dt} = \frac{\partial\hat{A}}{\partial t} + \left\{\hat{H}, \hat{A}\right\}_{\hat{p}_\alpha, \hat{q}_\alpha} \tag{20}$$

as the Hamiltonian is a canonical function where $\left\{\hat{H}, \hat{A}\right\}_{\hat{p}_\alpha, \hat{q}_\alpha}$ is the quantum mechanical poisson bracket. And getting its expectation value, I get

$$\left\langle\frac{d\hat{A}}{dt}\right\rangle = \left\langle\frac{\partial\hat{A}}{\partial t}\right\rangle + \left\langle\left\{\hat{H}, \hat{A}\right\}_{\hat{p}_\alpha, \hat{q}_\alpha}\right\rangle \tag{21}$$

and in the consideration of eq (11) and eq (21), I get

$$\left\langle\left\{\hat{H}, \hat{A}\right\}_{\hat{p}_\alpha, \hat{q}_\alpha}\right\rangle = 2\frac{i}{\hbar}\left\{\left\langle[\hat{T}, \hat{A}]_-\right\rangle + \hat{T}\left\langle\hat{A}\right\rangle\right\} \tag{22}$$

which is the *second quantization principle*.
and considering eigenvalue equation (17) for H and A or considering eq (47) for eq (22) (See appendix-1, eq (47)), I get

$$\left\langle\left\{H, A\right\}_{p_\alpha, q_\alpha}\right\rangle = 2\frac{i}{\hbar}\left\{\left\langle[\hat{T}, \hat{A}]_-\right\rangle + \hat{T}\left\langle\hat{A}\right\rangle\right\} \tag{23}$$

which is the classical form of second quantization principle.
and assuming the eigenvalue as the expectation value of its operator in eq (23), I get

$$\left\{H, A\right\}_{p_\alpha, q_\alpha} = 2\frac{i}{\hbar}\left\{[\hat{T}, \hat{A}]_- + A\hat{T}\right\} \tag{24}$$

which is another classical form of the second quantization principle.
and considering eq (47) for eq (24), I get

$$\left\{\hat{H}, \hat{A}\right\}_{\hat{p}_\alpha, \hat{q}_\alpha} = 2\frac{i}{\hbar}\left\{[\hat{T}, \hat{A}]_- + A\hat{T}\right\} \tag{25}$$

which is the standard form of the second quantization principle.
and considering eq (17) for A in eq (24) in the consideration of eq (23), I get

$$\{H, \mathrm{A}\}_{p_\alpha, q_\alpha} = 2\frac{i}{\hbar}\left\{[\hat{T}, \hat{\mathrm{A}}]_- + \hat{T}\hat{\mathrm{A}}\right\} \tag{26}$$

which is another form of the second quantization principle.

4. Second quantization principle-equation of continuity

We here consider the continuity equation for canonical probability density $\rho(q_\alpha, p_\alpha, t)$ (See my second paper, "A fundamental treatise on continuity equation".) to formulate the second quantization principle for canonical probability density.

Let we consider the *Heisenberg equation of motion* (13) for the canonical probability density, I get

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \{H, \rho\}_{p_\alpha, q_\alpha} \tag{27}$$

and considering eq (21) for canonical probability density, I get

$$\left\langle \frac{d\hat{\rho}}{dt} \right\rangle = \left\langle \frac{\partial\hat{\rho}}{\partial t} \right\rangle + \left\langle \{\hat{H}, \hat{\rho}\}_{\hat{p}_\alpha, \hat{q}_\alpha} \right\rangle \tag{28}$$

where $\hat{\rho}$ is the quantum mechanical canonical probability density (canonical probability density operator) defined as,

$$\hat{\rho} = \langle \psi | \psi \rangle \tag{29}$$

and considering the second quantization principle (22) for eq (28), I get

$$\left\langle \{\hat{H}, \hat{\rho}\}_{\hat{p}_\alpha, \hat{q}_\alpha} \right\rangle = 2\frac{i}{\hbar}\left\{ \left\langle [\hat{T}, \hat{\rho}]_- \right\rangle + \hat{T}\langle \hat{\rho} \rangle \right\} \tag{30}$$

which is the second quantization principle for canonical probability density.
and in the consideration of eq (28) and eq (30), I get

$$\left\langle \frac{d\hat{\rho}}{dt} \right\rangle = \left\langle \frac{\partial\hat{\rho}}{\partial t} \right\rangle + 2\frac{i}{\hbar}\left\{ \left\langle [\hat{T}, \hat{\rho}]_- \right\rangle + \hat{T}\langle \hat{\rho} \rangle \right\} \tag{31}$$

Now considering eq (20) for canonical probability density, I get

$$\frac{d\hat{\rho}}{dt}=\frac{\partial\hat{\rho}}{\partial t}+\left\{\hat{H},\hat{\rho}\right\}_{\hat{p}_\alpha,\hat{q}_\alpha} \qquad (32)$$

and considering the standard form of the second quantization principle (25) for eq (32), I get

$$\left\{\hat{H},\hat{\rho}\right\}_{\hat{p}_\alpha,\hat{q}_\alpha}=2\frac{i}{\hbar}\left\{[\hat{T},\hat{\rho}]_- + \rho\hat{T}\right\} \qquad (33)$$

which is the standard form of the second quantization principle for canonical probability density.
and in the consideration of eq (32) and eq (33), I get

$$\frac{d\hat{\rho}}{dt}=\frac{\partial\hat{\rho}}{\partial t}+2\frac{i}{\hbar}\left\{[\hat{T},\hat{\rho}]_- + \rho\hat{T}\right\} \qquad (34)$$

Now considering the second quantization principle (24) for the canonical probability density, I get

$$\left\{H,\rho\right\}_{p_\alpha,q_\alpha}=2\frac{i}{\hbar}\left\{[\hat{T},\hat{\rho}]_- + \rho\hat{T}\right\} \qquad (35)$$

which is the classical form of the second quantization principle for canonical probability density.
and in the consideration of eq (35) and eq (27), I get

$$\frac{d\rho}{dt}=\frac{\partial\rho}{\partial t}+2\frac{i}{\hbar}\left\{[\hat{T},\hat{\rho}]_- + \rho\hat{T}\right\} \qquad (36)$$

which is the second quantization principle for the canonical probability density.

Let we consider the continuity equation for canonical probability density;

$$2\frac{\partial\rho}{\partial t}+\left\{H,\rho\right\}_{p_\alpha,q_\alpha}=0 \qquad (37)$$

and

$$2\frac{d\rho}{dt}-\left\{H,\rho\right\}_{p_\alpha,q_\alpha}=0 \qquad (38)$$

and considering the classical form of the second quantization principle (35) for eq (37) and eq (38), I get

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}\left\{[\hat{T}, \hat{\rho}]_{-} + \rho \hat{T}\right\} \tag{39}$$

and

$$\frac{d\rho}{dt} = \frac{i}{\hbar}\left\{[\hat{T}, \hat{\rho}]_{-} + \rho \hat{T}\right\} \tag{40}$$

which are the quantum mechanical equations of motion.

The second quantization principle can be used for canonical probability density as well as to describe any canonical physical quantity.

Appendix

1. The hypothetical operators

We consider the poisson bracket as a classical operator which operates a canonical function in its defined form according to the operator analysis;

$$\Lambda \mathrm{A} = \left\{H, \mathrm{A}\right\}_{p_\alpha, q_\alpha} = \left\{\frac{\partial H}{\partial p_\alpha}\frac{\partial \mathrm{A}}{\partial q_\alpha} - \frac{\partial H}{\partial q_\alpha}\frac{\partial \mathrm{A}}{\partial p_\alpha}\right\} \tag{41}$$

where Λ is the classical poisson bracket operator defined as,

$$\Lambda = \left\{H, \quad \right\}_{p_\alpha, q_\alpha} \tag{42}$$

And considering the quantum mechanical poisson bracket as a quantum mechanical operator which operates a canonical operator in its defined form according to the operator analysis;

$$\hat{\Lambda} \hat{\mathrm{A}} = \left\{\hat{H}, \hat{\mathrm{A}}\right\}_{\hat{p}_\alpha, \hat{q}_\alpha} = \left\{\frac{\partial \hat{H}}{\partial \hat{p}_\alpha}\frac{\partial \hat{\mathrm{A}}}{\partial \hat{q}_\alpha} - \frac{\partial \hat{H}}{\partial \hat{q}_\alpha}\frac{\partial \hat{\mathrm{A}}}{\partial \hat{p}_\alpha}\right\} \tag{43}$$

where $\hat{\Lambda}$ is the quantum mechanical poisson bracket defined as,

$$\hat{\Lambda} = \left\{\hat{H}, \quad \right\}_{\hat{p}_\alpha, \hat{q}_\alpha} \tag{44}$$

Now we describe the *poisson bracket eigenoperator equation* to specify a relation between quantum mechanical and classical poisson brackets;

$$\hat{\Lambda}\hat{\Phi} = \Lambda\Phi \tag{45}$$

where Φ is a classical field.
and considering *poisson bracket eigenoperator equation* (45) for eq (17), I get

$$\hat{\Lambda}\hat{\mathrm{A}} = \Lambda\mathrm{A} \tag{46}$$

and in the consideration of eq (42), eq (44) and eq (46), I get

$$\left\{\hat{H}, \hat{\mathrm{A}}\right\}_{\hat{p}_\alpha, \hat{q}_\alpha} = \left\{H, \mathrm{A}\right\}_{p_\alpha, q_\alpha} \tag{47}$$

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