A Fundamental Treatise on Continuity Equation

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The work is based on the analysis and results of my previous paper "A Theoretical Study of ψ-waves".

KEY WORDS: Probability density; Saurav equation; Modern notations (Dirac notations); Complex conjugate; Hermitian operators; Current density; Continuity equation; Canonical transformation; Poisson bracket; Quantum mechanical Laplacian operator; Configuration space.

Let assuming

$$\rho = \psi^* \psi := \langle \psi | \psi \rangle \tag{1}$$

and

$$w = \int \psi^* \psi \, dq := \langle \psi | \hat{I} | \psi \rangle \tag{2}$$

where ρ is the probability density and w is the probability of getting a particle in q-configuration, having relation

$$w := \int \rho \, dq \tag{3}$$

And treating my equations (Saurav equations) (See my first paper, "A Theoretical Study of ψ-waves", eq (176) and eq (177)), I get

$$\psi^* \frac{d\psi}{dt} + \frac{i\hbar}{m} \psi^* \nabla^2 \psi - \psi^* \frac{\partial \psi}{\partial t} = 0 \tag{4}$$

and

$$\psi \frac{d\psi^*}{dt} - \frac{i\hbar}{m} \psi \nabla^2 \psi^* - \psi \frac{\partial \psi^*}{\partial t} = 0 \tag{5}$$

and in the consideration of eq (4) and eq (5), I get

$$\frac{d(\psi^*\psi)}{dt} = \frac{\partial(\psi^*\psi)}{\partial t} + \frac{i\hbar}{m}\left\{\psi\nabla^2\psi^* - \psi^*\nabla^2\psi\right\} \tag{6}$$

and integrating it w.r.t. q (configuration space), I get

$$\frac{d}{dt}\int \psi^*\psi\, dq = \frac{\partial}{\partial t}\int \psi^*\psi\, dq + \frac{i\hbar}{m}\int \left\{\psi\nabla^2\psi^* - \psi^*\nabla^2\psi\right\} dq \tag{7}$$

and considering eq (1) for eq (6) and eq (2) for eq (7), I get

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \frac{i\hbar}{m}\left\{\psi\nabla^2\psi^* - \psi^*\nabla^2\psi\right\} \tag{8}$$

and

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{i\hbar}{m}\int \left\{\psi\nabla^2\psi^* - \psi^*\nabla^2\psi\right\} dq \tag{9}$$

and in the consideration of eq (1) and eq (2), eq (8) and eq (9) may be written in their modern notations,

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \frac{i\hbar}{m}\left\{\nabla^2\langle\psi|\psi\rangle - \langle\psi|\nabla^2\psi\rangle\right\} \tag{10}$$

and

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{i\hbar}{m}\left\{\nabla^2\langle\psi|\hat{I}|\psi\rangle - \langle\psi|\nabla^2|\psi\rangle\right\} \tag{11}$$

(See appendix-2, eq (89) and eq (90)).
Now treating Schrödinger equation, I get

$$i\hbar\psi^*\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\psi^*\nabla^2\psi - \psi^*\hat{V}\psi = 0 \tag{12}$$

and getting its complex conjugate, I get

$$-i\hbar\psi\frac{\partial\psi^*}{\partial t} + \frac{\hbar^2}{2m}\psi\nabla^2\psi^* - \psi\hat{V}^*\psi^* = 0 \tag{13}$$

and assuming hermitian properties of $\hat{V}$, i.e.,

$$\psi^* \hat{V}\psi - \psi \hat{V}^* \psi^* = Б(\psi^* \psi - \psi\psi^*) = 0 \tag{14}$$

(See my first paper, “A Theoretical Study of ψ-waves”, eq (229) and eq (230) treating in the view to eq (12) and eq (13) assuming $\psi = \varphi$)
and in the consideration of eq (12), eq (13) and eq (14), I get

$$\frac{\partial(\psi^* \psi)}{\partial t} + \frac{i\hbar}{2m}\left\{\psi \nabla^2 \psi^* - \psi^* \nabla^2 \psi\right\} = 0 \tag{15}$$

and considering eq (1) for eq (15), I get

$$\frac{\partial \rho}{\partial t} + \frac{i\hbar}{2m}\left\{\psi \nabla^2 \psi^* - \psi^* \nabla^2 \psi\right\} = 0 \tag{16}$$

And now considering,

$$\psi \nabla^2 \psi^* - \psi^* \nabla^2 \psi = \mathrm{div}\left(\psi\, \mathrm{grad}\, \psi^* - \psi^* \mathrm{grad}\, \psi\right) \tag{17}$$

and assuming current density $\mathbf{j}$ define as,

$$\mathbf{j} := \frac{i\hbar}{2m}\left(\psi\, \mathrm{grad}\, \psi^* - \psi^*\, \mathrm{grad}\, \psi\right) \tag{18}$$

and in the consideration of eq (16), eq (17) and eq (18), I get

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\, \mathbf{j} = 0 \tag{19}$$

which is called the equation of continuity.
and considering eq (17), eq (18) and eq (8), I get

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + 2\, \mathrm{div}\, \mathbf{j} \tag{20}$$

and in the consideration of eq (19) and eq (20), I get

$$\frac{d\rho}{dt} - \mathrm{div}\, \mathbf{j} = 0 \tag{21}$$

and

$$\frac{d\rho}{dt} + \frac{\partial \rho}{\partial t} = 0 \tag{22}$$

Now assuming the canonical probability density,

$$\rho := \rho(q_{\alpha}, p_{\alpha}, t) \tag{23}$$

and getting the exact differential of ρ w.r.t. time, I get

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \{H, \rho\}_{p_{\alpha}, q_{\alpha}} \tag{24}$$

(See, appendix-1, eq (76) assuming $f = \rho$) where $\{H, \rho\}_{p_{\alpha}, q_{\alpha}}$ is poisson bracket for probability density defined as,

$$\{H, \rho\}_{p_{\alpha}, q_{\alpha}} = \left\{ \frac{\partial H}{\partial p_{\alpha}} \frac{\partial \rho}{\partial q_{\alpha}} - \frac{\partial H}{\partial q_{\alpha}} \frac{\partial \rho}{\partial p_{\alpha}} \right\} \tag{25}$$

and in the consideration of eq (24) and eq (20), I get

$$\text{div}\, \mathbf{j} = \frac{1}{2} \{H, \rho\}_{p_{\alpha}, q_{\alpha}} \tag{26}$$

and in the consideration of eq (26) and eq (19), I get

$$2 \frac{\partial \rho}{\partial t} + \{H, \rho\}_{p_{\alpha}, q_{\alpha}} = 0 \tag{27}$$

which is the continuity equation for canonical probability density.
and in the consideration of eq (26) and eq (21), I get

$$2 \frac{d\rho}{dt} - \{H, \rho\}_{p_{\alpha}, q_{\alpha}} = 0 \tag{28}$$

and considering eq (24) and eq (8), I get

$$\{H, \rho\}_{p_{\alpha}, q_{\alpha}} = \frac{i\hbar}{m} \{\psi \nabla^2 \psi^* - \psi^* \nabla^2 \psi\} \tag{29}$$

Now considering the canonical probability

$$w := w(q_\alpha, p_\alpha, t) \tag{30}$$

and getting its exact time derivative, I get

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \{H, w\}_{p_\alpha, q_\alpha} \tag{31}$$

(See, appendix-1, eq (76) assuming $f = w$) and in the consideration of eq (31) and eq (9), I get

$$\{H, w\}_{p_\alpha, q_\alpha} = \frac{i\hbar}{m} \int \{\psi \nabla^2 \psi^* - \psi^* \nabla^2 \psi\} dq \tag{32}$$

and eq (29) and eq (32) may be written in their modern notations, as

$$\{H, \rho\}_{p_\alpha, q_\alpha} = \frac{i\hbar}{m} \{\nabla^2 \langle \psi | \psi \rangle - \langle \psi | \nabla^2 \psi \rangle\} \tag{33}$$

and

$$\{H, w\}_{p_\alpha, q_\alpha} = \frac{i\hbar}{m} \{\nabla^2 \langle \psi | \hat{I} | \psi \rangle - \langle \psi | \nabla^2 | \psi \rangle\} \tag{34}$$

and putting eq (18) in eq (9) in the consideration of eq (17), I get

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + 2 \int (\operatorname{div} \mathbf{j})\, dq \tag{35}$$

and in the consideration of eq (35) and eq (31), I get

$$\{H, w\}_{p_\alpha, q_\alpha} = 2 \int (\operatorname{div} \mathbf{j})\, dq \tag{36}$$

Now integrating the continuity equation (16) with respect to q (configuration space) in the consideration of eq (1), eq (2) and eq (3), I get

$$\frac{\partial w}{\partial t} + \frac{i\hbar}{2m} \int \{\psi \nabla^2 \psi^* - \psi^* \nabla^2 \psi\} dq = 0 \tag{37}$$

and considering eq (18) in eq (37) in the consideration of eq (17), I get

$$\frac{\partial w}{\partial t} + \int (\text{div } \mathbf{j})\, dq = 0 \tag{38}$$

and considering eq (36) for eq (38), I get

$$2\frac{\partial w}{\partial t} + \{H, w\}_{p_\alpha, q_\alpha} = 0 \tag{39}$$

Let now considering the quantum mechanical Laplacian operator,

$$\hat{\nabla}_\alpha^2 = \frac{i}{\hbar}\left\{[\nabla_\alpha^2 s] + \frac{i}{\hbar}[\nabla_\alpha s]^2\right\} \tag{40}$$

(See my first paper, "A Theoretical Study of ψ-waves", eq (20))
and applying it to eq (17), I get

$$\begin{aligned}&\{\psi\nabla^2\psi^* - \psi^*\nabla^2\psi\} \\ &= \frac{i}{\hbar}\left[\{\psi[\nabla_\alpha^2 s]\psi^* - \psi^*[\nabla_\alpha^2 s]\psi\} + \frac{i}{\hbar}\{\psi[\nabla_\alpha s]^2\psi^* - \psi^*[\nabla_\alpha s]^2\psi\}\right]\end{aligned} \tag{41}$$

and considering eq (41) in eq (18) in the consideration of eq (17), I get

$$\text{div } \mathbf{j} = -\frac{1}{2m}\left[\{\psi[\nabla_\alpha^2 s]\psi^* - \psi^*[\nabla_\alpha^2 s]\psi\} + \frac{i}{\hbar}\{\psi[\nabla_\alpha s]^2\psi^* - \psi^*[\nabla_\alpha s]^2\psi\}\right] \tag{42}$$

where $\mathbf{j}$ is the current density and s is action (physical action).
and considering eq (41) for eq (8) or considering eq (42) for eq (20), I get

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} - \frac{1}{m}\left[\{\psi[\nabla_\alpha^2 s]\psi^* - \psi^*[\nabla_\alpha^2 s]\psi\} + \frac{i}{\hbar}\{\psi[\nabla_\alpha s]^2\psi^* - \psi^*[\nabla_\alpha s]^2\psi\}\right] \tag{43}$$

and considering eq (41) for eq (9) or considering eq (42) for eq (35), I get

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} - \frac{1}{m}\int\left[\{\psi[\nabla_\alpha^2 s]\psi^* - \psi^*[\nabla_\alpha^2 s]\psi\} + \frac{i}{\hbar}\{\psi[\nabla_\alpha s]^2\psi^* - \psi^*[\nabla_\alpha s]^2\psi\}\right]dq \tag{44}$$

and considering eq (41) for eq (16) or considering eq (42) for eq (19), I get

$$\frac{\partial \rho}{\partial t} - \frac{1}{2m}\left[\left\{\psi[\nabla_{\alpha}^{2} s]\psi^{*} - \psi^{*}[\nabla_{\alpha}^{2} s]\psi\right\} + \frac{i}{\hbar}\left\{\psi[\nabla_{\alpha} s]^{2}\psi^{*} - \psi^{*}[\nabla_{\alpha} s]^{2}\psi\right\}\right] = 0 \tag{45}$$

which is the continuity equation for quantum mechanical Laplacian operator (quantum mechanical continuity equation).
and considering eq (41) for eq (37) or considering eq (42) for eq (38), I get

$$\frac{\partial w}{\partial t} - \frac{1}{2m}\int\left[\left\{\psi[\nabla_{\alpha}^{2} s]\psi^{*} - \psi^{*}[\nabla_{\alpha}^{2} s]\psi\right\} + \frac{i}{\hbar}\left\{\psi[\nabla_{\alpha} s]^{2}\psi^{*} - \psi^{*}[\nabla_{\alpha} s]^{2}\psi\right\}\right] dq = 0 \tag{46}$$

Now considering eq (21) in the consideration of eq (17) and eq (18), I get

$$\frac{d\rho}{dt} - \frac{i\hbar}{2m}\left\{\psi\nabla^{2}\psi^{*} - \psi^{*}\nabla^{2}\psi\right\} = 0 \tag{47}$$

and integrating it w.r.t. q (configuration space) in the consideration of eq (1), eq (2) and eq (3), I get

$$\frac{dw}{dt} - \frac{i\hbar}{2m}\int\left\{\psi\nabla^{2}\psi^{*} - \psi^{*}\nabla^{2}\psi\right\} dq = 0 \tag{48}$$

and considering eq (48) in the consideration of eq (17) and eq (18), I get

$$\frac{dw}{dt} - \int (\operatorname{div} \mathbf{j})\, dq = 0 \tag{49}$$

and considering eq (36) for eq (49), I get

$$2\frac{dw}{dt} - \{H, w\}_{p_{\alpha}, q_{\alpha}} = 0 \tag{50}$$

and considering eq (41) for eq (47) or considering eq (42) for eq (21), I get

$$\frac{d\rho}{dt} + \frac{1}{2m}\left[\left\{\psi[\nabla_{\alpha}^{2} s]\psi^{*} - \psi^{*}[\nabla_{\alpha}^{2} s]\psi\right\} + \frac{i}{\hbar}\left\{\psi[\nabla_{\alpha} s]^{2}\psi^{*} - \psi^{*}[\nabla_{\alpha} s]^{2}\psi\right\}\right] = 0 \tag{51}$$

and considering eq (41) for eq (48) or considering eq (42) for eq (49), I get

$$\frac{dw}{dt}+\frac{1}{2m}\int\left[\left\{\psi[\nabla_{\alpha}^{2}s]\psi^{*}-\psi^{*}[\nabla_{\alpha}^{2}s]\psi\right\}+\frac{i}{\hbar}\left\{\psi[\nabla_{\alpha}s]^{2}\psi^{*}-\psi^{*}[\nabla_{\alpha}s]^{2}\psi\right\}\right]dq=0$$

(52)

Now integrating eq (22) with respect to q (configuration space) in the consideration of eq (1), eq (2) and eq (3) or considering eq (35) and eq (38) or considering eq (39) and eq (50), I get

$$\frac{dw}{dt}+\frac{\partial w}{\partial t}=0 \qquad (53)$$

and considering eq (41) for eq (29) or considering eq (42) for eq (26), I get

$$\{H,\rho\}_{p_{\alpha},q_{\alpha}}=-\frac{1}{m}\left[\left\{\psi[\nabla_{\alpha}^{2}s]\psi^{*}-\psi^{*}[\nabla_{\alpha}^{2}s]\psi\right\}+\frac{i}{\hbar}\left\{\psi[\nabla_{\alpha}s]^{2}\psi^{*}-\psi^{*}[\nabla_{\alpha}s]^{2}\psi\right\}\right]$$

(54)

and considering eq (41) for eq (32) or considering eq (42) for eq (36), I get

$$\{H,w\}_{p_{\alpha},q_{\alpha}}=-\frac{1}{m}\int\left[\left\{\psi[\nabla_{\alpha}^{2}s]\psi^{*}-\psi^{*}[\nabla_{\alpha}^{2}s]\psi\right\}+\frac{i}{\hbar}\left\{\psi[\nabla_{\alpha}s]^{2}\psi^{*}-\psi^{*}[\nabla_{\alpha}s]^{2}\psi\right\}\right]dq$$

(55)

and eq (54) and eq (55) may be written in their modern notations in the consideration of eq (33) and eq (34) considering eq (40) and eq (41), as

$$\{H,\rho\}_{p_{\alpha},q_{\alpha}}=-\frac{1}{m}\left[\left\{[\nabla_{\alpha}^{2}s]\langle\psi|\psi\rangle-\langle\psi|[\nabla_{\alpha}^{2}s]\psi\rangle\right\}+\frac{i}{\hbar}\left\{[\nabla_{\alpha}s]^{2}\langle\psi|\psi\rangle-\langle\psi|[\nabla_{\alpha}s]^{2}\psi\rangle\right\}\right]$$

(56)

and

$$\{H,w\}_{p_{\alpha},q_{\alpha}}=-\frac{1}{m}\left[\left\{[\nabla_{\alpha}^{2}s]\langle\psi|\hat{I}|\psi\rangle-\langle\psi|[\nabla_{\alpha}^{2}s]|\psi\rangle\right\}+\frac{i}{\hbar}\left\{[\nabla_{\alpha}s]^{2}\langle\psi|\hat{I}|\psi\rangle-\langle\psi|[\nabla_{\alpha}s]^{2}|\psi\rangle\right\}\right]$$

(57)

Now writing the continuity equation (16) in its modern notations, I get

$$\frac{\partial\rho}{\partial t}+\frac{i\hbar}{2m}\left\{\nabla^{2}\langle\psi|\psi\rangle-\langle\psi|\nabla^{2}\psi\rangle\right\}=0 \qquad (58)$$

and integrating it w.r.t. q (configuration space) in the consideration of eq (1), eq (2) and eq (3) or writing eq (37) in its modern notations, I get

$$\frac{\partial w}{\partial t}+\frac{i\hbar}{2m}\left\{\nabla^2\langle\psi|\hat{I}|\psi\rangle-\langle\psi|\nabla^2|\psi\rangle\right\}=0 \tag{59}$$

and considering eq (40) and eq (41) for eq (10) and eq (11) or writing eq (43) and eq (44) in their modern notations, I get

$$\frac{d\rho}{dt}=\frac{\partial\rho}{\partial t}-\frac{1}{m}\left[\left\{[\nabla_\alpha^2 s]\langle\psi|\psi\rangle-\left\langle\psi\middle|[\nabla_\alpha^2 s]\psi\right\rangle\right\}+\frac{i}{\hbar}\left\{[\nabla_\alpha s]^2\langle\psi|\psi\rangle-\left\langle\psi\middle|[\nabla_\alpha s]^2\psi\right\rangle\right\}\right] \tag{60}$$

and

$$\frac{dw}{dt}=\frac{\partial w}{\partial t}-\frac{1}{m}\left[\left\{[\nabla_\alpha^2 s]\langle\psi|\hat{I}|\psi\rangle-\langle\psi|[\nabla_\alpha^2 s]|\psi\rangle\right\}+\frac{i}{\hbar}\left\{[\nabla_\alpha s]^2\langle\psi|\hat{I}|\psi\rangle-\langle\psi|[\nabla_\alpha s]^2|\psi\rangle\right\}\right] \tag{61}$$

and considering eq (40) and eq (41) for eq (58) and eq (59) or writing eq (45) and eq (46) in their modern notations, I get

$$\frac{\partial\rho}{\partial t}-\frac{1}{2m}\left[\left\{[\nabla_\alpha^2 s]\langle\psi|\psi\rangle-\left\langle\psi\middle|[\nabla_\alpha^2 s]\psi\right\rangle\right\}+\frac{i}{\hbar}\left\{[\nabla_\alpha s]^2\langle\psi|\psi\rangle-\left\langle\psi\middle|[\nabla_\alpha s]^2\psi\right\rangle\right\}\right]=0 \tag{62}$$

which is the quantum mechanical continuity equation in modern notations.
and

$$\frac{\partial w}{\partial t}-\frac{1}{2m}\left[\left\{[\nabla_\alpha^2 s]\langle\psi|\hat{I}|\psi\rangle-\langle\psi|[\nabla_\alpha^2 s]|\psi\rangle\right\}+\frac{i}{\hbar}\left\{[\nabla_\alpha s]^2\langle\psi|\hat{I}|\psi\rangle-\langle\psi|[\nabla_\alpha s]^2|\psi\rangle\right\}\right]=0 \tag{63}$$

and writing eq (47) and eq (48) in their modern notations, I get

$$\frac{d\rho}{dt}-\frac{i\hbar}{2m}\left\{\nabla^2\langle\psi|\psi\rangle-\left\langle\psi\middle|\nabla^2\psi\right\rangle\right\}=0 \tag{64}$$

and

$$\frac{dw}{dt}-\frac{i\hbar}{2m}\left\{\nabla^2\langle\psi|\hat{I}|\psi\rangle-\langle\psi|\nabla^2|\psi\rangle\right\}=0 \tag{65}$$

and considering eq (40) and eq (41) for eq (64) and eq (65) or writing eq (51) and eq (52) in their modern notations, I get

$$\frac{d\rho}{dt}+\frac{1}{2m}\left[\left\{[\nabla_{\alpha}^{2}s]\langle\psi|\psi\rangle-\langle\psi|[\nabla_{\alpha}^{2}s]\psi\rangle\right\}+\frac{i}{\hbar}\left\{[\nabla_{\alpha}s]^{2}\langle\psi|\psi\rangle-\langle\psi|[\nabla_{\alpha}s]^{2}\psi\rangle\right\}\right]=0 \tag{66}$$

and

$$\frac{dw}{dt}+\frac{1}{2m}\left[\left\{[\nabla_{\alpha}^{2}s]\langle\psi|\hat{I}|\psi\rangle-\langle\psi|[\nabla_{\alpha}^{2}s]|\psi\rangle\right\}+\frac{i}{\hbar}\left\{[\nabla_{\alpha}s]^{2}\langle\psi|\hat{I}|\psi\rangle-\langle\psi|[\nabla_{\alpha}s]^{2}|\psi\rangle\right\}\right]=0$$

67)

and considering eq (29) and eq (32), I get

$$\{H,w\}_{p_\alpha,q_\alpha}=\int\{H,\rho\}_{p_\alpha,q_\alpha}\,dq \tag{68}$$

Appendix

1. The canonical function-poisson bracket

A canonical function is defined as,

$$f:=f(q_\alpha,p_\alpha,t) \tag{69}$$

and getting its exact differential w.r.t time, I get

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+\left\{\frac{\partial f}{\partial p_\alpha}\dot{p}_\alpha+\frac{\partial f}{\partial q_\alpha}\dot{q}_\alpha\right\} \tag{70}$$

and considering the canonical transformation (classical theoretical physics),

$$\dot{p}_\alpha=-\frac{\partial H}{\partial q_\alpha} \tag{71}$$

and

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \tag{72}$$

and in the consideration of eq (70), eq (71) and eq (72), I get

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \left\{ \frac{\partial H}{\partial p_\alpha} \frac{\partial f}{\partial q_\alpha} - \frac{\partial H}{\partial q_\alpha} \frac{\partial f}{\partial p_\alpha} \right\} \tag{73}$$

where the second term of eq (73) is the poisson bracket defined as,

$$\{H, f\}_{p_\alpha, q_\alpha} := \left\{ \frac{\partial H}{\partial p_\alpha} \frac{\partial f}{\partial q_\alpha} - \frac{\partial H}{\partial q_\alpha} \frac{\partial f}{\partial p_\alpha} \right\} \tag{74}$$

and

$$\{H, f\}_{p_\alpha, q_\alpha} = -\{H, f\}_{q_\alpha, p_\alpha} = \{f, H\}_{q_\alpha, p_\alpha} = -\{f, H\}_{p_\alpha, q_\alpha} \tag{75}$$

and in the consideration of eq (73) and eq (74), I get

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{H, f\}_{p_\alpha, q_\alpha} \tag{76}$$

Let assuming,

$$\{H, f\}_-^{PB} := \left\{ \frac{\partial H}{\partial p_\alpha} \frac{\partial f}{\partial q_\alpha} - \frac{\partial H}{\partial q_\alpha} \frac{\partial f}{\partial p_\alpha} \right\} \tag{77}$$

and

$$\{H, f\}_+^{PB} := \left\{ \frac{\partial H}{\partial p_\alpha} \frac{\partial f}{\partial q_\alpha} + \frac{\partial H}{\partial q_\alpha} \frac{\partial f}{\partial p_\alpha} \right\} \tag{78}$$

where $\{H, f\}_+^{PB}$ is the anti-poisson bracket.
and in the consideration of eq (77) and eq (78), I get

$$\frac{\partial H}{\partial p_\alpha} \frac{\partial f}{\partial q_\alpha} = \frac{1}{2}\{H, f\}_+ + \frac{1}{2}\{H, f\}_- \tag{79}$$

and

$$\frac{\partial H}{\partial q_\alpha}\frac{\partial f}{\partial p_\alpha} = \frac{1}{2}\{H, f\}_+ - \frac{1}{2}\{H, f\}_- \tag{80}$$

Now considering eq (77) and eq (78) for two corresponding canonical functions g and f, I get

$$\{g, f\}_-^{PB} := \left\{\frac{\partial g}{\partial p_\alpha}\frac{\partial f}{\partial q_\alpha} - \frac{\partial g}{\partial q_\alpha}\frac{\partial f}{\partial p_\alpha}\right\} \tag{81}$$

and

$$\{g, f\}_+^{PB} := \left\{\frac{\partial g}{\partial p_\alpha}\frac{\partial f}{\partial q_\alpha} + \frac{\partial g}{\partial q_\alpha}\frac{\partial f}{\partial p_\alpha}\right\} \tag{82}$$

(Remember that the Hamiltonian $H = H(q_\alpha, p_\alpha, t)$ in recent equations is a canonical function in the consideration of eq (69)).
and in the consideration of eq (81) and eq (82), I get

$$\frac{\partial g}{\partial p_\alpha}\frac{\partial f}{\partial q_\alpha} = \frac{1}{2}\{g, f\}_+ + \frac{1}{2}\{g, f\}_- \tag{83}$$

and

$$\frac{\partial g}{\partial q_\alpha}\frac{\partial f}{\partial p_\alpha} = \frac{1}{2}\{g, f\}_+ - \frac{1}{2}\{g, f\}_- \tag{84}$$

which holds for two corresponding canonical functions.

2. Modern notations

The multiplication of two different wave functions ψ^* and φ in modern notations is defined as,

$$\psi^* \varphi := \langle \psi | \varphi \rangle \tag{85}$$

where ψ and φ are two different state vectors defined in Hilbert space $\mathcal{H}$.
and the integration of $\psi^* \varphi$ w.r.t. q (configuration space) in modern notations is defined as,

$$\int \psi^* \varphi \, dq := \langle \psi | \hat{I} | \varphi \rangle \tag{86}$$

where $\hat{I}$ is the unitary operator defined as,

$$\hat{I}\psi = I\psi \tag{87}$$

where I is the unit matrix. Eq (87) is called the unitary eigenvalue equation (See, "A Theoretical Study of ψ-waves", eq (162) and eq (163)).
and in the consideration of eq (85) and eq (86), I get

$$\langle \psi | \hat{I} | \varphi \rangle := \int \langle \psi | \varphi \rangle \, dq \tag{88}$$

and the multiplication of ψ^* and $(\hat{f}\varphi)$ in its modern notations is defined as,

$$\psi^* (\hat{f}\varphi) := \left\langle \psi \middle| \hat{f}\varphi \right\rangle \tag{89}$$

where $\hat{f}$ is a classical or quantum mechanical operator.
and the integration of $\psi^*(\hat{f}\varphi)$ w.r.t. q (configuration space) in modern notations is defined as,

$$\int \psi^* (\hat{f}\varphi) \, dq := \langle \psi | \hat{f} | \varphi \rangle \tag{90}$$

and in the consideration of eq (89) and eq (90), I get

$$\langle \psi | \hat{f} | \varphi \rangle := \int \left\langle \psi \middle| \hat{f}\varphi \right\rangle dq \tag{91}$$

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